

ON SOLUTIONS TO TROPICAL MONGE–AMPÈRE EQUATIONS

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ABSTRACT. This paper develops convex analysis on tropical varieties and investigates the properties of the tropical Monge–Ampère equations.

We start by introducing notions of plurisubharmonic functions and Kähler forms on arbitrary tropical varieties. We show that many classical results of pluripotential theory extend to this setting, including an analogue of Demailly’s regularization theorem. Next, we study the solutions to the tropical Monge–Ampère equations. While we establish their uniqueness, we show that the existence is not guaranteed in general. More precisely, we prove that the tropical analogue of the Calabi–Yau theorem fails for any pair (X, θ) consisting of a compact tropical variety X of dimension at least two satisfying a mild genericity assumption, and a tropical Kähler form θ on X . In particular, there exist measures compactly supported in the open part of X that admit no solution. In contrast, for tropical varieties that are regular in codimension one, we show the existence of solutions. Finally, via strongly faithful tropicalizations whose existence is established in this paper, we relate the tropical solutions, when they exist, to the solutions of non-Archimedean Monge–Ampère equations on Berkovich spaces.

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1. INTRODUCTION

The study of convex analysis and Monge–Ampère equations on $\mathbb{R}^n$ has a long and fruitful history, with applications across many branches of mathematics. On the other hand, the theory of plurisubharmonic (psh) functions, viewed as the complex counterparts of convex functions, and of complex Monge–Ampère equations on complex manifolds has proved fundamental in complex and Kähler geometry. The purpose of this paper is to develop and study a theory of convex analysis and Monge–Ampère equations on tropical varieties.

We provide a broader background to our work before going into more details. Let M be a smooth projective complex variety of dimension d . Let ω be a Kähler form on M of volume $\deg(\omega^d) = \int_M \omega^d$, and let μ be a strictly positive smooth probability measure on M . The celebrated Calabi–Yau theorem asserts that the complex Monge–Ampère equation $(\omega + dd^c \varphi)^d = \deg(\omega^d) \mu$ has a unique solution $\varphi \in C^\infty(M)$, up to an additive constant, such that $\omega + dd^c \varphi \geq 0$. A function satisfying this positivity condition is called an ω -psh function on M . Geometrically, $\omega + dd^c \varphi$ constitutes a new Kähler form on M within the Kähler class $[\omega]$. Consequently, solving this Monge–Ampère equation is equivalent to finding a Kähler metric with a prescribed volume form, which in turn prescribes its Ricci curvature. A prominent consequence of the Calabi–Yau theorem is the existence of Ricci-flat Kähler metrics on Calabi–Yau manifolds (*i.e.* varieties with a trivial canonical bundle).

Understanding the degeneration of Kähler metrics, particularly Ricci-flat metrics, is a central problem in Kähler geometry. To make this precise, let $\mathfrak{X} \rightarrow \mathbb{C} \setminus \{0\}$ be a smooth projective morphism, so that each fiber $\mathfrak{X}_t$ is a d -dimensional projective manifold for $t \in \mathbb{C} \setminus \{0\}$. Let $\mathfrak{L}$ be an ample line bundle on $\mathfrak{X}$. For each t , we equip the fiber $\mathfrak{X}_t$ with a (Ricci-flat) Kähler metric ω_t in the class $c_1(\mathfrak{L}_t)$. A fundamental question is to determine the asymptotic behavior of ω_t as $t \rightarrow 0$.

A powerful approach to studying the limits of degenerating Kähler metrics relies on non-Archimedean geometry. This perspective was originally introduced by Kontsevich and Soibelman [KS06] in the context of the SYZ conjecture, and was subsequently developed by Boucksom, Favre, and Jonsson [BFJ15, BJ17]. To characterize these limits, Kontsevich and Soibelman proposed studying the non-Archimedean polarized Calabi–Yau variety $(\mathbf{X}^{\text{an}}, \mathbf{L}^{\text{an}})$ over the complete non-Archimedean field of Laurent series $\mathbb{K} = \mathbb{C}((t))$. This space is obtained as the Berkovich analytification of the polarized $\mathbb{K}$ -variety $(\mathbf{X}, \mathbf{L})$ induced by the generic fiber of the family $(\mathfrak{X}, \mathfrak{L})$. By a theorem of Payne [Pay09], the Berkovich space $\mathbf{X}^{\text{an}}$ can be realized as the inverse limit of all its tropicalizations, where each tropicalization is a finite integral polyhedral complex endowed with a weight structure. In this sense, $\mathbf{X}^{\text{an}}$ can be naturally viewed as an “infinite” polyhedral space.

Following an idea of Shou-Wu Zhang [Zha95], non-Archimedean psh functions are defined using the algebraic geometric notion of nefness, which, via the toric dictionary, serves as an analogue of convexity. A non-Archimedean Monge–Ampère operator was introduced by Chambert-Loir [CL06], and a non-Archimedean analogue of the Calabi–Yau theorem was subsequently established by Boucksom, Favre, and Jonsson [BFJ15]. Specifically, they proved that certain non-Archimedean Monge–Ampère equations admit unique solutions (up to additive constants) within the class of non-Archimedean psh functions. In particular, for a degenerating family $\mathfrak{X}$ of Calabi–Yau manifolds, there exists a non-Archimedean Calabi–Yau potential $\varphi_{\text{CY,NA}}$ on $\mathbf{X}^{\text{an}}$.

It was later shown by Yang Li [Li25] that the Archimedean Calabi–Yau potentials $\varphi_{\text{CY},t}$ converge in a suitable sense to $\varphi_{\text{CY,NA}}$. Furthermore, when $\mathfrak{X}$ is a maximally degenerating family, this non-Archimedean potential satisfies a “comparison property” [Li26], which ultimately leads to a version of the SYZ conjecture [Li23a].

However, because they are defined on an “infinite polyhedral space”, non-Archimedean psh functions are extremely difficult to describe explicitly. Therefore, despite the relatively complete non-Archimedean picture, it remains beneficial from a complex geometric perspective to study convex functions and Monge–Ampère equations on a single tropicalization, which is a finite polyhedral shadow of the Berkovich space, as a way to seek a more explicit description of the limit metrics. The tropical Monge–Ampère equations have previously been studied for specific examples of tropical hypersurfaces [HJMM24, Li23b, AH23] (although under disguised languages). The goal of this paper is to extend this study to general tropical varieties.

