EQUALITY OF TROPICAL RANK AND DIMENSION FOR TROPICAL LINEAR SERIES

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Abstract. The tropical rank of a semimodule of rational functions on a metric graph mirrors the concept of rank in linear algebra. Defined in terms of the maximal number of tropically independent elements within the semimodule, this quantity has remained elusive due to the challenges of computing it in practice. In this note, we establish that the tropical rank is, in fact, precisely equal to the topological dimension of the semimodule, one more than the dimension of the associated linear system of divisors. Moreover, we show that the equality of divisorial and tropical ranks in the definition of tropical linear series is equivalent to the pure dimensionality of the corresponding linear system. We conclude with complementary results and questions on combinatorial and topological properties of the tropical rank.

1. Introduction

Starting from the pioneering work by Baker–Norine [BN07] and the subsequent works on algebraic geometry of tropical curves, tropical methods have been quite successful in the study of the geometry of curves and their moduli spaces. We refer to the survey papers [BJ16, JP21] for a sample of results. The main results of this note are motivated by these developments.

Let Γ be a metric graph (see Section 2 for the precise definition). Denote by $\mathrm{Rat}(\Gamma)$ the union of the set of piecewise affine linear functions on Γ with integral slopes and the constant function on Γ with value ∞ everywhere. Endowed with the two operations of pointwise minimum, denoted by $\oplus$, and pointwise addition of constants, denoted by $\odot$, $\mathrm{Rat}(\Gamma)$ becomes a semimodule over the semifield of tropical numbers $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$.

Let D be a divisor of degree d on Γ. The Riemann–Roch space $\mathrm{Rat}(D) \subset \mathrm{Rat}(\Gamma)$ associated to D is defined by

$$\mathrm{Rat}(D) = \Big\{ f \in \mathrm{Rat}(\Gamma) \smallsetminus \{\infty\} \ \Big| \ \mathrm{div}(f) + D \geq 0 \Big\} \cup \{\infty\}.$$

Here, for $f \neq \infty$, the divisor of f, denoted by $\mathrm{div}(f)$, is given by

$$\mathrm{div}(f) = \sum_{x \in \Gamma} \mathrm{ord}_x(f)\,(x),$$

with the order of vanishing function defined by negative sum of the slopes of f along unit tangent directions to Γ at x,

$$\mathrm{ord}_x(f) = -\sum_{\nu \in \mathrm{T}_x\Gamma} \mathrm{sl}_\nu f.$$

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Endowed with the operations $\oplus$ and $\odot$, $\mathrm{Rat}(D)$ becomes a semimodule over $\mathbb{T}$.

Let M be a subsemimodule of $\mathrm{Rat}(D)$. The linear system associated to the pair (D, M), denoted by $|(D, M)|$, is defined by

$$|(D,M)| = \left\{E = \mathrm{div}(f) + D \;\middle|\; f \in M\right\}.$$

We naturally view $|(D, M)|$ as a subset of the symmetric product

$$\Gamma^{(d)} = \Gamma^d / \mathfrak{S}_d,$$

where the symmetric group $\mathfrak{S}_d$ of degree d acts on Γ^d by permuting the coordinates. In the case $M = \mathrm{Rat}(D)$, we get the complete linear system $|D|$.

The symmetric product $\Gamma^{(d)}$ has a natural polyhedral structure, see [Ami13, § 2.1]. In the case $M = \mathrm{Rat}(D)$, it was proved in [HMY12] that $\mathrm{Rat}(D)$ is finitely generated, and a polyhedral structure for the complete linear system $|D|$ was defined in *loc. cit.* We will prove in Section 2 that for any finitely generated subsemimodule $M \subseteq \mathrm{Rat}(D)$, the linear system $|(D, M)|$ inherits a polyhedral structure as a subspace of $\Gamma^{(d)}$.

For a finitely generated subsemimodule M in $\mathrm{Rat}(D)$, we define its *dimension* as the topological dimension of the associated linear system, increased by one:

$$\dim(M) = \dim(D, M) = \dim|(D, M)| + 1.$$

More generally, for any subsemimodule $M \subseteq \mathrm{Rat}(D)$, we define its dimension as the supremum (in fact, the maximum, by finite generation of $\mathrm{Rat}(D)$) of $\dim(N)$ over all finitely generated subsemimodules N of M:

$$\dim(M) = \sup\left\{\dim(N) \;\middle|\; N \text{ is a finitely generated subsemimodule of } M\right\}.$$

The dimension is an intrinsic numerical invariant of M, meaning that it does not depend on the choice of the divisor, see Proposition 2.2. The notation $\dim(M)$ is thus consistent. By analogy with the finitely generated case, we define, for any subsemimodule $M \subseteq \mathrm{Rat}(D)$, $\dim|(D, M)| = \dim(M) - 1$. Equivalently, this is the supremum of $\dim|(D, N)|$ over finitely generated subsemimodules $N \subseteq M$.

We recall from [JP14] that a family of functions $f_1, \dots, f_r$ in $\mathrm{Rat}(\Gamma) \smallsetminus \{\infty\}$ is called *tropically dependent* if there exist real numbers $c_1, \dots, c_r$ such that the minimum in

$$\min_{1 \le i \le r} (f_i(x) + c_i)$$

is achieved at least twice for every $x \in \Gamma$. Otherwise, the family is called *tropically independent*.

For a subsemimodule M of $\mathrm{Rat}(\Gamma)$, we define the *tropical rank* $r_{\mathrm{trop}}(M)$ as the maximum integer r for which there exist r tropically independent elements $f_1, \dots, f_r$ in M. Also, we define

$$r_{\mathrm{trop}}|(D, M)| = r_{\mathrm{trop}}(M) - 1.$$

Our first result is the following theorem (see Section 3 for the proof).

Theorem 1.1. *For each subsemimodule $M \subseteq \mathrm{Rat}(D)$, we have*

$$r_{\mathrm{trop}}(M) = \dim(M) \qquad \textit{and} \qquad r_{\mathrm{trop}}|(D, M)| = \dim|(D, M)|.$$

In particular, we have $r_{\text{trop}}|D| = \dim |D|$.

Motivated by applications, a combinatorial theory of (non-necessary complete) linear series was developed in recent works [AG22, JP22]. A combinatorial linear series of rank r in both of these works is a finitely generated subsemimodule $M \subseteq \text{Rat}(D)$ which verifies $r(D, M) = r_{\text{trop}}|(D, M)|$, with some extra constraints. Here, $r(D, M)$ is the divisorial rank of (D, M), introduced originally in [BN07] in the case $M = \text{Rat}(D)$. It is defined as the maximum integer $r \geq -1$ such that for any effective divisor E of degree r, there exists an element $f \in M$ with $\text{div}(f) + D - E \geq 0$.

Using Theorem 1.1, we prove the following reformulation of the equality $r(D, M) = r_{\text{trop}}|(D, M)|$.

