EQUALITY OF TROPICAL RANK AND DIMENSION FOR TROPICAL LINEAR SERIES

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Abstract. The tropical rank of a semimodule of rational functions on a metric graph mirrors the concept of rank in linear algebra. Defined in terms of the maximal number of tropically independent elements within the semimodule, this quantity has remained elusive due to the challenges of computing it in practice. In this note, we establish that the tropical rank is, in fact, precisely equal to the topological dimension of the semimodule, one more than the dimension of the associated linear system of divisors. Moreover, we show that the equality of divisorial and tropical ranks in the definition of tropical linear series is equivalent to the pure dimensionality of the corresponding linear system. We conclude with several complementary results and questions on combinatorial, topological, and computability properties of the tropical rank.

1. Introduction

Starting from the pioneering work by Baker–Norine [7] and the subsequent works on algebraic geometry of tropical curves, tropical methods have been quite successful in the study of the geometry of curves and their moduli spaces. We refer to the survey papers [6, 20] for a sample of results. The main results of this note are motivated by these developments.

Let Γ be a metric graph (see Section 2 for the precise definition). Denote by $\mathrm{Rat}(\Gamma)$ the union of the set of piecewise linear functions on Γ with integral slopes and the constant function on Γ with value ∞ everywhere. Endowed with the two operations of pointwise minimum, denoted by $\oplus$, and pointwise addition of constants, denoted by $\odot$, $\mathrm{Rat}(\Gamma)$ becomes a semimodule over the semifield of tropical numbers $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$.

Let D be a divisor of degree d on Γ. The Riemann–Roch space $R(D) \subset \mathrm{Rat}(\Gamma)$ associated to D is defined by

$$R(D) = \big\{ f \in \mathrm{Rat}(\Gamma) \smallsetminus \{\infty\} \;\big|\; \mathrm{div}(f) + D \geqslant 0 \big\} \cup \{\infty\}.$$

Here, for $f \neq \infty$, the divisor of f, denoted by $\mathrm{div}(f)$, is given by

$$\mathrm{div}(f) = \sum_{x \in \Gamma} \mathrm{ord}_x(f)\,(x),$$

with the order of vanishing function defined by negative sum of the slopes of f along unit tangent directions to Γ at x,

$$\mathrm{ord}_x(f) = -\sum_{\nu \in \mathrm{T}_x\Gamma} \mathrm{sl}_\nu f.$$

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Endowed with the operations $\oplus$ and $\odot$, $R(D)$ becomes a semimodule over $\mathbb{T}$.

Let M be a subsemimodule of $R(D)$. The linear system associated to the pair (D, M), denoted by $|(D, M)|$, is defined by

$$|(D,M)| = \bigl\{E = \mathrm{div}(f) + D \bigm| f \in M\bigr\}.$$

We naturally view $|(D, M)|$ as a subset of the symmetric product

$$\Gamma^{(d)} = \Gamma^d / \mathfrak{S}_d,$$

where the symmetric group $\mathfrak{S}_d$ of degree d acts on Γ^d by permuting the coordinates. In the case $M = R(D)$, we get the complete linear system $|D|$.

The symmetric product $\Gamma^{(d)}$ has a natural polyhedral structure, see [3, § 2.1]. In the case $M = R(D)$, it was proved in [17] that $R(D)$ is finitely generated, and a polyhedral structure for the complete linear system $|D|$ was defined in *loc. cit.* We will prove in Section 2 that for any finitely generated subsemimodule $M \subseteq R(D)$, the linear system $|(D, M)|$ inherits a polyhedral structure as a subspace of $\Gamma^{(d)}$.

For a finitely generated subsemimodule M in $R(D)$, we define its *dimension* as the topological dimension of the associated linear system, increased by one:

$$\dim(M) = \dim(D, M) = \dim|(D, M)| + 1.$$

More generally, for any subsemimodule $M \subseteq R(D)$, we define its dimension as the supremum (in fact, the maximum, by finite generation of $R(D)$) of $\dim(N)$ over all finitely generated subsemimodules N of M:

$$\dim(M) = \sup\bigl\{\dim(N) \bigm| N \text{ is a finitely generated subsemimodule of } M\bigr\}.$$

The dimension is an intrinsic numerical invariant of M, meaning that it does not depend on the choice of the divisor, see Proposition 2.2. The notation $\dim(M)$ is thus consistent. By analogy with the finitely generated case, we define, for any subsemimodule $M \subseteq R(D)$, $\dim|(D, M)| = \dim(M) - 1$. Equivalently, this is the supremum of $\dim|(D, N)|$ over finitely generated subsemimodules $N \subseteq M$.

We recall from [19] that a family of functions $f_1, \dots, f_r$ in $\mathrm{Rat}(\Gamma) \smallsetminus \{\infty\}$ is called *tropically dependent* if there exist real numbers $c_1, \dots, c_r$ such that the minimum in

$$\min_{1 \leqslant i \leqslant r} \bigl(f_i(x) + c_i\bigr)$$

is achieved at least twice for every $x \in \Gamma$. Otherwise, the family is called *tropically independent.*

For a subsemimodule M of $\mathrm{Rat}(\Gamma)$, we define the *tropical rank* $r_{\mathrm{trop}}(M)$ as the maximum integer r for which there exist r tropically independent elements $f_1, \dots, f_r$ in M. Also, we define

$$r_{\mathrm{trop}}|(D, M)| = r_{\mathrm{trop}}(M) - 1.$$

Our first result is the following theorem (see Section 3 for the proof).

Theorem 1.1. *For each subsemimodule $M \subseteq R(D)$, we have*

$$r_{\mathrm{trop}}(M) = \dim(M) \qquad \text{and} \qquad r_{\mathrm{trop}}|(D, M)| = \dim|(D, M)|.$$

In particular, we have $r_{\mathrm{trop}}|D| = \dim|D|$.

Motivated by applications, a combinatorial theory of linear series was developed in recent works [4, 21]. A combinatorial linear series of rank r in both of these works is a finitely generated subsemimodule $M \subseteq R(D)$ which verifies $r(D, M) = r_{\text{trop}}|(D, M)|$, with some extra constraints. Here, $r(D, M)$ is the divisorial rank of (D, M), introduced originally in [7] in the case $M = R(D)$. It is defined as the maximum integer $r \geqslant -1$ such that for any effective divisor E of degree r, there exists an element $f \in M$ with $\operatorname{div}(f) + D - E \geqslant 0$.

Using Theorem 1.1, we prove the following reformulation of the equality $r(D, M) = r_{\text{trop}}|(D, M)|$.

Theorem 1.2. *Let $M \subseteq R(D)$ be a finitely generated subsemimodule. The following statements are equivalent.*

(1) The equality $r(D, M) = r_{\text{trop}}|(D, M)|$ holds.
(2) The linear system $|(D, M)|$ is of pure dimension $r(D, M)$.

The proof is given in Section 4 and uses a general result about tropical linear systems formulated in Theorem 4.1.

The final section of this paper contains complementary results and questions on combinatorial and topological properties of semimodules related to the tropical rank, as well as its computability.

2. Preliminaries

In this section, we gather some background on metric graphs and their divisor theory. We refer to the survey paper [6] and [3, 17, 19] for more details. For each positive integer n, we denote by $[n]$ the set of positive integers i satisfying $1 \leqslant i \leqslant n$.

2.1. Linear series on metric graphs.

Let $G = (V, E)$ be a finite connected graph with vertex set V and edge set E. Let $\ell \colon E \to \mathbb{R}_{>0}$ be an edge length function, assigning a positive real number $\ell(e)$ to each edge e of the graph.

To the pair (G, ℓ), we associate a metric space Γ as follows. For each edge $e \in E$, we place a closed interval $I_e = [0, \ell(e)]$ of length $\ell(e)$ between the two vertices of e. The resulting space inherits a natural quotient topology from the topology on the disjoint union of the intervals I_e, identifying endpoints according to the adjacency relations in G.

Moreover, this topology is metrizable via the path metric, where the distance between any two points in Γ is defined as the length of the shortest path connecting them.

A metric space Γ obtained in this way is called a *metric graph*, and the pair (G, ℓ) is called a *model* of Γ. Note that a metric graph that is not a singleton has infinitely many different models.