1.1. Convex analysis on tropical varieties. In this paper, a tropical variety is defined as an extended polyhedral space endowed with a tropical weight function that satisfies the balancing condition (see Section 2.1). More precisely, a polyhedral chart is an open subset of the canonical compactification¹ of a simplicial rational fan, and an extended polyhedral space is a Hausdorff space obtained by gluing polyhedral charts via $\mathbb{Z}$ -linear maps. A tropical weight function on an extended polyhedral space assigns a positive integer to each top-dimensional face. It satisfies the balancing condition if the weighted sum of the normal vectors in the star fan of each codimension-one face vanishes. For a tropical variety X , we use X_o to denote its open part which is, by definition, the union of the open parts of the polyhedral charts of X ².

In order to be able to work on tropical Monge–Ampère equations, the first task is to specify a proper class of convex functions on tropical varieties. When the tropical variety X is embedded in $\mathbb{R}^n$, a natural choice to consider is the restriction of convex functions in the ambient space, the so-called class of “globally convex functions”. However, this class of convex functions turns out to be too restrictive for our purpose. Indeed, in the setting of the metric SYZ conjectures for Calabi–Yau hypersurfaces in toric Fano varieties, the optimal transport problem introduced and studied in [HJMM24, Li23b, AH23] turns out to be equivalent to solving a certain tropical Monge–Ampère equation inside the class of globally convex functions mentioned above. However, in [AH23], the authors showed that for many examples (actually, most examples they examined), this optimal transport problem does not admit any solution. This, along with recently developed Hodge theoretic results in the combinatorial and tropical geometric [AHK18, AP26, AP20a] settings, motivated us to introduce a larger class of convex functions.

Definition 1.1 (Definition 3.1). Let X be a tropical variety, and let U be an open subset of X . We say that a function $u: U \rightarrow \mathbb{R}$ is *locally convex* on U if

- (1) $u \in C^0(U)$.
- (2) For any $x \in U$, there exist an open neighborhood U' of x and an affine function $l: U' \rightarrow \mathbb{R}$ such that $u(x) = l(x)$ and $u(y) \geq l(y)$ for all $y \in U'$.

¹For a simplicial cone $\sigma \cong [0, +\infty)^k$, for the coordinates given by the generators of the rays of σ , its canonical compactification is isomorphic to $[0, +\infty]^k$, obtained by allowing the coordinates to take the value infinity. The canonical compactification of a simplicial fan is obtained by gluing the canonical compactifications of its cones along the common faces.

²For an open subset U in the canonical compactification $\overline{\Sigma}$ of a simplicial fan Σ , its open part is the intersection of U with the support of Σ .

We use $\text{LConv}(U)$ to denote the space of locally convex functions on U .

A locally convex function is called *strictly locally convex* if in addition we have that $u(y) > l(y)$ for all $y \in U' \setminus \{x\}$ in Definition 1.1 (2).

When $X = \mathbb{R}^n$, the above class of convex functions coincides with the class of usual convex functions (with finite values). In general, we show that over the open part X_o of the tropical variety, this class of convex functions is equivalent to a class of convex functions introduced in [BBS22] based on a notion of polyhedral convex combination.

In [APW26], we introduce a formalism of mixed differential forms. In this formalism, mixed $(0,0)$ -forms are just piecewise smooth functions³. In general, this differential calculus incorporates both discrete and continuous features. The discrete contribution is encoded in the Chow rings of local fans (isomorphic to the Chow rings of the associated toric varieties), while the continuous part arises from differential calculus of superforms on real vector spaces developed by Larserberg [Lag12]. The mixed differential operators $\partial, \bar{\partial}$, as well as the wedge products for mixed differential forms are defined, and enjoy the nice properties of their complex-geometric analogues. The framework of mixed differential calculus provides a generalization of smooth superforms of [Lag12, CLD12] and δ -forms of [GK17, Mih24, BGGK25]. In particular, the differential operators $\partial, \bar{\partial}$ are generalizations of the operators d', d'' in the setting of smooth superforms and δ -forms.

Based on the class of locally convex functions, and using the mixed differential calculus, we give the definition of Kähler forms and psh functions on tropical varieties.

A real closed mixed $(1,1)$ -form ω on X is a mixed $(1,1)$ -form satisfying the following condition: for every $x \in X$, there exist an open neighborhood U of x and a piecewise smooth function h on U such that $\omega = \partial\bar{\partial}h$ ⁴. In this case, h is called a *local potential* of ω .

Definition 1.2 (Definition 5.2). A *Kähler form* θ on a compact tropical variety X is a real closed mixed $(1,1)$ -form θ such that any local potential h of θ is strictly locally convex (Definition 5.2).

If we weaken the condition on the local potentials by requiring them to be locally convex (instead of strictly locally convex), we say that the real closed mixed $(1,1)$ -form θ is *semipositive* (Definition 4.1).

Definition 1.3 (Definition 4.3). Let θ be a Kähler form on X , and let φ be a (real-valued) continuous function on X . We say that φ is a *continuous θ -psh function* if for any $x \in X$, there exists a neighborhood U of x and a local potential h of θ on U such that $h + \varphi \in \text{LConv}(U)$. We use $\text{CPSH}_\theta(X)$ to denote the space of θ -psh functions on X .

The class of continuous θ -psh functions introduced above has many nice properties. For example, we show that it satisfies the so-called envelope property, *i.e.* the upper-semicontinuous regular upper envelope of any bounded above family of θ -psh functions is again a continuous θ -psh function. If X is a compact tropical variety, we also show that the class of continuous θ -psh functions is uniformly Lipschitz on any compact subset of X_o .

³A function f is called piecewise smooth if it is continuous, and there exists a polyhedral structure such that the restriction of f on each face is smooth.

⁴Readers who prefer not to delve into our formalism of mixed differential forms may instead view $\omega = d'd''h$ as a real closed $(1,1)$ -supercurrent (resp. $(1,1)$ - δ -current) in the sense of [CLD12] (resp. [GK17]). This does not affect the understanding of the main results presented in this paper.