Theorem 1.2. *Let $M \subseteq \text{Rat}(D)$ be a finitely generated subsemimodule. The following statements are equivalent.*

(1) The equality $r(D, M) = r_{\text{trop}}|(D, M)|$ holds.
(2) The linear system $|(D, M)|$ is of pure dimension $r(D, M)$.

The proof is given in Section 4 and uses a general result about tropical linear systems formulated in Theorem 4.1.

The final section of this paper contains complementary results and questions on combinatorial and topological properties of semimodules related to the tropical rank, as well as its computability.

2. Preliminaries

In this section, we gather some background on metric graphs and their divisor theory. We refer to the survey paper [BJ16] and [Ami13, HMY12, JP14] for more details. For each positive integer n, we denote by $[n]$ the set of positive integers i satisfying $1 \leq i \leq n$.

2.1. **Linear series on metric graphs.** Let $G = (V, E)$ be a finite connected graph with vertex set V and edge set E. Let $\ell \colon E \to \mathbb{R}_{>0}$ be an edge length function, assigning a positive real number $\ell(e)$ to each edge e of the graph.

To the pair (G, ℓ), we associate a metric space Γ as follows. For each edge $e \in E$, we place a closed interval $I_e = [0, \ell(e)]$ of length $\ell(e)$ between the two vertices of e. The resulting space inherits a natural quotient topology from the topology on the disjoint union of the intervals I_e, identifying endpoints according to the adjacency relations in G.

Moreover, this topology is metrizable via the path metric, where the distance between any two points in Γ is defined as the length of the shortest path connecting them.

A metric space Γ obtained in this way is called a *metric graph*, and the pair (G, ℓ) is called a *model* of Γ. Note that a metric graph that is not a singleton has infinitely many different models.

The group of divisors on Γ, denoted by $\text{Div}(\Gamma)$, is the free abelian group generated by the points of Γ. Explicitly, it consists of finite linear combinations of points of Γ:

$$\text{Div}(\Gamma) = \left\{ \sum_{x \in A \subset \Gamma} n_x(x) \;\middle|\; n_x \in \mathbb{Z} \text{ and } A \text{ a finite set} \right\}.$$

Here, we write (x) for the generator corresponding to the point $x \in \Gamma$. For a divisor $D \in \mathrm{Div}(\Gamma)$ and $x \in \Gamma$, the coefficient of (x) in D is denoted by $D(x)$. The *support* of D, denoted by $\mathrm{Supp}(D)$, is the set of points x with $D(x) \neq 0$. The *degree* of D, denoted by $\deg(D)$, is defined as the sum of its coefficients

$$\deg(D) = \sum_{x \in \Gamma} D(x).$$

The complete linear system $|D|$ is given, using the notation of the introduction, by

$$|D| = |(D, \mathrm{Rat}(D))| = \big\{\mathrm{div}(f) + D \;\big|\; f \in \mathrm{Rat}(D)\big\}.$$

We have a natural embedding

$$\eta \colon |D| \hookrightarrow \Gamma^{(d)}$$

which maps each divisor $E = (p_1) + \cdots + (p_d)$ in $|D|$ to the corresponding point in the symmetric product $\Gamma^{(d)}$, given by

$$\eta(E) = (p_1, \ldots, p_d).$$

Theorem 2.1. *Let D be a divisor of degree d and $M \subseteq Rat(D)$ be a finitely generated subsemimodule. Then, $|(D, M)|$ has the structure of a polyhedral space. Moreover, the embedding $|(D, M)| \subseteq \Gamma^{(d)}$ is piecewise affine.*

Proof. By [HMY12, Thm. 6], $R(D)$ is finitely generated. Let $g_1, \ldots, g_l$ be a set of generators for $R(D)$ and $f_1, \ldots, f_m$ be a set of generators for M. There exist real numbers λ_{ij} for $i \in [m]$ and $j \in [l]$ such that

$$f_i = \min_{j \in [l]} (g_j + \lambda_{ij}).$$

Now, consider the map

$$\Phi \colon \mathbb{R}^m \to \mathbb{R}^l$$

given by

$$\Phi(c_1, \ldots, c_m) = \left(\min_{i \in [m]} (c_i + \lambda_{ij})\right)_{j=1}^{l}.$$

In other words, Φ is the piecewise affine map from $\mathbb{R}^m$ to $\mathbb{R}^l$ given by the tropical matrix multiplication from the right by the $m \times l$ matrix $(\lambda_{ij})_{i,j}$.

Define the map

$$\Psi \colon \mathbb{R}^l \to |D|$$

by sending each $(x_1, \ldots, x_l)$ to the element $D + \mathrm{div}(f)$ in $|D|$ with

$$f = \min_{j \in [l]} (g_j + x_j).$$

It follows from the description of the polyhedral structure on $|D|$ given in [HMY12] that Ψ is a piecewise affine map.

Finally, the natural embedding

$$\eta \colon |D| \hookrightarrow \Gamma^{(d)}$$

is a piecewise affine map, see [Ami13].

The subset $|(D,M)|$ of $\Gamma^{(d)}$ is precisely the image of the composition of the maps we have constructed. More precisely, the subset $|(D,M)| \subseteq \Gamma^{(d)}$ is the image of the piecewise affine map $\eta \circ \Psi \circ \Phi$. This completes the proof of both statements in the theorem. ☐

Here are two observations for future use.

Proposition 2.2. *Let D and D' be divisors and M be a closed subsemimodule of $\mathrm{Rat}(\Gamma)$ included both in $\mathrm{Rat}(D)$ and $\mathrm{Rat}(D')$. Then $\dim(D,M) = \dim(D',M)$.*

Proof. The addition by $D' - D$ provides a homeomorphism (in fact, an isomorphism of polyhedral spaces) from $|(D,M)|$ to $|(D',M)|$, from which the proposition follows. ☐

Proposition 2.3. *For each closed subsemimodule M in $\mathrm{Rat}(\Gamma)$, $r_{\mathrm{trop}}(M)$ is the supremum of $r_{\mathrm{trop}}(N)$ over finitely generated subsemimodules $N \subseteq M$.*

Proof. The result follows directly from the observation that a tropically independent family $f_1, \dots, f_r$ in M remains independent in the finitely generated subsemimodule $\langle f_1, \dots, f_r \rangle$. ☐

We also state the following comparison result.

Proposition 2.4. *Given a point $x \in \Gamma$ and a unit tangent vector ν based at x, let n_ν be the number of distinct slopes along ν taken by functions $f \in M$. We have the inequalities*

$$r(D,M) + 1 \leq n_\nu \leq r_{\mathrm{trop}}|(D,M)| + 1.$$

Proof. The first inequality follows by choosing an effective divisor E made up of $r(D,M)$ distinct points close to x in the direction of ν and applying the definition of the divisorial rank. The second inequality comes from the fact that functions which coincide at x but have pairwise distinct slopes along ν are tropically independent (see [AG22, Rem. 6.6]). ☐

2.2. **Certificates of independence.** We will need the following result on certificates of independence, given in [FJP20, Thm. 1.6].