The group of divisors on Γ, denoted by $\operatorname{Div}(\Gamma)$, is the free abelian group generated by the points of Γ. Explicitly, it consists of finite linear combinations of points of Γ:

$$\operatorname{Div}(\Gamma) = \left\{ \sum_{x \in A \subset \Gamma} n_x (x) \;\middle|\; n_x \in \mathbb{Z} \text{ and } A \text{ a finite set} \right\}.$$

Here, we write (x) for the generator corresponding to the point $x \in \Gamma$. For a divisor $D \in \operatorname{Div}(\Gamma)$ and $x \in \Gamma$, the coefficient of (x) in D is denoted by $D(x)$. The *support* of D, denoted by $\operatorname{Supp}(D)$, is the set of points x with $D(x) \neq 0$. The *degree* of D, denoted by $\deg(D)$, is defined as the sum of its coefficients

$$\deg(D) = \sum_{x\in\Gamma} D(x).$$

The complete linear system $|D|$ is given, using the notation of the introduction, by

$$|D| = |(D, R(D))| = \bigl\{\operatorname{div}(f) + D \bigm| f \in R(D)\bigr\}.$$

We have a natural embedding

$$\eta\colon |D| \hookrightarrow \Gamma^{(d)}$$

which maps each divisor $E = (p_1) + \cdots + (p_d)$ in $|D|$ to the corresponding point in the symmetric product $\Gamma^{(d)}$, given by

$$\eta(E) = (p_1, \dots, p_d).$$

Theorem 2.1. *Let D be a divisor of degree d and $M \subseteq R(D)$ be a finitely generated subsemimodule. Then, $|(D, M)|$ has the structure of a polyhedral space. Moreover, the embedding $|(D, M)| \subseteq \Gamma^{(d)}$ is piecewise linear.*

Proof. By [17, Thm. 6], $R(D)$ is finitely generated. Let $g_1, \dots, g_l$ be a set of generators for $R(D)$ and $f_1, \dots, f_m$ be a set of generators for M. There exist real numbers λ_{ij} for $i \in [m]$ and $j \in [l]$ such that

$$f_i = \min_{j\in[l]} (g_j + \lambda_{ij}).$$

Now, consider the map

$$\Phi\colon \mathbb{R}^m \to \mathbb{R}^l$$

given by

$$\Phi(c_1, \dots, c_m) = \Bigl(\min_{i\in[m]} (c_i + \lambda_{ij})\Bigr)_{j=1}^{l}.$$

In other words, Φ is the piecewise linear map from $\mathbb{R}^m$ to $\mathbb{R}^l$ given by the tropical matrix multiplication from the right by the $m \times l$ matrix $(\lambda_{ij})_{i,j}$.

Define the map

$$\Psi\colon \mathbb{R}^l \to |D|$$

by sending each $(x_1, \dots, x_l)$ to the element $D + \operatorname{div}(f)$ in $|D|$ with

$$f = \min_{j\in[l]} (g_j + x_j).$$

It follows from the description of the polyhedral structure on $|D|$ given in [17] that Ψ is a piecewise linear map.

Finally, the natural embedding

$$\eta\colon |D| \hookrightarrow \Gamma^{(d)}$$

is a piecewise linear map, see [3].

The subset $|(D, M)|$ of $\Gamma^{(d)}$ is precisely the image of the composition of the maps we have constructed. More precisely, the subset $|(D, M)| \subseteq \Gamma^{(d)}$ is the image of the piecewise linear map $\eta \circ \Psi \circ \Phi$. This completes the proof of both statements in the theorem. □

Here are two observations for future use.

Proposition 2.2. *Let D and D' be divisors and M be a closed subsemimodule of* $\operatorname{Rat}(\Gamma)$ *included both in $R(D)$ and $R(D')$. Then,* $\dim(D, M) = \dim(D', M)$.

Proof. The addition by $D' - D$ provides a homeomorphism (in fact, an isomorphism of polyhedral spaces) from $|(D, M)|$ to $|(D', M)|$, from which the proposition follows. ☐

Proposition 2.3. *For each closed subsemimodule M in $\mathrm{Rat}(\Gamma)$, $r_{\mathrm{trop}}(M)$ is the supremum of $r_{\mathrm{trop}}(N)$ over finitely generated subsemimodules $N \subseteq M$.*

Proof. The result follows directly from the observation that a tropically independent family $f_1, \dots, f_r$ in M remains independent in the finitely generated subsemimodule $\langle f_1, \dots, f_r\rangle$. ☐

We also state the following comparison result.

Proposition 2.4. *Given a point $x \in \Gamma$ and a unit tangent vector ν based at x, let n_ν be the number of distinct slopes along ν taken by functions $f \in M$. We have the inequalities*

$$r(D, M) + 1 \leqslant n_\nu \leqslant r_{\mathrm{trop}}|(D, M)| + 1.$$

Proof. The first inequality follows by choosing an effective divisor E made up of $r(D, M)$ distinct points close to x in the direction of ν and applying the definition of the divisorial rank. The second inequality comes from the fact that functions which coincide at x but have pairwise distinct slopes along ν are tropically independent (see [4, Rem. 6.6]). ☐

2.2. **Certificates of independence.** We will need Theorem 2.5 below on certificates of independence; the equivalence of (1) and (4) is [12, Thm. 1.6].

Let $f_1, \dots, f_n$ denote real-valued functions defined on a set X such that for all pairs $i, j \in [n]$, the difference $f_i - f_j$ is bounded on X. For example, we can take $X = \Gamma$ and $f_1, \dots, f_n$ elements of $\mathrm{Rat}(\Gamma)$.

We define the map $T = (T_1, \dots, T_n)\colon \mathbb{R}^n \to \mathbb{R}^n$ by setting, for all $c = (c_1, \dots, c_n) \in \mathbb{R}^n$,

$$T_i(c) = \sup_{x\in X}\Big(\min_{j\in[n]\smallsetminus\{i\}} (f_j(x) - f_i(x) + c_j)\Big), \qquad \text{for all } i \in [n]. \tag{2.1}$$

The boundedness of the differences $f_i - f_j$ guarantees that T takes finite values.

Theorem 2.5. *The following assertions are equivalent.*

1. *The functions $f_1, \dots, f_n$ are tropically independent.*
2. *There exists a vector $c \in \mathbb{R}^n$ and a positive real number ρ such that*
$$T_i(c) = \rho + c_i, \qquad \text{for all } i \in [n].$$
3. *There exists a vector $c \in \mathbb{R}^n$ such that*
$$T_i(c) > c_i, \qquad \text{for all } i \in [n].$$
4. *There exist real numbers $c_1, \dots, c_n$ and $x_1, \dots, x_n \in X$ such that, for all $k \in [n]$, the minimum*
$$\min_{j\in[n]}(f_j(x_k) + c_j) \tag{2.2}$$
is achieved only for $j = k$.
5. *There exist points $x_1, \dots, x_n \in X$ such that the minimum*
$$\min_{\sigma\in\mathfrak{S}_n} \sum_{k\in[n]} f_{\sigma(k)}(x_k) \tag{2.3}$$
is achieved by a single permutation.

Remark 2.6. When X is finite, the equivalence between (1) and (5) follows from Proposition 4.1 and Theorem 5.5 of [11]. The same equivalence is established in Theorem 2.10 of [18] and Theorem 4.12 of [1] for finite X in more general settings. The equivalence of these two assertions with (4), again for finite X, can be found in the proof of Theorem 4.12, *ibid.* The equivalence between (1) and (4) is proved in [12]; the result is stated there for metric graphs, but the proof applies to the more general setting above. $\diamond$

Define the Hilbert seminorm (also called "Hopf's oscillation") $\|\cdot\|_{\mathrm{H}}$ on $\mathbb{R}^n$ by setting $\|x\|_{\mathrm{H}} = \max_i x_i - \min_j x_j$, $x \in \mathbb{R}^n$. Let $e = (1, \ldots, 1)$ be the point in $\mathbb{R}^n$ with coordinates all equal to one. Observe that $\|x\|_{\mathrm{H}} = 2 \cdot \inf\{\|x + \lambda e\|_\infty \mid \lambda \in \mathbb{R}\}$. Therefore, up to a factor of 2, the norm $\|\cdot\|_{\mathrm{H}}$ agrees with the norm induced by the supremum norm on the quotient space $\mathbb{R}^n/\mathbb{R}e$.

Proof of Theorem 2.5. We first prove (1) $\Rightarrow$ (2). Let

$$M^+ := \sup_{\substack{(i,j)\in[n]^2\\ x\in X}} (f_j(x) - f_i(x)) \qquad \text{and} \qquad M^- := \inf_{\substack{(i,j)\in[n]^2\\ x\in X}} (f_j(x) - f_i(x)).$$

We claim that for each $c \in \mathbb{R}^n$, the vector $d = T(c)$ satisfies $\|d\|_{\mathrm{H}} \leqslant M^+ - M^-$. Indeed, let $k \in [n]$ be such that $c_k = \min_{j\in[n]} c_j$. For each $i \in [n]$, we have $d_i \leqslant M^+ + c_k$ and $d_i \geqslant M^- + c_k$, and so $\|d\|_{\mathrm{H}} \leqslant M^+ - M^-$.