A tropical variety is called *prekähler* if it admits a Kähler form. A continuous function f on X is called *θ -piecewise linear* (Definition 4.20) if there exists a polyhedral structure on X such that, for any local potential h of θ , the restriction of $h + f$ on each face is affine. The class of functions on X that are both θ -psh and θ -piecewise linear plays the role of a “nice” subclass of tropical θ -psh functions, analogous to the class of smooth θ -psh functions in complex pluripotential theory ([Dem92]), or the class of θ -psh PL functions in non-Archimedean pluripotential theory ([BFJ16b, BJ22, BJ25]). Our first main theorem is a regularization theorem for θ -psh functions on compact prekähler tropical varieties, which can be viewed as a tropical analogue of Demailly’s regularization theorem ([Dem92, BK07]).

Theorem A (Theorem 5.13). *Let X be a compact prekähler tropical variety. Let θ be a Kähler form on X . Then, for any continuous θ -psh function φ on X , there exists a sequence of functions $\{\varphi_i\}_{i \geq 1}$ converging uniformly to φ such that φ_i is θ -piecewise linear and θ -psh for each $i \geq 1$.*

1.2. Tropical Monge–Ampère equations. Let X be a compact prekähler tropical variety of dimension d endowed with a Kähler form θ .

In [APW26], we use the formalism of mixed differential calculus mentioned above to define tropical Monge–Ampère measures for piecewise smooth functions. For a piecewise smooth θ -psh function φ , we define the Monge–Ampère measure $\text{MA}_\theta(\varphi)$ to be the nonnegative Radon measure associated to the top degree form $(\theta + \partial\bar{\partial}\varphi)^{\wedge d}$. Using the regularization theorem (Theorem A) and a tropical version of the Chern–Levine–Nirenberg inequality, we extend the definition of Monge–Ampère measures to all continuous θ -psh functions.

Let now μ be a compactly supported probability measure on X_o . The tropical Monge–Ampère equation is the following.

$$(1.1) \quad \text{MA}_\theta(\varphi) = \deg(\theta^d)\mu \text{ on } X, \quad \varphi \in \text{CPSH}_\theta(X).$$

Our second main theorem is the following.

Theorem B (Uniqueness, Theorem 8.1). *Let X be a compact prekähler tropical variety of dimension d equipped with a Kähler form θ ⁵. Assume that Equation (1.1) admits a solution. Then, the solution is unique up to adding a constant function.*

Regarding the existence of solutions, in dimension one, Equation (1.1) reduces to a Poisson equation on a metric graph, and it is well-known that it always admits a solution when X is locally matroidal (see for example [Thu05]). However, somehow surprisingly, we proved that it fails for almost every tropical variety of dimension at least 2.

Theorem C (Failure of existence in higher dimensions, Theorem 11.4). *Let X be a compact prekähler tropical variety of dimension d equipped with a Kähler form θ . Suppose $\dim X \geq 2$ and X satisfies some genericity condition⁶. Then, there exists a compactly supported probability measure μ on X_o such that Equation (1.1) does not admit any solution.*

We remark that if X is a projective tropical toric variety TP_Ξ , existence of the solutions holds (see for example [BFJ15, § 9.2]). Note that TP_Ξ does not satisfy the genericity condition

⁵Note that we actually prove Theorem B under the assumption that θ is a “strongly semipositive form” (see Definition 5.16 and Remark 5.17), which is a weaker condition than θ being a Kähler form. The same for Theorem C and Theorem D below.

⁶That is, the condition stated in Definition 11.1 is satisfied by the star fan of each codimension two face of some polyhedral structure of X .

stated in Definition 11.1. We conjecture that the tropical analogue of Calabi–Yau theorem holds for (X, θ) if, and only if, the open part X_o can be written locally as a product of $\mathbb{R}^{d-1}$ and a locally matroidal tropical curve (Conjecture 11.12). We establish the “if” direction of this conjecture in Theorem 10.4 by showing the existence of solutions on compact prekähler tropical varieties that are *regular in codimension one*.

Theorem D (Existence for tropical varieties regular in codimension one, Theorem 10.4). *Let X be a compact prekähler tropical variety of dimension d equipped with a Kähler form θ . Assume that X_o is locally modeled as a product of a matroidal tropical curve and $\mathbb{R}^{d-1}$. Then, Equation 1.1 has a solution for any probability measure compactly supported on X_o .*

We mention briefly the strategies for proving these theorems.

For the uniqueness theorem, the proof is similar to the proof of uniqueness for non-Archimedean Monge–Ampère equations given in [BFJ15] using a tropical Cauchy–Schwartz inequality. For the existence, we follow the variational approach for complex and non-Archimedean Monge–Ampère equations in [BBGZ13, BFJ15]. We enlarge the space $\text{CPSH}_\theta(X)$ to the space $\text{PSH}_\theta(X)$ equipped with the topology of pointwise convergence of X_o . Denote by $\text{PSH}_\theta(X)/\mathbb{R}$ the quotient space of $\text{PSH}_\theta(X)$ modulo additive constants. A uniform Lipschitz result for $\text{PSH}_\theta(X)$ (see Theorem 4.14 and Proposition 4.22) implies that $\text{PSH}_\theta(X)/\mathbb{R}$ is a compact space.

We construct an energy functional

$$E_\theta: \text{PSH}_\theta(X)/\mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$$

whose derivative is the Monge–Ampère operator, in the sense that

$$\left. \frac{d}{dt} \right|_{t=0+} E_\theta(\varphi + tf) = \int_X f \text{MA}_\theta(\varphi)$$

for every $\varphi \in \text{CPSH}_\theta(X)/\mathbb{R}$ and $f \in C^0(X)$ satisfying $\varphi + tf \in \text{CPSH}_\theta(X)/\mathbb{R}$ for any $t \geq 0$.