Theorem 2.5. *Let $f_1, f_2, \dots, f_r$ be elements of $\mathrm{Rat}(\Gamma)$. The following statements are equivalent.*

(1) The family $f_1, f_2, \dots, f_r$ is tropically independent.
(2) There exist points $x_1, x_2, \dots, x_r$ and real numbers $c_1, c_2, \dots, c_r$ such that for all $i \in [r]$, the minimum over j in

$$\min_{j \in [r]} \left(f_j(x_i) + c_j \right)$$

is achieved exactly once, at $j = i$.

This may be thought of as a "continuous" version of the following result of tropical linear algebra, characterizing tropical linear independence in terms of the non-vanishing of "tropical minors".

Theorem 2.6. *Let $A = \Big(a_{i,j} \;\Big|\; i \in [m], j \in [r] \Big) \in \mathbb{T}^{m \times r}$. The following statements are equivalent.*

(1) The columns $A_{\bullet,1},\dots,A_{\bullet,r}$ *are tropically independent.*
(2) There exist indices $i_1,\dots,i_r\in[m]$ *such that the minimum*

$$\min_{\sigma\in\mathfrak{S}_r}\sum_{k\in[r]} A_{i_k,\sigma(k)},\ \text{over the set of permutations of } r \text{ elements},$$

is achieved exactly once.
(3) There exist $c\in\mathbb{R}^n$ *and distinct elements* $i_1,\dots,i_r\in[m]$ *such that for all* $k\in[r]$*, the minimum*

$$\min_{j\in[r]}(A_{i_k,j}+c_j)$$

is achieved exactly once, for $j=k$*.*

In the special case in which all the entries of A are finite, the equivalence between (1) and (2) follows from Proposition 4.1 and Theorem 5.5 of [DSS05]. The same equivalence was established in more general settings, in Theorem 2.10 of [IR09] and Theorem 4.12 of [AGG12]. The equivalence with (3) can be found in the proof of Theorem 4.12, *ibid.* Alternatively, it can be derived from properties of the optimal assignment problem (linear programming formulation and strong duality theorem). The proofs of [DSS05, AGG12, FJP20] rely on topological fixed-point arguments (Sperner's lemma, or Collatz–Wielandt formula – a consequence of Brouwer's fixed-point theorem).

We refer to [AGG09, DSS05] for more background on the tropicalization of the notion of rank.

3. Proof of Theorem 1.1

Since both $\dim(M)$ and $r_{\text{trop}}(M)$ are given by the maximum of $\dim(N)$ and $r_{\text{trop}}(N)$, respectively, over all finitely generated subsemimodules N of M (see Proposition 2.3 for the tropical rank), it will be enough to consider only the case where M is finitely generated.

Thus, in what follows, we assume that M is finitely generated.

3.1. Proof of the inequality $\dim(M)\geq r_{\text{trop}}(M)$.

We will make use of the certificates of independence.

Let $r=r_{\text{trop}}(M)$, and consider a family of tropically independent elements $f_1,\dots,f_r$ in M. By Theorem 2.5, there exist points $x_1,\dots,x_r$ and real numbers $c_1,\dots,c_r$ such that for all $i\in[r]$, the minimum over j in

$$\min_{j\in[r]}\left(f_j(x_i)+c_j\right),$$

is achieved uniquely at $j=i$.

Let $c=(c_1,\dots,c_r)\in\mathbb{R}^r$ and define the subset $B_\varepsilon\subset\mathbb{R}^r$ as the set of points $p=(p_1,\dots,p_r)$ satisfying

- $p_1=c_1$, and
- $|p_j-c_j|<\varepsilon$ for each $1<j\leq r$.

We claim that for sufficiently small $\varepsilon > 0$, the elements f_p in M, defined for $p \in B_\varepsilon$ by

$$f_p(x) = \min_{1 \leq j \leq r} (f_j(x) + p_j) \quad \text{for all } x \in \Gamma, \tag{1}$$

are all distinct. To prove this, we choose $\varepsilon > 0$ sufficiently small so that, at $x = x_i$, the minimum in (1) is achieved uniquely at $j = i$ for all $p \in B_\varepsilon$. This ensures that the function f_p uniquely determines the value of p.

Since two elements of M which do not differ by a constant correspond to distinct elements in the linear system $|(D, M)|$, we obtain an embedding of B_ε into $|(D, M)|$. By the invariance of the dimension, it follows that $\dim |(D, M)| \geq r - 1$, which implies that $\dim(M) \geq r$. This establishes the inequality $\dim(M) \geq r_{\text{trop}}(M)$, as required. ☐

3.2. **Proof of the inequality** $\dim(M) \leq r_{\text{trop}}(M)$**.** Let $r = r_{\text{trop}}(M)$. Choose a sequence $x_1, x_2, \ldots$ forming a dense subset of Γ, and let $f_1, \ldots, f_m$ be a set of generators of M.

For each integer $K \geq 1$, define the evaluation map

$$\pi_K \colon M \longrightarrow \mathbb{T}^K$$

that sends each function $f \in M$ to its values at $x_1, \ldots, x_K$:

$$\pi_K(f) = (f(x_k))_{1 \leq k \leq K}$$

The semimodule

$$M_K = \pi_K(M) \subset \mathbb{T}^K$$

is finitely generated.

For each $K \geq 1$, we define a canonical section ρ_K of the projection π_K, given by

$$\begin{aligned} \rho_K \colon M_K &\longrightarrow M \\ g &\longmapsto \inf \{ f \in M \mid \pi_K(f) \geq g \} = \min \{ f \in M \mid \pi_K(f) \geq g \}. \end{aligned}$$

Since M is finitely generated, the infimum above is a minimum.

The following proposition provides an explicit expression of the section ρ_K.

Proposition 3.1. *For each $g \in M_K$, we have*

$$\rho_K(g) = \min_{i \in [m]} \big(f_i + c_i^K \big) \tag{2}$$

where

$$c_i^K = c_i^K(g) = \max_{k \in [K]} (g(x_k) - f_i(x_k)).$$

Proof. Let h denote the function on the right-hand side of Equation (2). By the definition of c_i^K, we have

$$f_i(x_k) + c_i^K \geq g(x_k) \qquad \text{for all } k \in [K].$$

From this, we deduce that

$$\pi_K \big(f_i + c_i^K \big) \geq g.$$

This implies the inequality $h \geq \rho_K(g)$.