Since T is order-preserving (for the coordinatewise partial order on $\mathbb{R}^n$) and commutes with the addition of a constant, it is nonexpansive (i.e., 1-Lipschitz) with respect to $\|\cdot\|_\infty$, see [10]. So it induces a quotient map $\varphi\colon \mathbb{R}^n/\mathbb{R}e \to \mathbb{R}^n/\mathbb{R}e$ that is nonexpansive with respect to $\|\cdot\|_{\mathrm{H}}$. Moreover, φ preserves the ball of radius $M^+ - M^-$ with respect to $\|\cdot\|_{\mathrm{H}}$ around 0. Since this ball is a compact convex subset of $\mathbb{R}^n/\mathbb{R}e$, it follows from Brouwer's fixed-point theorem that φ has a fixed point. Hence, there exists a real number ρ and a vector $c \in \mathbb{R}^n$ such that $T_i(c) = \rho + c_i$ for all $i \in [n]$.

We show that ρ is positive. For the sake of contradiction, suppose that $\rho \leqslant 0$. Then, for all $x \in X$, we have

$$c_i \geqslant c_i + \rho \geqslant \min_{j\in[n]\smallsetminus\{i\}} (f_j(x) - f_i(x) + c_j)$$

and thus

$$f_i(x) + c_i \geqslant \min_{j\in[n]\smallsetminus\{i\}} (f_j(x) + c_j),$$

implying that the minimum $\min_{j\in[n]}(f_j(x) + c_j)$ is achieved at least twice, contradicting the assumption in (1). Therefore, $\rho > 0$.

The implication (2) $\Rightarrow$ (3) is trivial. We next show (3) $\Rightarrow$ (4). Suppose that $T_i(c) > c_i$ holds for all $i \in [n]$. Then, for each $i \in [n]$, we can find a point $x_i \in X$ such that

$$\min_{j\in[n]\smallsetminus\{i\}} (f_j(x_i) - f_i(x_i) + c_j) > c_i,$$

and thus $\min_{j\in[n]\smallsetminus\{i\}}(f_j(x_i) + c_j) > f(x_i) + c_i$, as required.

In order to prove (4) $\Rightarrow$ (5), observe that the set of minimizing permutations in (2.2) is unchanged if we replace f_j by $f_j + c_j$. We can thus assume that $c_j = 0$ for all $j \in [n]$. Then, if (4) holds, we have that for all $k \in [n]$, the minimum $\min_{j\in[n]} f_j(x_k)$ is achieved only for $j = k$, which entails that the minimum in (2.2) is achieved only by the identity permutation, giving (5).

We prove the last implication (5) $\Rightarrow$ (1). Permuting the functions $f_1, \ldots, f_n$ if necessary, we may assume that the unique permutation giving the minimum in (2.3) is the identity. Suppose, for the sake of contradiction that $f_1, \dots, f_n$ are tropically linearly dependent. Then, translating each f_i by a constant, we may assume that for all $k \in [n]$, the minimum $\min_{i\in[n]} f_i(x_k)$ is attained at least twice. Consequently, for each $k \in [n]$, there exists an index $\alpha(k) \in [n] \smallsetminus \{k\}$ achieving the minimum in $\min_{i\in[n]} f_i(x_k)$. Fix an initial index k_0 and define $k_s = \alpha(k_{s-1})$ for $s \geqslant 1$. This produces an infinite sequence in $[n]$, so it must eventually repeat. Hence, it contains a cycle: there exist distinct elements $k_p, \dots, k_{p+q-1}$ such that $\alpha(k_{p+q-1}) = k_p$. This cycle defines a permutation σ, different from the identity, and we have

$$\sum_{k\in[n]} f_k(x_k) = \sum_{k\in[n]} f_{\sigma(k)}(x_k),$$

which contradicts (4). □

Remark 2.7. The introduction of T is inspired by the approach of [1, § 4.2] in which methods of non-linear fixed-point theory are applied to study the tropical rank of matrices. In particular, the number ρ in (1) may be interpreted as an *additive eigenvalue*, and it is unique. It coincides with the *escape rate* of the operator T, that is,

$$\rho = \lim_{k\to\infty} \left(k^{-1} \max_{i\in[n]} T_i(T^k(0)) \right) = \lim_{k\to\infty} \left(k^{-1} \min_{i\in[n]} T_i(T^k(0)) \right), \tag{2.4}$$

where $T^k = T \circ \cdots \circ T$ is the k-th iterate of T, see [13] and the introductory part of [2] for background. The existence of the additive eigenpair (c, ρ), needed for the implication (1) ⇒ (2), can be alternatively deduced by applying the general non-linear Perron–Frobenius theorem of [13]. It follows from [2, Thm. 20] that, for finite X, ρ can be interpreted as a quantitative measure of tropical independence: identifying $X = [m]$, each f_i gives a vector in $\mathbb{R}^m$, and ρ is the minimum over all tropical hyperplanes H in $\mathbb{R}^m$ of the maximum distance, in the Hilbert seminorm, of the vectors f_i to H. ⋄

3. Proof of Theorem 1.1

Since both $\dim(M)$ and $r_{\text{trop}}(M)$ are given by the maximum of $\dim(N)$ and $r_{\text{trop}}(N)$, respectively, over all finitely generated subsemimodules N of M (see Proposition 2.3 for the tropical rank), it will be enough to consider only the case where M is finitely generated.

Thus, in what follows, we assume that M is finitely generated.

3.1. Proof of the inequality $\dim(M) \geqslant r_{\text{trop}}(M)$.

We will make use of the certificates of independence.

Let $r = r_{\text{trop}}(M)$, and consider a family of tropically independent elements $f_1, \dots, f_r$ in M. By the equivalence of (1) and (4) in Theorem 2.5, there exist points $x_1, \dots, x_r$ and real numbers $c_1, \dots, c_r$ such that for all $i \in [r]$, the minimum over j in

$$\min_{j\in[r]} (f_j(x_i) + c_j),$$

is achieved uniquely at $j = i$.

Let $c = (c_1, \dots, c_r) \in \mathbb{R}^r$ and define the subset $B_\varepsilon \subset \mathbb{R}^r$ as the set of points $p = (p_1, \dots, p_r)$ satisfying

- $p_1 = c_1$, and
- $|p_j - c_j| < \varepsilon$ for each $1 < j \leqslant r$.

We claim that for sufficiently small $\varepsilon > 0$, the elements f_p in M, defined for $p \in B_\varepsilon$ by

$$f_p(x) = \min_{1 \leqslant j \leqslant r} (f_j(x) + p_j) \quad \text{for all } x \in \Gamma, \tag{3.1}$$

are all distinct. To prove this, we choose $\varepsilon > 0$ sufficiently small so that, at $x = x_i$, the minimum in (3.1) is achieved uniquely at $j = i$ for all $p \in B_\varepsilon$. This ensures that the function f_p uniquely determines the value of p.

Since two elements of M which do not differ by a constant correspond to distinct elements in the linear system $|(D, M)|$, we obtain a piecewise linear embedding of B_ε into $|(D, M)|$, from which we deduce that $\dim |(D, M)| \geqslant r - 1$. This implies that $\dim(M) \geqslant r$, establishing the inequality $\dim(M) \geqslant r_{\text{trop}}(M)$, as required. □

3.2. Proof of the inequality $\dim(M) \leqslant r_{\text{trop}}(M)$.

Let $r = r_{\text{trop}}(M)$. Choose a sequence $x_1, x_2, \dots$ forming a dense subset of Γ, and let $f_1, \dots, f_m$ be a set of generators of M.

For each integer $K \geqslant 1$, define the evaluation map

$$\vartheta_K \colon M \longrightarrow \mathbb{T}^K$$

that sends each function $f \in M$ to its values at $x_1, \dots, x_K$:

$$\vartheta_K(f) = (f(x_k))_{1 \leqslant k \leqslant K}$$

The semimodule

$$M_K = \vartheta_K(M) \subset \mathbb{T}^K$$

is finitely generated.

For each $K \geqslant 1$, we define a canonical section ρ_K of the projection ϑ_K, given by

$$\begin{aligned} \rho_K \colon M_K &\longrightarrow M \\ g &\longmapsto \inf \bigl\{ f \in M \bigm| \vartheta_K(f) \geqslant g \bigr\} = \min \bigl\{ f \in M \bigm| \vartheta_K(f) \geqslant g \bigr\}. \end{aligned}$$

Since M is finitely generated, the infimum above is a minimum.

The following proposition provides an explicit expression of the section ρ_K.

Proposition 3.1. *For each $g \in M_K$, we have*

$$\rho_K(g) = \min_{i \in [m]} \bigl(f_i + c_i^K \bigr) \tag{3.2}$$

where

$$c_i^K = c_i^K(g) = \max_{k \in [K]} (g(x_k) - f_i(x_k)).$$

Proof. Let h denote the function on the right-hand side of Equation (3.2). By the definition of c_i^K, we have

$$f_i(x_k) + c_i^K \geqslant g(x_k) \qquad \text{for all } k \in [K].$$

From this, we deduce that

$$\vartheta_K \bigl(f_i + c_i^K \bigr) \geqslant g.$$

This implies the inequality $h \geqslant \rho_K(g)$.