Given a compactly supported probability measure μ on X_o , consider the Lagrangian functional $F_\mu: \text{PSH}_\theta(X)/\mathbb{R} \rightarrow \mathbb{R}$ defined by

$$F_\mu(\varphi) := E_\theta(\varphi) - \int_X \varphi \mu.$$

The upper semicontinuity of F_μ together with the compactness of $\text{PSH}_\theta(X)/\mathbb{R}$ guarantee the existence of a maximizer $\varphi_\mu \in \text{PSH}_\theta(X)/\mathbb{R}$. Furthermore, we show that the maximizer φ_μ belongs to $\text{CPSH}_\theta(X)/\mathbb{R}$. However, contrary to the setting of complex and non-Archimedean pluripotential theory, φ_μ is not necessarily a solution to Equation 1.1. Actually, in our tropical setting, we show that the maximizer of F_μ is a solution of Equation 1.1 for any compactly supported probability measure on X_o if and only if the orthogonality property holds for (X, θ) . Following [BBGZ13, BFJ15], we say that (X, θ) satisfies the *orthogonality property* if

$$\int_X (f - P_\theta(f)) \text{MA}_\theta(P_\theta(f)) = 0 \quad \text{for any } f \in C^0(X).$$

Here, $P_\theta(f)$ is the θ -psh envelope defined by

$$P_\theta(f) := \left(\sup \left\{ \varphi \mid \varphi \in \text{PSH}_\theta(X) \text{ and } \varphi \leq f \right\} \right)^*.$$

The *envelope property*, that $P_\theta(f)$ belongs to $\text{CPSH}_\theta(X)$, is proved in the tropical setting (Corollary 4.17).

When X is locally modeled by a product of a matroidal curve and R^{d-1} , we show that the orthogonality property holds for (X, θ) . In contrast, if $\dim X \geq 2$ and X satisfies the genericity condition alluded to above, the orthogonality property fails. The obstruction is that, for such X , one can construct, using polyhedral-geometric methods, a function $\varphi \in \text{CPSH}_\theta(X)$ such that $\text{MA}_\theta(\varphi)(\{x\}) > 0$ for some $x \in X_o$, while $\varphi + h$ is not strictly locally convex at x , for a local potential h of θ around x . An explicit example in dimension 2 is given in Section 11.3. We show that the orthogonality property holds in codimension one (Theorem 10.2).

1.3. Connections to non-Archimedean solutions. Let $(\mathfrak{X}_t, \mathfrak{L}_t)_{t \in \mathbb{D}^*}$ be a maximally degenerating family of smooth polarized Calabi–Yau varieties over the punctured disc $\mathbb{D}^* \subset \mathbb{C} \setminus \{0\}$. Let $(\mathbf{X}, \mathbf{L})$ be the generic fibre of the family over $\mathbb{K} = \mathbb{C}((t))$. We assume that $\mathbf{L}$ is very ample on $\mathbf{X}$, and $\iota: \mathbf{X} \hookrightarrow \mathbb{P}^n$ is a closed embedding satisfying $\iota^* \mathcal{O}(1) = \mathbf{L}$. Then, the trivial model of $(\mathbb{P}^n, \mathcal{O}(1))$ over $\mathbb{K}^\circ$ gives a semipositive $(1, 1)$ -form on $(\mathbb{P}^n)^{\text{an}}$. We denote by ω_{NA} its restriction on $\mathbf{X}^{\text{an}}$. Let $\mu_{\text{CY,NA}}$ be the normalized non-Archimedean Calabi–Yau measure supported on the essential skeleton $\text{Sk}(\mathbf{X}) \subset \mathbf{X}^{\text{an}}$ defined in [KS06, BJ17], and let $\varphi_{\text{CY,NA}} \in \text{CPSH}_{\omega, \text{NA}}(\mathbf{X})$ be the Calabi–Yau potential given by the non-Archimedean Calabi–Yau theorem ([BFJ15], see also Theorem 12.2).

We view $\mathbb{P}^n$ as a toric variety with the open dense torus $\mathbf{T}$. Let $\mathbf{Y} := \mathbf{X} \cap \mathbf{T}$, and let

$$\text{Trop}: \mathbf{X}^{\text{an}} \rightarrow \mathbb{TP}^n$$

be the (extended) tropicalization map. We denote by $X := \text{Trop}(\mathbf{X}, \iota)$ and $Y := \text{Trop}(\mathbf{Y}, \iota)$ the tropicalizations of $\iota: \mathbf{X} \hookrightarrow \mathbb{P}^n$ and $\iota: \mathbf{Y} \rightarrow \mathbf{T}$, respectively. Then Y is a tropical variety embedded in a real vector space of dimension n . However, X is not necessarily a tropical variety, since X does not necessarily satisfy the regularity along the boundary of $\mathbb{TP}^n$ (i.e. it might develop “singularities” at infinity).

Therefore, instead of considering tropical Monge–Ampère equations on X , we consider the canonical compactification $\bar{Y}$ of Y with respect to a rational simplicial polyhedral structure on Y , such that $\bar{Y}$ behaves nicely at infinity and moreover admits a closed semipositive $(1, 1)$ -form ω induced by the Fubini–Study potential $h = \sup\{0, -x_1, \dots, -x_n\}$ in $\mathbb{R}^n$. (For these to happen, it is enough to ensure that the recession cones of the polyhedra in the polyhedral structure form a fan that refines the fan of the projective space; this can be always achieved, see Corollary 5.9.)

Recall that $\mu_{\text{CY,NA}}$ is a measure supported on the dual complex $\Delta_{\mathfrak{X}}$, where $\mathfrak{X}$ is an SNC model of $\mathbf{X}$. We say that the tropicalization map is *strongly faithful* (with respect to $\Delta_{\mathfrak{X}}$) if $\text{Trop}|_{\Delta_{\mathfrak{X}}}$ is injective, and any maximal dimensional face contained in $\text{Trop}(\Delta_{\mathfrak{X}})$ is of weight 1. This is a refinement of the existing notion of faithful tropicalization in tropical geometry [BPR13, GRW16]. Refining [GRW16, Thm. 9.5], we show that there always exists a projective embedding of $\mathbf{X}$ given by a basis of $H^0(\mathbf{X}, \mathbf{L}^{\otimes k})$ with k large enough such that the associated tropicalization to this embedding is strongly faithful (with respect to $\Delta_{\mathfrak{X}}$).

We prove the following result.