The other way around, since $f_1, \dots, f_m$ is a generating set for M, there exist real numbers $p_1, \dots, p_m$ such that

$$\rho_K(g) = \min_{1 \le i \le m} (f_i + p_i).$$

Evaluating at x_k, we obtain

$$g(x_k) \le f_i(x_k) + p_i \qquad \text{for all } k \in [K] \text{ and } i \in [m],$$

which implies

$$c_i^K \le p_i \qquad \text{for all } i \in [m].$$

Thus, we conclude that $h \le \rho_K(g)$, proving the proposition. $\square$

Proposition 3.2. *For each positive integer K, the following equalities hold:*

$$\pi_K \circ \rho_K = \mathrm{Id}_{M_K} \qquad\qquad \pi_K \circ \rho_K \circ \pi_K = \pi_K,$$

and

$$\rho_K \circ \pi_K \le \mathrm{Id}_M \qquad\qquad \rho_K \circ \pi_K \circ \rho_K = \rho_K.$$

Proof. This is straightforward. $\square$

Proposition 3.3. *We have*

$$\bigcup_{K \ge 1} \rho_K(M_K) = M.$$

Proof. The inclusion $\subseteq$ holds by definition. We prove the reverse inclusion $\supseteq$.

Let f be an element of M. We show that $f \in \rho_K(M_K)$ for some $K \ge 1$. There exist real numbers λ_i such that

$$f = \min_{i \in [m]} (f_i + \lambda_i).$$

Replacing each f_i with $f_i + \lambda_i$ and removing the unnecessary terms, we may assume without loss of generality that all λ_i are zero and that each f_i contributes to the minimum.

This means that for each $i = 1, \dots, m$, there exists a point y_i of Γ such that

$$f_i(y_i) < f_j(y_i) \quad \text{for all } j \ne i.$$

Thus, for each i, there exists an open set $U_i \subset \Gamma$ such that for all $x \in U_i$, we have

$$f_i(x) < \min_{j \ne i} f_j(x).$$

Since the sequence (x_k) is dense in Γ, we can choose an index $k_i \ge 1$ such that $x_{k_i} \in U_i$. Let $K = \max_i k_i$.

Now, consider $g = \pi_K(f) \in M_K$. We will show that $c_i^K(g) = 0$ for each $i = 1, \dots m$, from which it follows that $f = \rho_K(g)$.

For a given element $i \in [m]$, we have, for each $k \in [K]$,

$$g(x_k) = f(x_k) \le f_i(x_k),$$

where equality holds for $k = k_i$. By Proposition 3.1, this implies that $c_i^K(g) = 0$, as required. $\square$

Proposition 3.4. *We have*

$$\dim(\rho_K(M_K)) \le r.$$

Proof. Recall that r denotes the tropical rank of M. Since M_K is a projection of M, we have

$$r_{\text{trop}}(M_K) \le r_{\text{trop}}(M).$$

Thus, to prove the proposition, it suffices to show that

$$\dim(\rho_K(M_K)) \le r_{\text{trop}}(M_K).$$

The space M_K consists of the column space of the matrix

$$A_K = \Big(f_1(x_k), \ldots, f_m(x_k)\Big)_{1\le k\le K}.$$

By the results in [DSS05], M_K is a polyhedral complex of dimension

$$r_{\text{trop}}(A_K) = r_{\text{trop}}(M_K).$$

To conclude, observe that

$$\rho_K\colon M_K \to \rho_K(M_K) \hookrightarrow M$$

is a piecewise affine isomorphism, as given by the explicit formula for $c_i^K(g)$ (see Proposition 3.1). Consequently,

$$\dim(\rho_K(M_K)) = \dim(M_K),$$

which completes the proof. □

Using the results stated above, we are now in the position to prove the inequality

$$\dim(M) \le r_{\text{trop}}(M),$$

finishing the proof of Theorem 1.1.

As previously stated, we have an injective piecewise affine map

$$\begin{aligned}\varphi\colon M/\mathbb{R} &\longrightarrow \Gamma^{(d)}\\ [f] &\longmapsto [\operatorname{div}(f) + D].\end{aligned}$$

By Proposition 3.3, we obtain

$$\varphi(M/\mathbb{R}) = \bigcup_{K\ge 1} \varphi(\rho_K(M_K)/\mathbb{R}).$$

Moreover, by Proposition 3.4, we have

$$\dim(\varphi(\rho_K(M_K)/\mathbb{R})) \le r - 1$$

for every $K \ge 1$.

Now, let E be a cell of $\varphi(M/\mathbb{R})$ of maximal dimension. Proceeding by contradiction, assume that $\dim(E) \ge r$. Consider the intersection

$$F_K = \varphi(\rho_K(M_K)/\mathbb{R}) \cap E.$$

Since each $\varphi(\rho_K(M_K)/\mathbb{R})$ has dimension at most $r-1$, it follows that F_K has empty relative interior in E.

By the Baire category theorem, the countable union $\bigcup_K F_K$ has empty relative interior in E. However, this contradicts the fact that $\bigcup_K F_K = E$. Thus, we conclude that $\dim(M) \leq r_{\text{trop}}(M)$, which completes the proof. □

4. Proof of Theorem 1.2

We first establish a general result, Theorem 4.1, regarding the connection between the dimension and the divisorial rank of linear systems, from which we deduce Theorem 1.2. In the case where $M = \mathrm{Rat}(D)$, this might be already known to experts, but we could not find it in the literature.

Let D be a divisor of degree d on a metric graph Γ, and $M \subseteq \mathrm{Rat}(D)$ be a finitely generated subsemimodule. By Theorem 2.1, $|(D, M)|$ has a polyhedral structure. Denote by $r(D, M)$ the divisorial rank of (D, M).

Theorem 4.1. *Keeping the same notation as above, all the maximal faces of the polyhedral structure on $|(D, M)|$ have dimension at least $r(D, M)$.*

Using this result and Theorem 1, we easily deduce Theorem 1.2.

Proof of Theorem 1.2. The implication (1) ⇒ (2) follows directly from Theorem 1.1.

For the reverse implication (2) ⇒ (1), note that the equality $r_{\text{trop}}(D, M) = \dim |(D, M)|$, given by Theorem 1.1 again, implies that $\dim |(D, M)| = r(D, M)$. By applying Theorem 4.1, we conclude that all the maximal faces of $|(D, M)|$ have dimension exactly $r(D, M)$, which implies that $|(D, M)|$ is of pure dimension $r(D, M)$. □

We now come to the proof of Theorem 4.1.

Proof of Theorem 4.1. We assume that $|(D, M)|$ is nonempty and $r(D, M) \geq 1$; otherwise, the statement holds trivially. Fix an element $E \in |(D, M)|$, and express it as $E = D + \mathrm{div}(h)$ for $h \in M$. We claim that every closed neighborhood $\mathcal{U}$ of E in $|(D, M)|$ has dimension at least $r(D, M)$, which will prove the theorem.

For each positive integer $r \leq r(D, M)$, let S_r denote the subset of Γ^r consisting of all tuples $\underline{x} = (x_1, \dots, x_r) \in \Gamma^r$ for which there exists a tuple $\underline{y} = (y_1, \dots, y_{d-r}) \in \Gamma^{d-r}$ such that the sum $(x_1) + \cdots + (x_r) + (y_1) + \cdots + (y_{d-r})$ belongs to $\mathcal{U}$. We claim that S_r is a closed subset of Γ^r.