The other way around, since $f_1, \dots, f_m$ is a generating set for M, there exist real numbers $p_1, \dots, p_m$ such that

$$\rho_K(g) = \min_{1 \leqslant i \leqslant m} (f_i + p_i).$$

Evaluating at x_k, we obtain

$$g(x_k) \leqslant f_i(x_k) + p_i \qquad \text{for all } k \in [K] \text{ and } i \in [m],$$

which implies

$$c_i^K \leqslant p_i \qquad \text{for all } i \in [m].$$

Thus, we conclude that $h \leqslant \rho_K(g)$, proving the proposition. ☐

Proposition 3.2. *For each positive integer K, the following equalities hold:*

$$\vartheta_K \circ \rho_K = \mathrm{Id}_{M_K} \qquad\qquad \vartheta_K \circ \rho_K \circ \vartheta_K = \vartheta_K,$$

and

$$\rho_K \circ \vartheta_K \leqslant \mathrm{Id}_M \qquad\qquad \rho_K \circ \vartheta_K \circ \rho_K = \rho_K.$$

Proof. This is straightforward. ☐

Proposition 3.3. *We have*

$$\bigcup_{K \geqslant 1} \rho_K(M_K) = M.$$

Proof. The inclusion $\subseteq$ holds by definition. We prove the reverse inclusion $\supseteq$.

Let f be an element of M. We show that $f \in \rho_K(M_K)$ for some $K \geqslant 1$. There exist real numbers λ_i such that

$$f = \min_{i \in [m]} (f_i + \lambda_i).$$

Replacing each f_i with $f_i + \lambda_i$ and removing the unnecessary terms, we may assume without loss of generality that all λ_i are zero and that each f_i contributes to the minimum.

This means that for each $i = 1, \dots, m$, there exists a point y_i of Γ such that

$$f_i(y_i) < f_j(y_i) \quad \text{for all } j \neq i.$$

Thus, for each i, there exists an open set $U_i \subset \Gamma$ such that for all $x \in U_i$, we have

$$f_i(x) < \min_{j \neq i} f_j(x).$$

Since the sequence (x_k) is dense in Γ, we can choose an index $k_i \geqslant 1$ such that $x_{k_i} \in U_i$. Let $K = \max_i k_i$.

Now, consider $g = \vartheta_K(f) \in M_K$. We will show that $c_i^K(g) = 0$ for each $i = 1, \dots m$, from which it follows that $f = \rho_K(g)$.

For a given element $i \in [m]$, we have, for each $k \in [K]$,

$$g(x_k) = f(x_k) \leqslant f_i(x_k),$$

where equality holds for $k = k_i$. By Proposition 3.1, this implies that $c_i^K(g) = 0$, as required. ☐

Proposition 3.4. *We have*

$$\dim(\rho_K(M_K)) \leqslant r.$$

Proof. Recall that r denotes the tropical rank of M. Since M_K is a projection of M, we have

$$r_{\mathrm{trop}}(M_K) \leqslant r_{\mathrm{trop}}(M).$$

Thus, to prove the proposition, it suffices to show that

$$\dim(\rho_K(M_K)) \leqslant r_{\mathrm{trop}}(M_K).$$

The space M_K consists of the column space of the matrix

$$A_K = \Big(f_1(x_k), \dots, f_m(x_k)\Big)_{1 \leqslant k \leqslant K}.$$

By [11, Thm. 4.2], M_K is a polyhedral complex of dimension

$$r_{\mathrm{trop}}(A_K) = r_{\mathrm{trop}}(M_K).$$

To conclude, observe that

$$\rho_K \colon M_K \to \rho_K(M_K) \hookrightarrow M$$

is a piecewise linear isomorphism, as given by the explicit formula for $c_i^K(g)$ (see Proposition 3.1). Consequently,

$$\dim(\rho_K(M_K)) = \dim(M_K),$$

which completes the proof. □

Using the results stated above, we are now in the position to prove the inequality

$$\dim(M) \leqslant r_{\mathrm{trop}}(M),$$

finishing the proof of Theorem 1.1.

As previously stated, we have an injective piecewise linear map

$$\varphi \colon M/\mathbb{R} \longhookrightarrow \Gamma^{(d)}, \qquad [f] \longmapsto [\mathrm{div}(f) + D].$$

By Proposition 3.3, we obtain

$$\varphi(M/\mathbb{R}) = \bigcup_{K \geqslant 1} \varphi(\rho_K(M_K)/\mathbb{R}).$$

Moreover, by Proposition 3.4, we have

$$\dim(\varphi(\rho_K(M_K)/\mathbb{R})) \leqslant r - 1 \qquad \text{for every } K \geqslant 1.$$

Now, let E be a cell of $\varphi(M/\mathbb{R})$ of maximal dimension. Proceeding by contradiction, assume that $\dim(E) \geqslant r$. Consider the intersection

$$F_K = \varphi(\rho_K(M_K)/\mathbb{R}) \cap E.$$

Since each $\varphi(\rho_K(M_K)/\mathbb{R})$ has dimension at most $r-1$, it follows that F_K has empty relative interior in E.

By the Baire category theorem, the countable union $\bigcup_K F_K$ has empty relative interior in E. However, this contradicts the fact that $\bigcup_K F_K = E$. Thus, we conclude that $\dim(M) \leqslant r_{\mathrm{trop}}(M)$, which completes the proof. □

4. PROOF OF THEOREM 1.2

We first establish a general result, Theorem 4.1, regarding the connection between the dimension and the divisorial rank of linear systems, from which we deduce Theorem 1.2. In the case where $M = R(D)$, this might be already known to some experts, but we could not find it in the literature.

Let D be a divisor of degree d on a metric graph Γ, and $M \subseteq R(D)$ be a finitely generated subsemimodule. By Theorem 2.1, $|(D, M)|$ has a polyhedral structure. Denote by $r(D, M)$ the divisorial rank of (D, M).

Theorem 4.1. *Keeping the same notation as above, all the maximal faces of the polyhedral structure on $|(D, M)|$ have dimension at least $r(D, M)$.*

The proof is given below. Using this result and Theorem 1.1, we deduce Theorem 1.2.

Proof of Theorem 1.2. The implication (1) ⇒ (2) follows directly from Theorem 1.1.

For the reverse implication (2) ⇒ (1), note that the equality $r_{\mathrm{trop}}(D, M) = \dim |(D, M)|$, given by Theorem 1.1 again, implies that $\dim |(D, M)| = r(D, M)$. By applying Theorem 4.1, we conclude that all the maximal faces of $|(D, M)|$ have dimension exactly $r(D, M)$, which implies that $|(D, M)|$ is of pure dimension $r(D, M)$. □

Proof of Theorem 4.1. We assume that $|(D, M)|$ is nonempty and $r(D, M) \geqslant 1$; otherwise, the statement holds trivially. Fix an element $E \in |(D, M)|$, and express it as $E = D + \mathrm{div}(h)$ for $h \in M$. We claim that every closed neighborhood $\mathcal{U}$ of E in $|(D, M)|$ has dimension at least $r(D, M)$, which will prove the theorem.

For each positive integer $r \leqslant r(D, M)$, let S_r denote the subset of Γ^r consisting of all tuples $\underline{x} = (x_1, \dots, x_r) \in \Gamma^r$ for which there exists a tuple $\underline{y} = (y_1, \dots, y_{d-r}) \in \Gamma^{d-r}$ such that the sum $(x_1) + \cdots + (x_r) + (y_1) + \cdots + (y_{d-r})$ belongs to $\mathcal{U}$. We claim that S_r is a closed subset of Γ^r.

To see this, suppose we have a sequence $\underline{x}_t = (x_{1,t}, \dots, x_{r,t})$ of points of S_r converging to $\underline{x} \in \Gamma^r$. For each t, there is a corresponding tuple $\underline{y}_t = (y_{1,t}, \dots, y_{d-r,t}) \in \Gamma^{d-r}$ such that the sum $(x_{1,t}) + \cdots + (x_{r,t}) + (y_{1,t}) + \cdots + (y_{d-r,t})$ lies in $\mathcal{U}$. By compactness of Γ^{d-r}, and passing to a subsequence if necessary, we can ensure that $\underline{y}_t$ converges to some point $\underline{y} = (y_1, \dots, y_{d-r}) \in \Gamma^{d-r}$. Since $\mathcal{U}$ is closed, the limit $(x_1) + \cdots + (x_r) + (y_1) + \cdots + (y_{d-r})$ of the divisors $(x_{1,t}) + \cdots + (x_{r,t}) + (y_{1,t}) + \cdots + (y_{d-r,t})$ belongs to $\mathcal{U}$, implying that $\underline{x} \in S_r$. Therefore, S_r is closed in Γ^r.