Theorem E (Theorem 12.11). *Suppose that Y is a strongly faithful tropicalization with respect to $\Delta_{\mathfrak{X}}$ for some SNC model $\mathfrak{X}$. Choose a polyhedral structure on Y and let $\bar{Y}$ be the corresponding canonical compactification of Y . Suppose that $\mu_{\text{CY,Trop}}$ admits a tropical Monge–Ampère solution $\varphi_{\text{CY,Trop}} \in \text{CPSH}_\omega(\bar{Y})$. Then, $(\varphi_{\text{CY,Trop}})|_Y \circ \text{Trop}$ coincides with $(\varphi_{\text{CY,NA}})|_{\mathbf{Y}^{\text{an}}}$ after adding a constant.*

Therefore, an alternative approach to the metric SYZ conjecture is to try to solve the corresponding tropical Monge–Ampère equations on some tropicalization of $\mathbf{X}$ which is strongly faithful with respect to $\Delta_{\mathfrak{X}}$. Also, the structure of the non-Archimedean Calabi–Yau potential would be easier to understand in this setting.

At last, we sketch the strategy to prove Theorem 12.11. We first prove the theorem in the case when X is the canonical compactification of Y with respect to some polyhedral structure. More precisely, applying the techniques of tropical compactifications and tropical models, we show that for any $\varphi \in \text{PSH}_{\omega} \cap \text{PL}(X)$, one has

$$\varphi \circ \text{Trop} \in \text{PSH}_{\omega, \text{NA}} \cap \text{PL}(\mathbf{X}^{\text{an}}), \quad \text{and} \quad \text{Trop}_*(\text{MA}_{\omega, \text{NA}})(\varphi \circ \text{Trop}) = \text{MA}_{\omega, \text{Trop}}(\varphi).$$

This together with the regularization theorem for tropical psh functions, as well as the Bedford–Taylor theory in both tropical and non-Archimedean setting, indicates that

$$\varphi_{\text{CY}, \text{Trop}} \circ \text{Trop} \in \text{CPSH}_{\omega, \text{NA}}(\mathbf{X}^{\text{an}}), \quad \text{and} \quad \text{Trop}_*(\text{MA}_{\omega, \text{NA}})(\varphi_{\text{CY}, \text{Trop}} \circ \text{Trop}) = \mu_{\text{CY}, \text{Trop}}.$$

On the other hand, the strongly faithful tropicalization condition implies that

$$\text{Supp}(\text{MA}_{\omega, \text{NA}}(\varphi_{\text{CY}, \text{Trop}} \circ \text{Trop})) \subset \text{Sk}(\mathbf{X}).$$

Therefore, we have $\varphi_{\text{CY}, \text{Trop}} \circ \text{Trop} = \varphi_{\text{CY}, \text{NA}}$ after adding a constant, which proves the theorem. We also note that our proof of the compatibility results for PL functions gives a new proof of similar results in the setting of [CLD12] and [GK17].

In the general case when X is not isomorphic to a canonical compactification of Y , we take a birational modification $\tilde{\mathbf{X}}$ of $\mathbf{X}$ such that the extended tropicalization of $\tilde{\mathbf{X}}$ equals the canonical compactification $\bar{Y}$. By the special case treated before, we get $\varphi_{\text{CY}, \text{Trop}} \circ \text{Trop} \in \text{CPSH}_{\omega, \text{NA}}(\tilde{\mathbf{X}}^{\text{an}})$. Then, applying Proposition 12.3, we can descend $\varphi_{\text{CY}, \text{Trop}} \circ \text{Trop}$ to $\text{CPSH}_{\omega, \text{NA}}(\mathbf{X}^{\text{an}})$. A similar argument based on strongly faithful tropicalization concludes.

1.4. Related work and further directions. Potential theory on finite and compact metric graphs (also known as abstract tropical curves and quantum graphs) has a long history, with applications ranging across multiple disciplines, see e.g. [Pos12, KN23, Kur24], and [Zha93, Thu05, BR10] for connections to arithmetic geometry.

Tropical Monge–Ampère equation on tropical toric varieties and tropical hypersurfaces have been studied in [BGJK25] and [HJMM24, Li23b, AH23], respectively, where also connection to optimal transport appears.

A work [BMPS26] parallel to the work of the authors contains results overlapping partially with the material presented in this article. For tropical varieties embedded in projective tropical spaces, the authors formulate a tropical Monge–Ampère equation by associating Monge–Ampère measures to PL functions via intersection products of tropical cycles, and extend it via Bedford–Taylor theory. Derivability of the energy functional is established in *loc. cit.* assuming the regularity property (Theorem A). Theorem D is established there when $\dim X = 1$. Theorem E is established in *loc. cit.* in the case of toric degenerations under the additional assumption of compatibility between tropical and non-Archimedean psh functions. We also note that a special case of Theorem E appears in a recent work [Lin26].

The most natural problem arising from our work is the question of classifying the measures μ for which Equation (1.1) has a solution. Extending the framework of optimal transport theory to the tropical setting is also an interesting direction of research. Such an extension should allow to address the classification problem, in particular, for discrete measures (*i.e.* finite sums of dirac masses).

Finally, the formalism of mixed differential calculus and tropical pluripotential theory allows to formulate and study a tropical analogue of the Kähler–Einstein equation, an interesting direction for future research.

1.5. Organization of the paper. In Section 2, we review background on tropical and toric varieties, as well as the formalism of mixed differential calculus developed in [APW26]. Section 3 introduces the notion of locally convex functions on tropical varieties, and provides some of their basic properties. In Section 4, we define ω -psh functions for a semipositive form ω , and prove the compactness of the space $\text{PSH}_\omega(X)/\mathbb{R}$. In Section 5, we introduce the notion of Kähler forms and prekähler tropical varieties, and prove the regularization theorem (Theorem A). We establish the Bedford–Taylor theory on tropical varieties in Section 6. In Section 7, we define and study the tropical energy functional. In Section 8, we formulate the tropical Monge–Ampère equations, and prove the uniqueness of solutions (Theorem B). The equivalence between the existence of solutions and the orthogonality property is established in Section 9. In Section 10, we prove the orthogonality property in codimension 1 and existence of solutions for regular-in-codimension-one tropical varieties (Theorem D). The failure of existence of the solutions in general (Theorem C) is proved in Section 11. In Section 12, we prove a comparison theorem between tropical and non-Archimedean Monge–Ampère solutions (Theorem E). In Appendix A, we prove a uniform Lipschitz property for tropical psh functions. Appendix B contains four lemmas in polyedral geometry used in the proof of Theorem C.