To see this, suppose we have a sequence $\underline{x}_t = (x_{1,t}, \dots, x_{r,t})$ of points of S_r converging to $\underline{x} \in \Gamma^r$. For each t, there is a corresponding tuple $\underline{y}_t = (y_{1,t}, \dots, y_{d-r,t}) \in \Gamma^{d-r}$ such that the sum $(x_{1,t}) + \cdots + (x_{r,t}) + (y_{1,t}) + \cdots + (y_{d-r,t})$ lies in $\mathcal{U}$. By compactness of Γ^{d-r}, and passing to a subsequence if necessary, we can ensure that $\underline{y}_t$ converges to some point $\underline{y} = (y_1, \dots, y_{d-r}) \in \Gamma^{d-r}$. Since $\mathcal{U}$ is closed, the limit $(x_1) + \cdots + (x_r) + (y_1) + \cdots + (y_{d-r})$ of the divisors $(x_{1,t}) + \cdots + (x_{r,t}) + (y_{1,t}) + \cdots + (y_{d-r,t})$ belongs to $\mathcal{U}$, implying that $\underline{x} \in S_r$. Therefore, S_r is closed in Γ^r.

We proceed by induction on r and demonstrate that for each closed neighborhood $\mathcal{U}$ of an element $E \in |(D, M)|$ as above with corresponding sets S_j, $j \leq r(D, M)$, the set S_r contains a nonempty open subset of Γ^r, i.e., it has a nonempty interior in Γ^r. By the definition of $S_{r(D,M)}$, this implies that each $\mathcal{U}$ is of dimension at least $r(D, M)$, as claimed.

We begin by considering the case $r = 1$. Since $r(D, M) \geq 1$, there exists an element $F = D + \mathrm{div}(f)$ in $|(D, M)|$, distinct from E, with $f \in M$. For each $t \in \mathbb{R}$, define $f_t = \min\{t + h, f\}$ and let $E_t = D + \mathrm{div}(f_t)$. We observe that $f_t \in M$ and $E_t \in |(D, M)|$.

For t approaching $-\infty$, we have $E_t = E$, and for t approaching $+\infty$, we have $E_t = F$. Thus, we obtain a one-dimensional segment in $\mathcal{U}$, with one endpoint at E, completing the proof in this case.

Assume $r \geq 2$ and that the claim has been proved for $r - 1$. Consider the projection map $\pi \colon S_r \to S_{r-1}$, given by $(x_1, \dots, x_{r-1}, x_r) \mapsto (x_1, \dots, x_{r-1})$. For each point $\underline{x} = (x_1, \dots, x_{r-1})$ in S_{r-1}, the fiber $\pi^{-1}(\underline{x})$ is nonempty. We claim that there exists a nonempty open subset $\mathcal{W}$ of Γ^{r-1} contained in S_{r-1} over which each fiber of π contains a one-dimensional segment of Γ (possibly different for distinct points).

To prove this, let $\mathcal{U}'$ be a closed neighborhood of E included in the interior of $\mathcal{U}$. The subset S'_{r-1} associated to $\mathcal{U}'$, defined analogously to the previous construction, is a subset of S_{r-1}, and by the induction hypothesis, it has a nonempty interior, which we will show to be the desired nonempty open subset $\mathcal{W}$. We need to show that for each $\underline{x} \in S'_{r-1}$, the fiber $\pi^{-1}(\underline{x})$ contains a one-dimensional segment.

Let $\underline{y} \in \Gamma^{d-r+1}$ be a point such that $E' = (x_1) + \cdots + (x_{r-1}) + (y_1) + \cdots + (y_{d-r+1}) \in \mathcal{U}'$. Since $\mathcal{U}'$ is contained in the interior of $\mathcal{U}$, $\mathcal{U}$ is a closed neighborhood of E'. Write $E' = D + \mathrm{div}(f)$ for some $f \in M$ and let $M' = M - f$ be the set consisting of all elements of the form $g - f$ for $g \in M$. Then, M' is a finitely generated subsemimodule of $\mathrm{Rat}(E')$.

Moreover, the pair $((y_1) + \cdots + (y_{d-r+1}), M')$ has divisorial rank at least one. Therefore, repeating the argument from the case $r = 1$, there exists a segment of dimension one in $\mathcal{U}$ of the form $(x_1) + \cdots + (x_{r-1}) + (y_{1,t}) + \cdots + (y_{d-r+1,t})$ with one endpoint at E'. This implies that $\pi^{-1}(\underline{x})$ contains a one-dimensional segment.

Recall that $\mathcal{W}$ is a nonempty open subset of Γ^{r-1} contained in S_{r-1} over which each fiber of π contains a one-dimensional segment of Γ. Let $(I_j)_{j=1}^{\infty}$ be a countable collection of one-dimensional segments in Γ such that each one-dimensional segment of Γ contains at least one of the I_j's, for some $j \in \mathbb{N}$. In particular, each fiber of π over $\mathcal{W}$ contains one of the segments I_j.

For each I_j, define $\mathcal{A}_j$ as the subset of $\mathcal{W}$ consisting of all the points $\underline{x}$ with $I_j \subseteq \pi^{-1}(\underline{x})$. By continuity of the projection map π and the closedness of S_r, each $\mathcal{A}_j$ is a closed subset of $\mathcal{W}$. Moreover, the union of the sets $\mathcal{A}_j$ covers the full open set $\mathcal{W}$. Therefore, by the Baire category theorem, there exists some $j \in \mathbb{N}$ such that $\mathcal{A}_j$ has a nonempty interior.

We conclude that the product $\mathcal{A}_j \times I_j$ is contained in S_r. Moreover, it has a nonempty interior in Γ^r, as required. □

5. Further discussion

In this section, we discuss some complementary results and a possible extension of our main theorem to higher dimension.

5.1. **Finite evaluation maps on semimodules of rational functions.** Given a subsemimodule M of $\mathrm{Rat}(D)$, it is natural to ask whether functions $f \in M$ can be fully characterized by the set of their values on a well-chosen finite set of points in Γ. More precisely:

Question 5.1. *Given a subsemimodule M of $\mathrm{Rat}(D)$, do there exist a positive integer K and points $x_1, \dots, x_K \in \Gamma$ such that the evaluation map $\pi\colon M \to \mathbb{T}^K$, defined by*

$$\pi(f) = (f(x_k))_{1 \le k \le K},$$

is injective?

This is motivated by the following observation.