We proceed by induction on r and demonstrate that for each closed neighborhood $\mathcal{U}$ of an element $E \in |(D, M)|$ as above with corresponding sets S_j, $j \leqslant r(D, M)$, the set S_r contains a nonempty open subset of Γ^r, i.e., it has a nonempty interior in Γ^r. By the definition of $S_{r(D,M)}$, this implies that each $\mathcal{U}$ is of dimension at least $r(D, M)$, as claimed.

We begin by considering the case $r = 1$. Since $r(D, M) \geqslant 1$, there exists an element $F = D + \mathrm{div}(f)$ in $|(D, M)|$, distinct from E, with $f \in M$. For each $t \in \mathbb{R}$, define $f_t = \min\{t + h, f\}$ and let $E_t = D + \mathrm{div}(f_t)$. We observe that $f_t \in M$ and $E_t \in |(D, M)|$.

For t approaching $-\infty$, we have $E_t = E$, and for t approaching ∞, we have $E_t = F$. Thus, we obtain a one-dimensional segment in $\mathcal{U}$, with one endpoint at E, completing the proof in this case.

Assume $r \geqslant 2$ and that the claim has been proved for $r - 1$. Consider the projection map $\pi \colon S_r \to S_{r-1}$, given by $(x_1, \dots, x_{r-1}, x_r) \mapsto (x_1, \dots, x_{r-1})$. For each point $\underline{x} = (x_1, \dots, x_{r-1})$ in S_{r-1}, the fiber $\pi^{-1}(\underline{x})$ is nonempty. We claim that there exists a nonempty open subset $\mathcal{W}$ of Γ^{r-1} contained in S_{r-1} over which each fiber of π contains a one-dimensional segment of Γ (possibly different for distinct points).

To prove this, let $\mathcal{U}'$ be a closed neighborhood of E included in the interior of $\mathcal{U}$. The subset S'_{r-1} associated to $\mathcal{U}'$, defined analogously to the previous construction, is a subset of S_{r-1}, and by the induction hypothesis, it has a nonempty interior, which we will show to be the desired nonempty open subset $\mathcal{W}$. We need to show that for each $\underline{x} \in S'_{r-1}$, the fiber $\pi^{-1}(\underline{x})$ contains a one-dimensional segment.

Let $\underline{y} \in \Gamma^{d-r+1}$ be a point such that $E' = (x_1) + \cdots + (x_{r-1}) + (y_1) + \cdots + (y_{d-r+1}) \in \mathcal{U}'$. Since $\mathcal{U}'$ is contained in the interior of $\mathcal{U}$, $\mathcal{U}$ is a closed neighborhood of E'. Write $E' = D + \mathrm{div}(f)$ for some $f \in M$ and let $M' = M - f$ be the set consisting of all elements of the form $g - f$ for $g \in M$. Then, M' is a finitely generated subsemimodule of $R(E')$.

Moreover, the pair $((y_1)+\cdots+(y_{d-r+1}), M')$ has divisorial rank at least one. Therefore, repeating the argument from the case $r=1$, there exists a segment of dimension one in $\mathcal{U}$ of the form $(x_1)+\cdots+(x_{r-1})+(y_{1,t})+\cdots+(y_{d-r+1,t})$ with one endpoint at E'. This implies that $\pi^{-1}(\underline{x})$ contains a one-dimensional segment.

Recall that $\mathcal{W}$ is a nonempty open subset of Γ^{r-1} contained in S_{r-1} over which each fiber of π contains a one-dimensional segment of Γ. Let $(I_j)_{j=1}^{\infty}$ be a countable collection of one-dimensional segments in Γ such that each one-dimensional segment of Γ contains at least one of the I_j's, for some $j\in\mathbb{N}$. In particular, each fiber of π over $\mathcal{W}$ contains one of the segments I_j.

For each I_j, define $\mathcal{A}_j$ as the subset of $\mathcal{W}$ consisting of all the points $\underline{x}$ with $I_j\subseteq\pi^{-1}(\underline{x})$. By continuity of the projection map π and the closedness of S_r, each $\mathcal{A}_j$ is a closed subset of $\mathcal{W}$. Moreover, the union of the sets $\mathcal{A}_j$ covers the full open set $\mathcal{W}$. Therefore, by the Baire category theorem, there exists some $j\in\mathbb{N}$ such that $\mathcal{A}_j$ has a nonempty interior.

We conclude that the product $\mathcal{A}_j\times I_j$ is contained in S_r. Moreover, it has a nonempty interior in Γ^r, as required. ☐

5. Complementary results and questions

In this section, we discuss several complementary results and questions, including a possible extension to higher dimension.

5.1. Finite evaluation maps on semimodules of rational functions. Given a subsemimodule M of $R(D)$, it is natural to ask whether functions $f\in M$ can be fully characterized by the set of their values on a well-chosen finite set of points in Γ. More precisely:

Question 5.1. *Given a subsemimodule M of $R(D)$, do there exist a positive integer K and points $x_1,\dots,x_K\in\Gamma$ such that the evaluation map $\vartheta\colon M\to\mathbb{T}^K$, defined by*

$$\vartheta(f)=(f(x_k))_{1\leqslant k\leqslant K},$$

is injective?

This is motivated by the following observation.

Proposition 5.2. *Suppose that M is a module for which the answer to Question 5.1 is positive. Then, $r_{\mathrm{trop}}(M)=r_{\mathrm{trop}}(\vartheta(M))$.*

Proof. A family $\vartheta(f_1),\dots,\vartheta(f_r)$ with $f_1,\dots,f_r\in M$ is tropically dependent if and only if there are real numbers $\lambda_1,\dots,\lambda_r$ such that for all $k\in[K]$, the minimum in $\min_{j\in[r]}(\lambda_j+f_j(x_k))$ is achieved at least twice. This condition can be rewritten as

$$\vartheta(g)=\vartheta(g_j)\text{ for all } j,\text{ with } g=\min_{s\in[r]}(\lambda_s+f_s)\text{ and } g_j=\min_{s\in[r]\smallsetminus\{j\}}(\lambda_s+f_s).$$

If ϑ is injective, if follows that $g=g_j$ holds for all $j\in[r]$. This means that $f_1,\dots,f_r$ are tropically dependent. We infer that $r_{\mathrm{trop}}(M)\leqslant r_{\mathrm{trop}}(\vartheta(M))$. The other inequality is trivial. ☐

The answer to Question 5.1 depends on the structure of M, namely, on the number of distinct slopes that functions in M realize along any edge of the metric graph. To formulate this dependence, we refine the combinatorial model of Γ so that the set of slopes taken by functions $f\in M$ along unit tangent directions becomes constant on each edge. More precisely, for each pair (e,v) consisting of an edge e and an extremity v of e, and any point x of Γ lying on the half-closed interval $e\smallsetminus\{v\}$, let ν_x denote the unit tangent vector at x directed toward v. Then, we require that the set $\mathrm{sl}_{(e,v)}(M) \coloneqq \left\{\mathrm{sl}_{\nu_x} f \;\middle|\; f \in M\right\}$ be independent of x. A compactness argument, using the closedness of $R(D)$, ensures that such a model always exists.

Let us first assume that $\mathrm{sl}_{(e,v)}(M)$ has at most two elements for every pair (e, v) in the graph. This is the case, for instance, if $r_{\mathrm{trop}}(M) \leqslant 1$ (see Proposition 2.4). Since, on each edge e with endpoints u and v and for every $f \in M$, the function f changes slope at most once along e, the values of $f(u)$ and $f(v)$ completely determine f on the entire edge e. Consequently, if $x_1, \dots, x_K$ are chosen as the vertices of Γ, the corresponding evaluation map ϑ is injective.

On the contrary, let us assume that $\mathrm{sl}_{(e,v)}(M)$ contains at least three elements for some pair (e, v). In this case, for any choice of points $x_1, \dots, x_K \in \Gamma$, the associated evaluation map ϑ is generally not injective. This means that no finite evaluation map can fully characterize the functions $f \in M$.

To illustrate this, we further refine the combinatorial model of Γ and assume the following:

1. there exists an edge e' (in this new model) contained in the original edge e such that there are functions $f_1, f_2, f_3 \in M$ with constant slopes s_1, s_2, s_3 on e' satisfying $s_1 < s_2 < s_3$; and
2. no point x_i lies in the interior of e', i.e., $x_i \notin \mathring{e}'$ for all i.

It is easy to see that if M contains sufficiently many functions – for instance, if $M = R(D)$ – then, using the functions f_i, we can construct infinitely many distinct functions $f \in M$ that coincide outside $\mathring{e}'$ (see Figure 1). Since these functions share the same value at all the chosen points x_i, they have the same image under the evaluation map. Since this argument can be applied to any choice of the points $x_1, \dots, x_K$, for any K, this proves that the finite evaluation maps cannot be injective in general.

FIGURE 1. Construction of infinitely many functions using three different slopes.