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AI statement. The mathematical ideas, results, and proofs presented in this paper are all work of the authors. LLM was used occasionally for English grammatical assistance.

1.6. Notation and Convention.

- Unless otherwise specified, all the fans and polyhedral structures (complexes) in this paper are assumed to be rational and simplicial.
- For σ a rational polytope or a polyhedron, we denote by $\text{relint}(\sigma)$ its relative interior.
- For a fan Σ , we use $|\Sigma|$ to denote its support, and use $\bar{\Sigma}$ to denote its *canonical compactification* (see Section 2.1.2). A fanfold is the support of a rational fan.
- We use X for tropical varieties, and X_o for their open parts (the minimum sedentarity stratum). We use bold letters (*e.g.* $\mathbf{X}, \mathbf{Y}, \dots$) to denote algebraic varieties.
- For a tropical variety X , we denote by Aff the sheaf of affine functions on X (see Section 2.1.4), and denote by LConv the sheaf of locally convex functions on X (see Section 3).

- A simplicial rational polyhedral complex (*resp.* structure) on a tropical variety is defined in Section 2.1.6. Throughout this paper, unless stated otherwise, the term polyhedral complexes (*resp.* structures) refers to simplicial rational polyhedral complexes (*resp.* structures).
- We use $\overline{\Pi}$ to denote a polyhedral structure on X , and Π to denote the restriction of $\overline{\Pi}$ on X_o (see Section 2.1.6). We denote by Π_f the polyhedral subcomplex of Π formed by compact faces in Π .
- For a natural number k , $\overline{\Pi}_k$ (*resp.* Π_k) denote the set of k -dimensional polytopes in $\overline{\Pi}$ (*resp.* Π).
- For $\sigma \in \Pi$, we denote by T_σ the tangent space at σ (see Section 2.1), and denote by Π^σ the star fan of Π at σ . The integer lattice in T_σ is denoted by N_σ .
- For a tropical variety X with open part X_o , we denote by PL the sheaf of piecewise linear functions on X , and denote by PC^∞ the sheaf of piecewise smooth functions on X_o (see Section 2.1.5).
- For a fan Σ , we use $\text{PL}_{\text{hom}}(\Sigma)$ to denote the set of homogeneous PL functions on $|\Sigma|$ (*i.e.* PL functions on $|\Sigma|$ satisfying $f(tx) = tf(x)$).
- The mixed differential operators are denoted by ∂ and $\overline{\partial}$ (see Section 2.4). They are generalizations to the piecewise smooth setting of the tropical differential operators d', d'' for smooth forms, introduced in [Lag12, CLD12].
- Let E be a finite dimensional Euclidean space. For a subset $S \subset E$, we denote by $\text{conv}(S)$ the convex hull of S in E , and we denote by $\text{Cone}(S)$ the convex cone generated by S in E .
- We use the standard abbreviations *usc* for “upper semicontinuous”, *lsc* for “lower semicontinuous”. We also use *SNC* for “simple normal crossing”, *PL* for “piecewise linear”, and *psh* for “plurisubharmonic”.
- Let W be a locally compact Hausdorff space. The notation $\subseteq$ indicates being compactly contained, *i.e.* $V \subseteq W$ if the closure of V is compact and contained in W .
- If W is a Hausdorff topological space, and $\varphi: W \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is any function, then, the *usc envelope* $\varphi^\star$ of φ is the smallest usc function with $\varphi^\star \geq \varphi$. Concretely, $\varphi^\star(x) = \limsup_{y \rightarrow x} \varphi(y)$.

2. PRELIMINARIES

We recall basic definitions and results on tropical varieties, closely following [JSS19, AP20a, ACM25], to which we refer for further details. We also provide a brief account of the formalism of mixed differential forms from [APW26].

2.1. Tropical varieties. Let N be a free abelian group of finite rank, with dual $M = N^\star = \text{Hom}(N, \mathbb{Z})$. Denote by $N_{\mathbb{Q}}, N_{\mathbb{R}}, M_{\mathbb{Q}}, M_{\mathbb{R}}$ the associated rational and real vector spaces, so that $M_{\mathbb{Q}} = N_{\mathbb{Q}}^\star$ and $M_{\mathbb{R}} = N_{\mathbb{R}}^\star$. The pairing between M and N is denoted by $\langle \cdot, \cdot \rangle$.

An affine half-space in $N_{\mathbb{R}}$ is a set of the form $\{v \in N_{\mathbb{R}} \mid \langle u, v \rangle \leq b\}$ with $u \in M_{\mathbb{R}}$ and $b \in \mathbb{R}$. It is called rational if $u \in M$. It is linear if $b = 0$.

A finite intersection of affine (*resp.* linear) half-spaces is called a polyhedron (*resp.* polyhedral cone). It is called rational if the half-spaces can be chosen rational. A compact polyhedron is called a polytope.

The tangent space T_Q of a polyhedron Q is the linear span of the differences of points in Q . It is rational if $N \cap T_Q$ is a full-rank lattice in T_Q . The dimension of Q denoted d_Q is the dimension of T_Q .

Each polyhedron Q can be written as a Minkowski sum $Q' + \tau$ of a polytope Q' and a unique polyhedral cone τ called the recession cone of Q and denoted by $\text{rec}(Q)$. A polyhedral cone τ with exactly d_τ rays is called simplicial. A polyhedron Q is called simplicial if it is the sum of a simplex δ and a simplicial cone τ such that the intersection of the two tangent spaces T_δ and T_τ is $\underline{0}$. In this case, δ is also unique with this property.