Proposition 5.2. *Suppose that M is a module for which the answer to Question 5.1 is positive. Then, $r_{\mathrm{trop}}(M) = r_{\mathrm{trop}}(\pi(M))$.*

Proof. A family $\pi(f_1), \dots, \pi(f_r)$ with $f_1, \dots, f_r \in M$ is tropically dependent if and only if there are real numbers $\lambda_1, \dots, \lambda_r$ such that for all $k \in [K]$, the minimum in $\min_{j \in [r]}(\lambda_j + f_j(x_k))$ is achieved at least twice. This condition can be rewritten as

$$\pi(g) = \pi(g_j) \text{ for all } j, \text{ with } g = \min_{s \in [r]}(\lambda_s + f_s) \text{ and } g_j = \min_{s \in [r] \smallsetminus \{j\}}(\lambda_s + f_s).$$

If π is injective, if follows that $g = g_j$ holds for all $j \in [r]$. This means that $f_1, \dots, f_r$ are tropically dependent. We infer that $r_{\mathrm{trop}}(M) \le r_{\mathrm{trop}}(\pi(M))$. The other inequality is trivial. $\square$

The answer to Question 5.1 depends on the structure of M, namely, on the number of distinct slopes that functions in M realize along any edge of the metric graph. To formulate this dependence, we refine the combinatorial model of Γ so that the set of slopes taken by functions $f \in M$ along unit tangent directions becomes constant on each edge. More precisely, for each pair (e, v) consisting of an edge e and an extremity v of e, and any point x of Γ lying on the half-closed interval $e \smallsetminus \{v\}$, let ν_x denote the unit tangent vector at x directed toward v. Then, we require that the set $\mathrm{sl}_{(e,v)}(M) \coloneqq \left\{\mathrm{sl}_{\nu_x} f \;\middle|\; f \in M\right\}$ be independent of x. A compactness argument, using the closedness of $\mathrm{Rat}(D)$, ensures that such a model always exists.

Let us first assume that $\mathrm{sl}_{(e,v)}(M)$ has at most two elements for every pair (e, v) in the graph. This is the case, for instance, if $r_{\mathrm{trop}}(M) \le 1$ (see Proposition 2.4). Since, on each edge e with endpoints u and v and for every $f \in M$, the function f changes slope at most once along e, the values of $f(u)$ and $f(v)$ completely determine f on the entire edge e. Consequently, if $x_1, \dots, x_K$ are chosen as the vertices of Γ, the corresponding evaluation map π is injective.

On the contrary, let us assume that $\mathrm{sl}_{(e,v)}(M)$ contains at least three elements for some pair (e, v). In this case, for any choice of points $x_1, \dots, x_K \in \Gamma$, the associated evaluation map π is generally not injective. This means that no finite evaluation map can fully characterize the functions $f \in M$.

To illustrate this, let us further refine the combinatorial model of Γ and assume the following:

(1) there exists an edge e' (in this new model) contained in the original edge e such that there are functions $f_1, f_2, f_3 \in M$ with constant slopes s_1, s_2, s_3 on e' satisfying $s_1 < s_2 < s_3$; and
(2) no point x_i lies in the interior of e', i.e., $x_i \notin \mathring{e}'$ for all i.

It is easy to see that if M contains sufficiently many functions – for instance, if $M = \mathrm{Rat}(D)$ – then, using the functions f_i, we can construct infinitely many distinct functions $f \in M$ that coincide outside $\mathring{e}'$ (see Figure 1). Since these functions share the same value at all the chosen points x_i, they have the same image under the evaluation map. Since this argument can be applied to any choice of the points $x_1, \dots, x_K$, for any K, this proves that the finite evaluation maps cannot be injective in general.

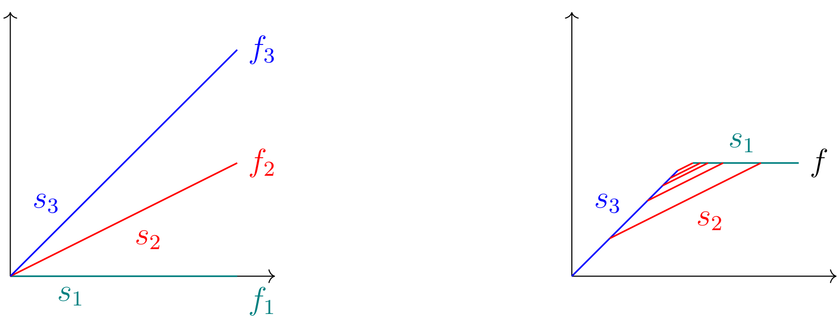

FIGURE 1. Construction of infinitely many distinct functions using three different slopes.

5.2. **Closedness versus finite generation of subsemimodules.** In general, $\mathrm{Rat}(D)$ contains closed subsemimodules that are not finitely generated (see the example below). However, subsemimodules of $\mathrm{Rat}(D)$ arising from the tropicalization of linear series on curves are always finitely generated, see [AG22, § 9.4].

Example 5.3. Let Γ be a metric graph with model $(G = (V, E), \ell)$, and let x_1, x_2, x_3 be three distinct points on an edge $e = \{u, v\}$ of G in Γ, such that x_2 is the midpoint of the edge and lies at the midpoint of the segment joining x_1 and x_3. Consider the divisor $D = n\,(u) + n\,(v)$ for a sufficiently large positive integer n, and define $M \subset \mathrm{Rat}(D)$ to be the set of all functions $f \in \mathrm{Rat}(D)$ satisfying the inequality $-\varepsilon + 2f(x_2) \geq f(x_1) + f(x_3)$, for $\varepsilon > 0$ small. Then, M forms a closed subsemimodule of $\mathrm{Rat}(D)$. However, for $\varepsilon > 0$ small enough, M is not finitely generated.

To see this, consider the evaluation map

$$\Phi\colon \mathrm{Rat}(D) \to \mathbb{T}^3, \qquad \Phi(f) = (f(x_1), f(x_2), f(x_3)).$$

If M were finitely generated, then $\Phi(M)$ would also be finitely generated, implying that it would have only finitely many extreme points.

Now, we shall use the following characterization of tropical extreme points, which can be found in [GK07] (proof of Theorem 3.1): if N is a closed subsemimodule of $\mathbb{T}^n$, then a point $z \in N$ is extreme if and only if there is an index $i \in [n]$ such that $z - z_i\, e_i$ is a maximal

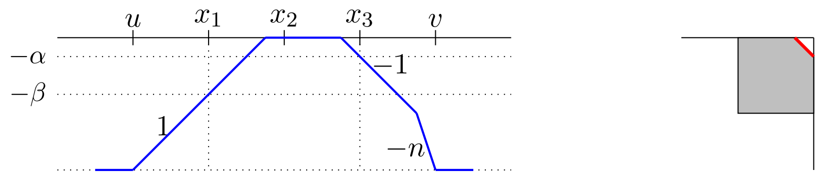

FIGURE 2. A parametric family of functions $f \in \mathrm{Rat}(D)$ (left). Fragment of the cross section $\{(f(x_1), f(x_3)) \mid f \in M, f(x_2) = 0\}$ with an infinite upper Pareto set depicted in red (right).

element of the cross section $S_i(N)$ of N given by $S_i(N) = \{w \in N \mid w_i = 0\}$. Coming back to our example, it suffices to check that $S_2(\Phi(M))$ has an infinite set of maximal elements.