5.2. **Closedness versus finite generation of subsemimodules.** In general, $R(D)$ contains closed subsemimodules that are not finitely generated (see the example below). However, subsemimodules of $R(D)$ arising from the tropicalization of linear series on curves are always finitely generated, see [4, § 9.4].

Example 5.3. Let Γ be a metric graph with model $(G = (V, E), \ell)$, and let x_1, x_2, x_3 be three distinct points on an edge $e = \{u, v\}$ of G in Γ, such that x_2 is the midpoint of the edge and lies at the midpoint of the segment joining x_1 and x_3. Consider the divisor $D = n\,(u) + n\,(v)$ for a sufficiently large positive integer n, and define $M \subset R(D)$ to be the set of all functions $f \in R(D)$ satisfying the inequality $-\varepsilon + 2f(x_2) \geqslant f(x_1) + f(x_3)$, for $\varepsilon > 0$ small. Then, M forms a closed subsemimodule of $R(D)$. However, for $\varepsilon > 0$ small enough, M is not finitely generated.

To see this, consider the evaluation map

$$\vartheta\colon R(D)\to\mathbb{T}^3,\qquad \vartheta(f)=(f(x_1),f(x_2),f(x_3)).$$

If M were finitely generated, then $\vartheta(M)$ would also be finitely generated, implying that it would have only finitely many extreme points.

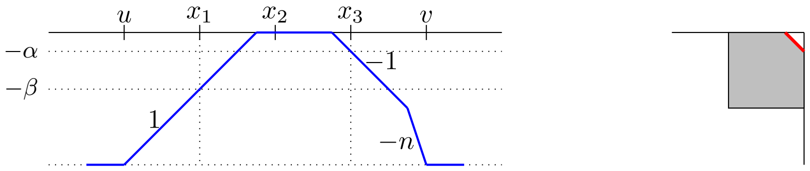

FIGURE 2. A parametric family of functions $f\in R(D)$ (left). Fragment of the cross section $\{(f(x_1),f(x_3))\mid f\in M, f(x_2)=0\}$ with an infinite upper Pareto set depicted in red (right).

Now, we shall use the following characterization of tropical extreme points, which can be found in [14] (proof of Theorem 3.1): if N is a closed subsemimodule of $\mathbb{T}^n$, then a point $z\in N$ is extreme if and only if there is an index $i\in[n]$ such that $z-z_i\,e_i$ is a maximal element of the cross section $S_i(N)$ of N given by $S_i(N)=\{w\in N\mid w_i=0\}$. Coming back to our example, it suffices to check that $S_2(\vartheta(M))$ has an infinite set of maximal elements.

To show this, consider the function f in $\mathrm{Rat}(\Gamma)$, constant on $\Gamma\smallsetminus e$, and given on the edge e by

$$f(x)=\max(u-x_1-\beta,\min(x-x_1-\beta,x_3-\alpha-x,u-x_1-\beta+n(v-x)))\quad\text{for all } x \text{ in } e,$$

with x a parameter on e and α,β real numbers satisfying $0\leqslant\alpha\leqslant\beta\leqslant L\coloneqq x_3-x_2$. For n large enough, this function has the shape shown in Fig. 2 (the slopes are indicated on the figure), with $\mathrm{ord}_u(f)=-1$, $\mathrm{ord}_v(f)=-n$, and $\mathrm{ord}_x(f)\geqslant 0$ elsewhere, thus $f\in R(D)$. Observe that $f(x_1)=-\beta$ and $f(x_3)=-\alpha$, where all the values $0\leqslant\alpha\leqslant\beta\leqslant L$ are realizable.

This entails that the set $\widetilde{S}_2=\{(y_1,y_3)\mid(y_1,0,y_3)\in\vartheta(R(D))\}$ contains every element of the form $(-\beta,-\alpha)$. By symmetry, we deduce that $\widetilde{S}_2\supset[-L,0]^2$. It follows that

$$S_2(\vartheta(M))\cap[-L,0]^2=\{(y_1,y_3)\mid(y_1+y_3)/2\leqslant-\varepsilon \text{ and } -L\leqslant y_1,y_2\leqslant 0\}.$$

Hence, $S_2(\vartheta(M))\cap[-L,0]^2$ has an infinite set of maximal elements, consisting of its upper boundary, defined by the equality $(y_1+y_3)/2=-\varepsilon$. All the elements of this boundary are maximal elements of $S_2(\vartheta(M))$ as well, showing that the set of extreme points of $\vartheta(M)$ is infinite. This is illustrated in Figure 2 (right). ◇

When M is finitely generated, by Theorem 2.1, the linear system $|(D,M)|$ admits a polyhedral structure. In Section 1, for a more general subsemimodule M, we defined the dimension of the corresponding linear system as a supremum over finitely generated subsemimodules because as we show below (see Proposition 5.5), in general, M can have pathological behavior, and we do not know the answer to the following question.

Question 5.4. *Let $M\subset R(D)$ be closed. Is it possible to describe the geometric structure of $|(D,M)|$? In particular, does it admit a good notion of dimension?*

Our next result shows the subtlety behind this question. Recall that o-minimal structures capture a notion of "tameness" for geometric objects; e.g., semialgebraic sets are the simplest examples of sets definable in an o-minimal structure. See [25] for background.

Proposition 5.5. *There exists a closed subsemimodule $M \subset R(D)$ such that $|(D, M)|$ is not definable in any o-minimal structure.*

Proof. We consider the metric graph Γ with distinguished points x_1, x_2, x_3, and the divisor D defined in Example 5.3. Let $\vartheta \colon R(D) \to \mathbb{T}^3$ be the evaluation map on these three points.

Let $\kappa \colon \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function such that $0 \leqslant \kappa'(x) \leqslant 1$ holds almost everywhere. Now define the function $\psi \colon \mathbb{R}^2 \to \mathbb{R}$ by setting $\psi(y, z) = y + \kappa(z - y)$. Let M_κ be the union of $\{\infty\}$ and the set of finite-valued functions $f \in R(D)$ which satisfy the inequality $f(x_2) \geqslant \psi(f(x_1), f(x_3))$. Given that $0 \leqslant \kappa'(x) \leqslant 1$, it follows that

$$\kappa(y) \leqslant \kappa(y + z) \leqslant \kappa(y) + z \quad \text{for all } y \in \mathbb{R},\ z \in \mathbb{R}_{\geqslant 0}.$$

This implies that ψ is order-preserving. Moreover, it commutes with the addition of constants. Consequently, M_κ is a closed subsemimodule of $R(D)$. For instance, the semimodule M of Example 5.3 corresponds to the special choice $\kappa(y) = \varepsilon/2 + y/2$.

Now, we choose κ so that $\kappa'(x)$ takes only the values 0 and 1 almost everywhere, and makes an infinite number of switches between the values 0 and 1 in a neighborhood of 0. To make this example concrete, we may choose a sequence of positive real numbers $(\alpha_k)_k$ decreasing to zero, define the open interval $I_k = (\alpha_{k+1}, \alpha_k)$, and set $\kappa'(x) = 0$ for all $x \in I_{2k} \cup (-I_{2k})$ and $\kappa'(x) = 1$ for all $x \in I_{2k+1} \cup (-I_{2k+1})$. Here, $-I_k$ is the interval $(-\alpha_k, -\alpha_{k+1})$. We fix $\kappa(0) = -\varepsilon$ with $\varepsilon > 0$ small enough. Then, the cross section

$$S_2(\vartheta(M_\kappa)) = \{(f(x_1), f(x_2)) \mid f \in M_\kappa, f(x_2) = 0\}$$

is a set similar to the one depicted in Figure 2 (right), but now with an upper right boundary consisting of a staircase with an infinite number of stairs accumulating at the point $(-\varepsilon, -\varepsilon)$ of $\mathbb{R}^2$ with coordinates $-\varepsilon$. This implies that $S_2(\vartheta(M_\kappa))$ is not definable in any o-minimal structure.

We claim that $|(D, M_\kappa)|$ is not a definable topological space in any o-minimal structure. Suppose for the sake of a contradiction that $|(D, M_\kappa)|$ is definable in some o-minimal structure.