2.1.1. *Tropical fans.* For a polyhedral cone $\sigma \subset N_{\mathbb{R}}$, let $N_{\sigma, \mathbb{R}}$ be the real span of σ and set

$$N_{\mathbb{R}}^\sigma := N_{\mathbb{R}} / N_{\sigma, \mathbb{R}}.$$

We have $d_\sigma = \dim N_{\sigma, \mathbb{R}}$. If σ is rational, this induces full-rank lattices $N_\sigma \subset N_{\sigma, \mathbb{R}}$ and $N^\sigma \subset N_{\mathbb{R}}^\sigma$.

A polyhedral cone that contains no line is called strongly convex. A fan Σ in $N_{\mathbb{R}}$ is a finite, nonempty collection of strongly convex polyhedral cones with the property that every face of a cone $\sigma \in \Sigma$ lies in Σ , and moreover, the intersection of any two cones in Σ is a common face. We refer to elements of Σ as its faces.

A fan is called rational if all cones are rational. It is called simplicial if all its cones are. All the fans appearing in this paper will be rational and simplicial.

For cones σ, τ , we write $\tau \leq \sigma$ if τ is a face of σ , and $\tau < \sigma$ if it has codimension one. This induces a partial order on Σ . We denote by Σ_k the set of k -dimensional cones. The cone $\{0\}$ is written $\underline{0}$ and one-dimensional elements of Σ are called rays. Maximal faces of Σ are called facets. A fan is pure-dimensional if all its facets have the same dimension.

The support of Σ , written $|\Sigma|$, is the union of its cones in $N_{\mathbb{R}}$. For Σ of dimension one, there is a unique fan structure on its support, so we simply use Σ in place of $|\Sigma|$. A fanfold $X \subset N_{\mathbb{R}}$ is a closed subset that is the support of some rational fan. A point x in a fanfold of dimension d is regular if x admits a neighborhood in X homeomorphic to an open subset of $\mathbb{R}^d$.

For a cone $\sigma \in \Sigma$ and a codimension-one face $\tau < \sigma$, the subspace $N_{\tau, \mathbb{R}}$ divides $N_{\sigma, \mathbb{R}}$ into two closed half-spaces, one of which contains σ . A primitive normal vector to τ in σ is any $\nu \in N_\sigma$ lying in this half-space and satisfying $N_\tau + \mathbb{Z}\nu = N_\sigma$; we denote such a choice by $\mathbf{n}_{\sigma/\tau}$. Its image in the quotient lattice $N_\sigma^\tau := N_\sigma / N_\tau$ is then well-defined and denoted by $\mathbf{e}_\sigma^\tau$. If $\tau = \underline{0}$, and σ is a ray of Σ , we simply write $\mathbf{e}_\sigma$. We denote by $\mathbf{n}_{\sigma/\tau}^* \in N_\sigma^*$ the unique integral linear form that is zero on N_τ and takes value one on $\mathbf{n}_{\sigma/\tau}$.

A tropical fan is a pair (Σ, ϖ) consisting of a rational fan Σ in $N_{\mathbb{R}}$ of pure dimension $d \in \mathbb{Z}_{\geq 0}$ and a function

$$\varpi: \Sigma_d \rightarrow \mathbb{N}$$

called tropical weight that satisfies the balancing condition: for every cone $\tau \in \Sigma_{d-1}$,

$$(2.1) \quad \sum_{\sigma > \tau} \varpi(\sigma) \mathbf{n}_{\sigma/\tau} \in N_\tau, \quad \text{equivalently} \quad \sum_{\sigma > \tau} \varpi(\sigma) \mathbf{e}_\sigma^\tau = 0 \quad \text{in } N^\tau.$$

The tropical weight function ϖ defines a weight function on the set of regular points of $|\Sigma|$. The support of a tropical fan endowed with this weight function is called a tropical fanfold.

For a cone $\sigma \in \Sigma$, the star fan of σ in Σ denoted Σ^σ is a fan in the quotient space $N_{\mathbb{R}}^\sigma$ defined as follows. Consider the projection map $\pi_\sigma: N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}^\sigma$. The star fan Σ^σ consists of the projected cones $\eta^\sigma := \pi_\sigma(\eta)$ for $\eta \geq \sigma$ in Σ . For Σ a rational fan of pure dimension d , the star fan Σ^σ is of pure dimension $d - d_\sigma$ and rational with respect to the quotient lattice N^σ .

A tropical weight function ϖ on Σ naturally induces a tropical weight on Σ^σ that we denote by $\varpi^\sigma: \Sigma_{d-d_\sigma}^\sigma \rightarrow \mathbb{N}$. It is defined by setting $\varpi^\sigma(\eta^\sigma) := \varpi(\eta)$ for each cone $\eta \in \Sigma_d$ with $\eta \geq \sigma$. The balancing condition around a face η^σ of codimension one in Σ^σ reduces to the balancing

condition around η in Σ . If there is no risk of confusion, we drop σ in ϖ^σ , and use ϖ for the induced weight function on the star fan.

A *generalized tropical line* is a tropical fan $\mathcal{L}$ of dimension one in some $N_{\mathbb{R}}$ such that the space of linear relations between the primitive vectors $\mathbf{e}_\varrho$, $\varrho \in \mathcal{L}_1$, is of dimension one, generated by the balancing condition $\sum_{\varrho} \varpi(\varrho) \mathbf{e}_\varrho = 0$. A tropical curve locally modeled by generalized tropical lines is called *locally matroidal*.

2.1.2. Tropical toric varieties and canonical compactifications of fans. A rational fan Σ in $N_{\mathbb{R}}$ defines the tropical toric variety $\mathbb{TP}_\Sigma$, a partial compactification of $N_{\mathbb{R}}$ that coincides with the tropicalization of the toric variety $\mathbb{P}_\Sigma$ [BGJK20, Kaj08, OR11, Pay09, Thu07]. The canonical compactification of Σ denoted $\bar{\Sigma}$ is obtained by taking the closure of cones of Σ in $\mathbb{TP}_\Sigma$, whose support also denoted $\bar{\Sigma}$, is called the canonical compactification induced by Σ of the fanfold $|\Sigma|$. We give a brief account of the constructions.