To show this, consider the function f in $\mathrm{Rat}(\Gamma)$, constant on $\Gamma \smallsetminus e$, and given on the edge e by

$$f(x) = \max(u - x_1 - \beta, \min(x - x_1 - \beta, x_3 - \alpha - x, u - x_1 - \beta + n(v - x))) \quad \text{for all } x \text{ in } e,$$

with x a parameter on e and α, β real numbers satisfying $0 \leq \alpha \leq \beta \leq L \coloneqq x_3 - x_2$. For n large enough, this function has the shape shown in Fig. 2 (the slopes are indicated on the figure), with $\mathrm{ord}_u(f) = -1$, $\mathrm{ord}_v(f) = -n$, and $\mathrm{ord}_x(f) \geq 0$ elsewhere, thus $f \in \mathrm{Rat}(D)$. Observe that $f(x_1) = -\beta$ and $f(x_3) = -\alpha$, where all the values $0 \leq \alpha \leq \beta \leq L$ are realizable.

This entails that the set $\widetilde{S}_2 = \{(y_1, y_3) \mid (y_1, 0, y_3) \in \Phi(\mathrm{Rat}(D))\}$ contains every element of the form $(-\beta, -\alpha)$. By symmetry, we deduce that $\widetilde{S}_2 \supset [-L, 0]^2$. It follows that

$$S_2(\Phi(M)) \cap [-L, 0]^2 = \{(y_1, y_3) \mid (y_1 + y_3)/2 \leq -\varepsilon \text{ and } -L \leq y_1, y_2 \leq 0\}.$$

Hence, $S_2(\Phi(M)) \cap [-L, 0]^2$ has an infinite set of maximal elements, consisting of its upper boundary, defined by the equality $(y_1 + y_3)/2 = -\varepsilon$. All the elements of this boundary are maximal elements of $S_2(\Phi(M))$ as well, showing that the set of extreme points of $\Phi(M)$ is infinite. This is illustrated in Figure 2 (right). ◇

When M is finitely generated, by Theorem 2.1, the linear system $|(D, M)|$ admits a polyhedral structure. In Section 1, for a more general subsemimodule M, we defined the dimension of the corresponding linear system as a supremum over finitely generated subsemimodules because we do not know the answer to the following question.

Question 5.4. *Let $M \subset \mathrm{Rat}(D)$ be closed. Does $|(D, M)|$ admit a polyhedral structure?*

5.3. **Effective computation of the rank.** Checking whether the columns of a matrix are tropically independent reduces to solving a "mean payoff game", a well-studied example of a deterministic repeated game, see [AGG12, Thm. 4.12]. The converse also holds: solving a mean payoff game reduces to the problem of checking tropical linear independence [GP15]. The question of the existence of a polynomial-time algorithm to solve mean payoff games, first raised in [GKK88], remains unsettled. When formulated as a decision problem, mean payoff games belong to the complexity class NP ∩ coNP [ZP96], making them unlikely to be NP-complete. Several practical algorithms have been developed to solve mean payoff games, see [AGQS23] for a recent discussion in relation to tropical geometry.

Computing the tropical rank of a matrix turns out to be a harder problem than checking tropical linear independence. In fact, this problem is known to be NP-hard, even for tropical matrices with entries in $\{0,1\}$, see [KR05, Thm. 13]. We use this result to deduce the following analogue for semimodules of rational functions.

Proposition 5.5. *Computing the tropical rank of a subsemimodule $M \subseteq \mathrm{Rat}(D)$ is NP-hard.*

Some technical details on the encoding of M and the metric graph are in order. We assume that the metric graph is connected and given by a model with rational edge lengths. We also assume that the divisor D is supported on vertices of the graph. Furthermore, we assume that the subsemimodule $M \subseteq \mathrm{Rat}(D)$ is given explicitly by a finite set of generators, each described by a collection of intervals on which it is affine, as well as by the integral slope. This uniquely determines the affine functions up to an additive constant, which is required to be rational.

Proof of Proposition 5.5. Consider a matrix $A \in \mathbb{T}^{m\times n}$ with entries in $\{0,1\}$. First, we define a metric graph Γ of model $(G = (V,E), \ell)$, with $G = (V,E)$ the complete graph on m vertices and $\ell(e) = 2$ for all edges of G. Let $V = \{v_1,\dots,v_m\}$ be the vertex set and $E = \big\{\{v_i, v_s\} \;\big|\; i,s \in [m], i \neq s\big\}$ the edge set of G. For every edge $\{v_i, v_s\}$, denote by w_{is} the midpoint of the edge.

We associate to A a module M of rational functions on Γ, generated by the following rational functions $f_1,\dots,f_n$. For each $j \in [n]$, set

$$f_j(v_i) = A_{ij} \text{ and } f_j(w_{is}) = \min(A_{ij}, A_{sj}) \text{ for all pair of distinct elements } i,s \in [m].$$

We extend f_j to Γ by linear interpolation. It is easy to see that $f_j \in \mathrm{Rat}(\Gamma)$.

Let

$$D = \sum_{i\in[m]} (m-1)(v_i) + \sum_{i<s, i,s\in[m]} (w_{is}).$$

Since the entries of the matrix A belong to $\{0,1\}$, for each edge $\{v_i, v_s\}$ of G, one of the following four cases occurs:

- $f_j(v_i) = f_j(w_{ij}) = f_j(v_s) = 0$,
- $f_j(v_i) = f_j(w_{is}) = 0 < f_j(v_s) = 1$,
- $f_j(w_{is}) = f_j(v_s) = 0 < f_j(v_i) = 1$,
- $f_j(v_i) = f_j(w_{ij}) = f_j(v_s) = 1$.

A simple verification then shows that $\mathrm{div}(f_j) + D \geq 0$, and therefore $M \subseteq \mathrm{Rat}(D)$.

Now consider the family of points $x_1,\dots,x_K$ consisting of the vertices $v_1,\dots,v_m$ together will all the midpoints w_{is} with $i < s$ and $i,s \in [m]$. We deduce by checking the above four cases that every f_j satisfies the "two-slopes" condition stated in Subsection 5.1, so that the restriction map $\pi\colon M \to \mathbb{T}^K$ is injective.

By Proposition 5.2, $r_{\text{trop}}(M) = r_{\text{trop}}(\pi(M)) = r_{\text{trop}}(B)$ where the matrix $B \in \mathbb{T}^{K\times n}$ is given explicitly by $B_{kj} = A_{ij}$ if x_k is equal to some vertex v_i, and $B_{kj} = \min(A_{ik}, A_{sj})$ if x_k is equal to some midpoint w_{is}. Therefore, the rows of the matrix B comprise all the rows of A together with additional rows each of which is a tropical sum of pair of rows in A. It follows that $r_{\text{trop}}(B) = r_{\text{trop}}(A)$.