First, we embed $|(D, M_\kappa)|$ in M_κ by sending each $E \in |(D, M_\kappa)|$ to the unique function $f \in M_\kappa$ with $E = D + \operatorname{div}(f)$ and $f(x_2) = 0$. The evaluation map $\vartheta \colon M_\kappa \to \mathbb{T}^3$ defines a map $|(D, M_\kappa)| \to S_2(\vartheta(M_\kappa))$, and identifies $S_2(\vartheta(M_\kappa))$ with a topological quotient $|(D, M_\kappa)|/\mathcal{R}$ for a definable relation $\mathcal{R} \subset |(D, M_\kappa)| \times |(D, M_\kappa)|$. Note that $\mathcal{R}$ is definably proper in the sense that the preimage of any compact subset of $S_2(\vartheta(M_\kappa))$ under the projection map $|(D, M_\kappa)| \to S_2(\vartheta(M_\kappa))$ is compact. A theorem of van den Dries [25, Chap. 10, Thm. 2.15] then implies that $S_2(\vartheta(M_\kappa))$ is itself definable in the same o-minimal structure, leading to a contradiction. □

5.3. **Verification of tropical independence and effective computation of the rank.**

Checking whether a finite family of tropical vectors is tropically independent reduces to solving a deterministic mean-payoff game, a well-studied example of a repeated game, see [1, Thm. 4.12]. The converse also holds: solving a deterministic mean-payoff game reduces to the problem of checking tropical linear independence [15]. The question of the existence of a polynomial-time algorithm to solve deterministic mean-payoff games, first raised in [16], remains unsettled. When formulated as a decision problem, deterministic mean-payoff games belong to the complexity class NP ∩ coNP [26], making them unlikely to be NP-complete. Note that tropical vectors with finite entries can be identified to rational functions over a trivial metric graph – with connected components reduced to isolated vertices. In contrast, the metric graphs we consider here are non-trivial and connected.

The following result shows that for rational functions over metric graphs, checking tropical linear independence reduces to solving a more expressive class of games, with *stochastic* transitions.

Theorem 5.6. *Checking whether rational functions $f_1, \dots, f_n$ over a metric graph are tropically independent reduces in polynomial time to solving a stochastic turn-based mean-payoff game.*

Some technical details on the encoding of $f_1, \dots, f_n$ and the metric graph are in order. We assume that the metric graph is given by a model with rational edge lengths. The input comprises one distinguished vertex. Furthermore, we assume that each of the function $f_1, \dots, f_n$ is described by a collection of intervals on which it is affine, as well as by the collection of integral slopes on these intervals, together with a rational value at the distinguished vertex. In this way, each function is uniquely determined.

We prove Theorem 5.6 in Appendix A, after providing more information on game models and algorithmic aspects. It follows from Theorem 5.6 that the decision problem associated with tropical independence of rational functions belongs to NP $\cap$ coNP. This theorem also leads to effective methods for checking tropical independence, see the discussion which follows the proof in the appendix.

Computing the tropical rank of a matrix turns out to be a harder problem. In fact, this problem is known to be NP-hard, even for matrices with entries in $\{0, 1\}$, see [22, Thm. 13]. We use this to deduce the following analogue for rational functions.

Theorem 5.7. *Computing the tropical rank of finitely generated subsemimodules of rational functions on metric graphs is NP-hard.*

Proof. Consider a matrix $A \in \mathbb{T}^{m\times n}$ with entries in $\{0, 1\}$. First, we define a metric graph Γ of model $(G = (V, E), \ell)$, with $G = (V, E)$ the complete graph on m vertices and $\ell(e) = 2$ for all edges of G. Let $V = \{v_1, \dots, v_m\}$ be the vertex set and $E = \big\{\{v_i, v_s\} \bigm| i, s \in [m], i \neq s\big\}$ the edge set of G. For every edge $\{v_i, v_s\}$, denote by w_{is} the midpoint of the edge.

We associate to A a semimodule M of rational functions on Γ, generated by the following rational functions $f_1, \dots, f_n$. For each $j \in [n]$, set

$$f_j(v_i) = A_{ij} \text{ and } f_j(w_{is}) = \min(A_{ij}, A_{sj}) \text{ for all pair of distinct elements } i, s \in [m].$$

We extend f_j to Γ by linear interpolation. It is easy to see that $f_j \in \mathrm{Rat}(\Gamma)$.

Since the entries of the matrix A belong to $\{0, 1\}$, for each edge $\{v_i, v_s\}$ of G, one of the following four cases occurs:

- $f_j(v_i) = f_j(w_{ij}) = f_j(v_s) = 0$,
- $f_j(w_{is}) = f_j(v_s) = 0 < f_j(v_i) = 1$,
- $f_j(v_i) = f_j(w_{is}) = 0 < f_j(v_s) = 1$,
- $f_j(v_i) = f_j(w_{ij}) = f_j(v_s) = 1$.

Now consider the family of points $x_1, \dots, x_K$ consisting of the vertices $v_1, \dots, v_m$ together will all the midpoints w_{is} with $i < s$ and $i, s \in [m]$. We deduce by checking the above four cases that every f_j satisfies the "two-slopes" condition stated in Subsection 5.1, so that the restriction map $\vartheta \colon M \to \mathbb{T}^K$ is injective.

By Proposition 5.2, $r_{\text{trop}}(M) = r_{\text{trop}}(\vartheta(M)) = r_{\text{trop}}(B)$, where the matrix $B \in \mathbb{T}^{K\times n}$ is given explicitly by $B_{kj} = A_{ij}$ if x_k is equal to some vertex v_i, and $B_{kj} = \min(A_{ik}, A_{sj})$ if x_k is equal to some midpoint w_{is}. Therefore, the rows of the matrix B comprise all the rows of A together with additional rows each of which is a tropical sum of pair of rows in A. It follows that $r_{\text{trop}}(B) = r_{\text{trop}}(A)$.

By [22, Thm. 13], checking whether $r_{\text{trop}}(A) \geqslant r$ is an NP-hard problem. We infer that checking whether $r_{\text{trop}}(M) \geqslant r$ for a finitely generated subsemimodule $M \subseteq \mathrm{Rat}(\Gamma)$ is also NP-hard. □

5.4. **Higher dimension.** Let $Y \subseteq \mathbb{R}^d$ be a polyhedral subspace (e.g., a tropical subvariety). Let $\mathrm{Rat}(Y, \mathbb{R})$ be the union of ∞ and the set of piecewise linear functions on Y (with non-necessarily integral slopes). Endowed with the operation of tropical addition and tropical multiplication by constants, $\mathrm{Rat}(Y, \mathbb{R})$ is a semimodule over $\mathbb{T}$. Let M be a finitely generated subsemimodule of $\mathrm{Rat}(Y, \mathbb{R})$. For example, if Y is a tropical subvariety, and D is a divisor on Y, then M may be a finitely generated subsemimodule of $R(D)$, where $R(D)$ is the union of $\{\infty\}$ and the set of piecewise linear functions on Y with integral slopes. We define $r_{\text{trop}}(M)$ as the maximum integer r such that there exist tropically independent elements $f_1, \dots, f_r \in M$.

Let $g_1, \dots, g_l$ be a generating set for M. Consider the map

$$\Psi \colon \mathbb{R}^l \to M, \qquad (c_1, \dots, c_l) \mapsto \min_{j \in [l]} \big(g_j + c_j\big).$$

We define a notion of dimension for M as follows. Consider an element $f \in M$.

Proposition 5.8. *The subset $\Psi^{-1}(f) \subset \mathbb{R}^l$ is polyhedral.*

Proof. Choose a polyhedral structure Δ on Y such that the generators $g_1, \dots, g_l$ and f are affine on each face of Δ. For each face $\sigma \in \Delta$ and $(c_1, \dots, c_l) \in \Psi^{-1}(f)$, since $\min_{j \in [l]}(g_j + c_j)_{|\sigma} = f_{|\sigma}$, and $f_{|\sigma}$ is affine, we get the existence of $j = \mu(\sigma) \in [l]$ such that

$$f_{|\sigma} = (g_{\mu(\sigma)} + c_{\mu(\sigma)})_{|\sigma} \leqslant (g_i + c_i)_{|\sigma}, \text{ for all } i \in [l].$$

For each function $\mu \colon \Delta \to [l]$, let C_μ be the set of all points $(c_1, \dots, c_l) \in \mathbb{R}^l$ such that the inequality above is satisfied for all $\sigma \in \Delta$. This is a polyhedral subset of $\mathbb{R}^l$. Moreover, $\Psi^{-1}(f)$ is the union of the sets C_μ. We infer the result. □

We define the dimension of M, denoted by $\dim(M)$, as

$$\dim(M) = \max_{f \in M} \big[l - \dim\big(\Psi^{-1}(f)\big)\big].$$

Question 5.9. *Let M be a finitely generated subsemimodule of* $\mathrm{Rat}(Y, \mathbb{R})$. *Do we have the equality $r_{\text{trop}}(M) = \dim(M)$?*

A positive answer would generalize Theorem 1.1. Note that we may reduce to the case where Y is compact in order to use Theorem 2.5.