By [KR05, Thm. 13], checking whether $r_{\text{trop}}(A) \geq r$ is an NP-hard problem. We infer that checking whether $r_{\text{trop}}(M) \geq r$ for a finitely generated subsemimodule $M \subseteq \mathrm{Rat}(D)$ is also NP-hard. □

5.4. **Higher dimension.** Let $Y \subseteq \mathbb{R}^n$ be a polyhedral subspace (e.g., a tropical subvariety). Let $\mathrm{Rat}(Y, \mathbb{R})$ be the union of ∞ and the set of piecewise linear affine functions on Y (with non-necessarily integral slopes). Endowed with the operation of tropical addition and tropical multiplication by constants, $\mathrm{Rat}(Y, \mathbb{R})$ is a semimodule over $\mathbb{T}$. Let M be a finitely generated subsemimodule of $\mathrm{Rat}(Y, \mathbb{R})$. For example, if Y is a tropical subvariety, and D is a divisor on Y, then M may be a finitely generated subsemimodule of $\mathrm{Rat}(D)$, where $\mathrm{Rat}(D)$ is the union of $\{\infty\}$ and the set of piecewise affine functions on Y with integral slopes. We define $r_{\text{trop}}(M)$ as the maximum integer r such that there exist tropically independent elements $f_1, \dots, f_r \in M$.

Let $g_1, \dots, g_l$ be a generating set for M. Consider the map

$$\Psi \colon \mathbb{R}^l \to M, \qquad (c_1, \dots, c_l) \mapsto \min_{j \in [l]} \bigl(g_j + c_j\bigr).$$

We define a notion of dimension for M as follows. Consider an element $f \in M$.

Proposition 5.6. *The subset $\Psi^{-1}(f) \subset \mathbb{R}^l$ is polyhedral.*

Proof. Choose a polyhedral structure Δ on Y such that the generators $g_1, \dots, g_l$ and f are affine on each face of Δ. For each face $\sigma \in \Delta$ and $(c_1, \dots, c_l) \in \Psi^{-1}(f)$, since $\min_{j\in[l]}(g_j + c_j)_{|\sigma} = f_{|\sigma}$, and $f_{|\sigma}$ is affine, we get the existence of $j = \mu(\sigma) \in [l]$ such that

$$f_{|\sigma} = (g_{\mu(\sigma)} + c_{\mu(\sigma)})_{|\sigma} \leq (g_i + c_i)_{|\sigma}, \text{ for all } i \in [l].$$

For each function $\mu \colon \Delta \to [l]$, let C_μ be the set of all points $(c_1, \dots, c_l) \in \mathbb{R}^l$ such that the inequality above is satisfied for all $\sigma \in \Delta$. This is a polyhedral subset of $\mathbb{R}^l$. Moreover, $\Psi^{-1}(f)$ is the union of the sets C_μ. We infer the result. □

We define the dimension of M, denoted by $\dim(M)$, as

$$\dim(M) = \max_{f \in M} \bigl[l - \dim\bigl(\Psi^{-1}(f)\bigr)\bigr].$$

Question 5.7. *Let M be a finitely generated subsemimodule of $\mathrm{Rat}(Y, \mathbb{R})$. Do we have the equality $r_{\text{trop}}(M) = \dim(M)$?*

A positive answer would generalize Theorem 1.1. The proof would require extending Theorem 2.5 to arbitrary dimensions, and adapting our proof of Theorem 1.1.

References

[AG22] Omid Amini and Lucas Gierczak. Limit linear series: combinatorial theory. *arXiv:2209.15613*, 2022.

[AGG09] Marianne Akian, Stéphane Gaubert, and Alexander Guterman. Linear independence over tropical semirings and beyond. In *Tropical and idempotent mathematics*, volume 495 of *Contemp. Math.*, pages 1–38. Amer. Math. Soc., Providence, RI, 2009.

[AGG12] Marianne Akian, Stéphane Gaubert, and Alexander Guterman. Tropical polyhedra are equivalent to mean payoff games. *Internat. J. Algebra Comput.*, 22(1):1250001, 43, 2012.

[AGQS23] Marianne Akian, Stéphane Gaubert, Yang Qi, and Omar Saadi. Tropical linear regression and mean payoff games: or, how to measure the distance to equilibria. *SIAM J. Discrete Math.*, 37(2):632–674, 2023.

[Ami13] Omid Amini. Reduced divisors and embeddings of tropical curves. *Trans. Amer. Math. Soc.*, 365(9):4851–4880, 2013.

[BJ16] Matthew Baker and David Jensen. Degeneration of linear series from the tropical point of view and applications. In *Nonarchimedean and tropical geometry*, Simons Symp., pages 365–433. Springer, [Cham], 2016.

[BN07] Matthew Baker and Serguei Norine. Riemann–Roch and Abel–Jacobi theory on a finite graph. *Adv. Math.*, 215(2):766–788, 2007.

[DSS05] Mike Develin, Francisco Santos, and Bernd Sturmfels. On the rank of a tropical matrix. In *Combinatorial and computational geometry*, volume 52 of *Math. Sci. Res. Inst. Publ.*, pages 213–242. Cambridge Univ. Press, Cambridge, 2005.

[FJP20] Gavril Farkas, David Jensen, and Sam Payne. The Kodaira dimensions of $\mathcal{M}_{22}$ and $\mathcal{M}_{23}$. *arXiv:2005.00622*, 2020.

[GK07] Stéphane Gaubert and Ricardo D. Katz. The Minkowski theorem for max-plus convex sets. *Linear Algebra Appl.*, 421(2-3):356–369, 2007.

[GKK88] V. A. Gurvich, A. V. Karzanov, and L. G. Khachiyan. Cyclic games and finding minimax mean cycles in digraphs. *Zh. Vychisl. Mat. i Mat. Fiz.*, 28(9):1407–1417, 1439, 1988.

[GP15] Dima Grigoriev and Vladimir V. Podolskii. Complexity of tropical and min-plus linear prevarieties. *Comput. Complexity*, 24(1):31–64, 2015.

[HMY12] Christian Haase, Gregg Musiker, and Josephine Yu. Linear systems on tropical curves. *Math. Z.*, 270(3-4):1111–1140, 2012.

[IR09] Zur Izhakian and Louis Rowen. The tropical rank of a tropical matrix. *Comm. Algebra*, 37(11):3912–3927, 2009.

[JP14] David Jensen and Sam Payne. Tropical independence I: Shapes of divisors and a proof of the Gieseker-Petri theorem. *Algebra & Number Theory*, 8(9):2043–2066, 2014.

[JP21] David Jensen and Sam Payne. Recent developments in Brill–Noether theory. *arXiv:2111.00351*, 2021.

[JP22] David Jensen and Sam Payne. Tropical linear series and tropical independence. *arXiv:2209.15478*, 2022.

[KR05] Ki H. Kim and Fred W. Roush. Factorization of polynomials in one variable over the tropical semiring. *arXiv:math/0501167v2*, 2005.

[ZP96] Uri Zwick and Mike Paterson. The complexity of mean payoff games on graphs. *Theoret. Comput. Sci.*, 158(1-2):343–359, 1996.

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