Appendix A. Stochastic Game Model and Proof of Theorem 5.6

We first provide some background on the stochastic game appearing in Theorem 5.6. It is a special case of the model introduced by Shapley [24]. There are two players, called Min and Max. The game is played on a finite state space $[n]$. In each state $i \in [n]$, a finite set of possible actions of Max, A_i, is specified. Moreover, for each choice of state $i \in [n]$ and for each choice of action $\alpha \in A_i$ of Max, a finite set of possible actions of Min, $B_{i,\alpha}$, is specified. For every pair of actions $\alpha \in A_i$ and $\beta \in B_{i,\alpha}$, we associate a real number $r_i^{\alpha\beta}$ and a probability measure on the set $[n]$, $P_i^{\alpha\beta} = (P_{i,1}^{\alpha\beta}, \dots, P_{i,n}^{\alpha\beta}) \in \mathbb{R}^n_{\geqslant 0}$; so we have $\sum_{j \in [n]} P_{i,j}^{\alpha\beta} = 1$.

We consider the *turn-based* game, also known as the *perfect information game*, in which the two players choose their actions sequentially, being informed of the current state and previous actions of the other player. Every stage of the game is played as follows. At a given stage, the current state being $i \in [n]$, Max selects an action $\alpha \in A_i$, and Min, aware of this choice, selects an action $\beta \in B_{i,\alpha}$. Then, Min pays $r_i^{\alpha\beta}$ to Max, and with probability $P_{i,j}^{\alpha\beta}$ the next state becomes $j \in [n]$.

In the finite horizon game, starting from a given initial state, Max seeks to maximize the expected payment received from Min over a given number of consecutive stages, whereas Min seeks to minimize the same payment.

The dynamic programming operator of this game, known as the *Shapley operator*, is the map $T = (T_1, \dots, T_n)\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$T_i(c) \coloneqq \max_{\alpha \in A} \min_{\beta \in B_{i,\alpha}} \Big(r_i^{\alpha\beta} + \sum_{j \in [n]} P_{i,j}^{\alpha\beta} c_j \Big). \tag{A.1}$$

The game started from state i and played over N consecutive stages has a value denoted by v_i^N, and the value vector $v^N = (v_1^N, \dots, v_n^N) \in \mathbb{R}^n$ satisfies $v^N = T^N(0)$, with T^N being the N-th iterate of T, see Lem. VII.1.3 and Rem. VII.1.1 in [23]. Moreover, the *mean payoff* of this stochastic game is defined as $\lim_{N\to\infty} v_i^N/N$. The existence of this limit can be derived as a special case of a result by Bewley and Kohlberg [8], which applies to concurrent games.

In order to prove Theorem 5.6, we first show that the operator T defined in Equation (2.1) in Section 2.2 can be rewritten in the form (A.1). To see this, we consider a finite family $(X_\alpha)_{\alpha \in A}$ of closed segments on edges of Γ on which every map f_j is affine and such that $\cup_{\alpha \in A} X_\alpha = \Gamma$. Thus, after identifying the segment X_α with an interval of the form $[u_\alpha, v_\alpha]$ in $\mathbb{R}$, the restriction of f_j on the interval X_α is of the form $f_j(x) = m_{\alpha,j}x + \eta_{\alpha,j}$ for all $x \in X_\alpha$, for some integer $m_{\alpha,j}$ and some rational number $\eta_{\alpha,j}$. For each $\alpha \in A$, define the operator $T_\alpha = (T_{\alpha,1}, \dots, T_{\alpha,n})\colon \mathbb{R}^n \to \mathbb{R}^n$ by setting

$$T_{\alpha,i}(c) \coloneqq \sup_{x \in [u_\alpha, v_\alpha]} \Big(\min_{j \in [n] \smallsetminus \{i\}} \big((m_{\alpha,j} - m_{\alpha,i})x + \eta_{\alpha,j} - \eta_{\alpha,i} + c_j \big) \Big), \tag{A.2}$$

so that $T_i(c) = \max_{\alpha \in A} T_{\alpha,i}(c)$.

Evaluating the operators $T_{\alpha,i}$, for $\alpha \in A$ and $i \in [n]$ involves solving an optimization problem of the form

$$\sup_{x \in [u,v]} \Phi(x) \quad \text{where} \quad \Phi(x) = \min_{j \in J} (\gamma_j x + d_j) \text{ for } x \in [u,v], \tag{A.3}$$

with $(\gamma_j)_{j \in J}$ and $(d_j)_{j \in J}$ finite families of real numbers, $J \subseteq [n]$, and $u, v \in \mathbb{R}$.

In fact, this optimization problem has an explicit solution given by the following lemma.

Lemma A.1. *Keeping the same notation as above, the value of the optimization problem (A.3) is given by*

$$\min \left(\min_{j \in J^-} (\gamma_j u + d_j),\ \min_{j \in J^+} (\gamma_j v + d_j),\ \min_{\substack{j \in J^+, k \in J^- \\ \gamma_j > \gamma_k}} \Big(\frac{-\gamma_k}{\gamma_j - \gamma_k} d_j + \frac{\gamma_j}{\gamma_j - \gamma_k} d_k \Big) \right) \tag{A.4}$$

where $J^+ = \{ j \in J \mid \gamma_j \geqslant 0 \}$ *and* $J^- = \{ j \in J \mid \gamma_j \leqslant 0 \}$.

Proof. The function Φ is piecewise linear and concave. Hence, the maximum locus of Φ forms a segment I possibly reduced to a point. If $u \in I$, the minimum in (A.4) is achieved by the first term, which is the value of Φ at u. Similarly, if $v \in I$, the minimum in (A.4) is achieved by the second term, which is the value of Φ at v. It remains to treat the case where $u, v \notin I$. In this case, I is exactly the locus of points of $[u,v]$ around which both outgoing slopes of

Φ are nonpositive. Then, the minimum is achieved by the third term in (A.4), which is the (constant) value of Φ on I. □

We now define a game such that $T_i(c)$ can be rewritten in the form (A.1). The set of states is $[n]$. The set of actions of Max is $A_i = A$ independently of the state $i \in [n]$. Now, for each $\alpha \in A_i$, to express $T_{\alpha,i}(c)$ in the form (A.3), we take $u = u_\alpha$, $v = v_\alpha$, $J = [n] \setminus \{i\}$, $\gamma_j = m_{\alpha,j} - m_{\alpha,i}$ and $d_j = \eta_{\alpha,j} - \eta_{\alpha,i} + c_j$, and we define $J^+_{\alpha,i} = \{j \in J \mid \gamma_j \geqslant 0\}$ and $J^-_{\alpha,i} = \{j \in J \mid \gamma_j \leqslant 0\}$, as in Lemma A.1. The set of actions $B_{i,\alpha}$ is the disjoint union of the sets $J^+_{\alpha,i}$, $J^-_{\alpha,i}$ and the set of ordered pairs $(j,k) \in J^+_{\alpha,i} \times J^-_{\alpha,i}$ with $\gamma_j > \gamma_k$. The instantaneous payments and transition probabilities are defined as follows, for all $i \in [n]$ and $\alpha \in A$.

- If $\beta = j$ with $j \in J^+_{\alpha,i}$, we set $r_i^{\alpha,\beta} = (m_{\alpha,j} - m_{\alpha,i})u + \eta_{\alpha,j} - \eta_{\alpha,i}$ and $P_i^{\alpha,\beta} = e_j$, where e_j denotes the j-th vector of the canonical basis of $\mathbb{R}^n$.
- If $\beta = j$ with $j \in J^-_{\alpha,i}$, we set $r_i^{\alpha,\beta} = (m_{\alpha,j} - m_{\alpha,i})v + \eta_{\alpha,j} - \eta_{\alpha,i}$ and $P_i^{\alpha,\beta} = e_j$.
- If $\beta = (j,k)$ with $(j,k) \in J^+_{\alpha,i} \times J^-_{\alpha,i}$ and $\gamma_j > \gamma_k$, we define $\pi_j = -\gamma_k/(\gamma_j - \gamma_k)$ and $\pi_k = \gamma_j/(\gamma_j - \gamma_k)$, so that π_j and π_k are nonnegative and satisfy $\pi_j + \pi_k = 1$. We set $r_i^{\alpha,\beta} = \pi_j(\eta_j - \eta_i) + \pi_k(\eta_k - \eta_i)$.

As noted in Remark 2.7, the number ρ such that $\rho + c = T(c)$ for some $c \in \mathbb{R}^n$ is unique. Moreover, by (2.4), ρ coincides with the value of the turn-based stochastic mean-payoff game, independently of the choice of the initial state $i \in [n]$. By Theorem 2.5, the family $f_1, \dots, f_n$ is tropically linearly independent if, and only if, ρ is strictly positive.

The number of states and actions, as well as the bit-sizes of the instantaneous payments and transition probabilities of this game are polynomially bounded in the input size. It follows that verifying whether $f_1, \dots, f_n$ are tropically linearly independent reduces in polynomial time to checking whether the value of a turn-based mean-payoff game is strictly positive. This finishes the proof of Theorem 5.6. □

As a final remark, we note that, to solve the game here, it suffices to solve the additive eigenproblem $T(c) = \rho + c$. This can be done using several algorithms, see e.g. [9] and the discussion in Section 6 of [2]. We refer the reader to [5] for more information on the complexity of turn-based games.

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