Let $\mathbb{T} := \mathbb{R} \cup \{+\infty\}$ endowed with the topology generated by intervals $(a, +\infty]$, $a \in \mathbb{R}$, and denote $\mathbb{T}_+ = \mathbb{R}_+ \cup \{+\infty\}$. The operation of addition and multiplication by positive scalars extend continuously to $\mathbb{T}$ and $\mathbb{T}_+$. If there is no risk of confusion, we write $\infty \in \mathbb{T}$ for $+\infty$.

Let $\mathcal{M}$ be the category of $\mathbb{R}_+$ -modules. Objects of $\mathcal{M}$ are commutative monoids $(M, +)$ endowed with an action of $(\mathbb{R}_+, \times)$ that is distributive under addition of scalars. In particular, $\mathcal{M}$ contains $\mathbb{T}$, $\mathbb{T}_+$, and all polyhedral cones.

For a cone $\sigma \subset N_{\mathbb{R}}$, define its dual $\sigma^\vee$ and orthogonal $\sigma^\perp$ by

$$\sigma^\vee = \{u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0 \ \forall v \in \sigma\}, \quad \sigma^\perp = \{u \in M_{\mathbb{R}} \mid \langle u, v \rangle = 0 \ \forall v \in \sigma\},$$

and set $\mathbb{U}_\sigma = \text{Hom}_{\mathcal{M}}(\sigma^\vee, \mathbb{T})$. The tropical toric variety $\mathbb{TP}_\Sigma$ is obtained by gluing the $\mathbb{U}_\sigma$ for $\sigma \in \Sigma$, as in the classical setting. The tropical torus orbit of a cone $\tau \in \Sigma$ is

$$\mathbb{O}(\tau) = \{v \in \mathbb{U}_\tau \mid \langle u, v \rangle = \infty \text{ exactly for all } u \in \tau^\vee \setminus \tau^\perp\} \simeq N_{\mathbb{R}}^\tau.$$

Then, $\mathbb{TP}_\Sigma$ is the union of the tropical torus orbits of its cones, and for each σ , we have $\mathbb{U}_\sigma = \bigsqcup_{\tau \leq \sigma} \mathbb{O}(\tau)$.

The canonical compactification $\bar{\sigma}$ of a cone $\sigma \in \Sigma$ is $\bar{\sigma} = \text{Hom}_{\mathcal{M}}(\sigma^\vee, \mathbb{T}_+)$, and coincides with the closure of σ in $\mathbb{U}_\sigma$. We have $\bar{\Sigma} = \bigcup_{\sigma \in \Sigma} \bar{\sigma}$.

The polyhedral face structure on $\bar{\Sigma}$ is described as follows. For each $\tau \in \Sigma$, let ∞_τ be the element of $\mathbb{O}(\tau)$ that vanishes on $\tau^\perp$. For $\sigma \geq \tau$, set

$$\sigma_\infty^\tau := \infty_\tau + \sigma = \{v \in \bar{\sigma} \mid \langle u, v \rangle = \infty \text{ iff } u \in \tau^\vee \setminus \tau^\perp\} \subset \mathbb{O}(\tau).$$

Via the identification $\mathbb{O}(\tau) \simeq N_{\mathbb{R}}^\tau$, the cone σ_∞^τ is the image of σ under the projection map $\pi: N_{\mathbb{R}} \rightarrow N_{\mathbb{R}}^\tau$. The family $\{\sigma_\infty^\tau \mid \sigma \geq \tau\}$ forms a fan Σ_∞^τ in $\mathbb{O}(\tau)$ with origin ∞_τ . Under $N_{\mathbb{R}}^\tau \simeq \mathbb{O}(\tau)$, this fan can be identified with the star fan Σ^τ of τ in Σ . For $\underline{0} \in \Sigma$, we have $\Sigma_\infty^{\underline{0}} = \Sigma$. The closure in $\bar{\Sigma}$ of each cone σ_∞^τ coincides with the canonical compactification $\bar{\sigma}_\infty^\tau$. A face of $\bar{\Sigma}$ is a compactified cone $\bar{\sigma}_\infty^\tau$ for a pair $\tau \leq \sigma$ of elements of Σ .

For $x \in \mathbb{TP}_\Sigma$, the sedentarity $\text{sed}(x) \in \Sigma$ is the unique cone τ such that $x \in \mathbb{O}(\tau)$. The sedentarity fan of a subset $U \subset \bar{\Sigma}$ denoted Σ_U is the subfan of Σ consisting of the sedentarities $\text{sed}(x)$ for $x \in U$. (We omit the proof that this is indeed a subfan of Σ).

Let δ be a simplicial cone with rays $\varrho_1, \dots, \varrho_k$, $k = d_\delta$. Then, in terms of the primitive vectors $\mathbf{e}_{\varrho_i}$, we have $\bar{\delta} = \mathbb{T}_{\geq 0} \mathbf{e}_{\varrho_1} + \dots + \mathbb{T}_{\geq 0} \mathbf{e}_{\varrho_k}$. Under this identification, each $x \in \bar{\delta}$ gets a set of k coordinates, and the sedentarity of $x \in \bar{\delta}$ corresponds to the subset of $[k]$ given by the ∞ -coordinates of x .

Suppose Σ is simplicial. For $x \in \bar{\Sigma}$, let $\delta = \text{sed}(x)$ and let x_0 be its image under the natural isomorphism $\Sigma_\infty^\delta \simeq \Sigma^\delta$. The local structure of $\bar{\Sigma}$ around x is modeled by an open subset of the point $(\infty_\delta, x_0) \in \bar{\delta} \times \Sigma^\delta$, given by a set of $k = d_\delta$ coordinates in $\mathbb{T}_{\geq 0}$ corresponding to the rays $\varrho_1, \dots, \varrho_k$ of δ , and an open neighborhood V of x_0 in Σ^δ . Under this identification, a basic open neighborhood of x is an open neighborhood U of (∞_δ, x_0) of the form

$$\left((t, \infty] \cdot \mathbf{e}_{\varrho_1} + \dots + (t, \infty] \cdot \mathbf{e}_{\varrho_k} \right) \times V.$$

A basic open subset of $\bar{\Sigma}$ is a basic open neighborhood of some point $x \in \bar{\Sigma}$. Basic open sets form a basis of open sets for the topology of $\bar{\Sigma}$.

Let U be a basic open subset of $\bar{\Sigma}$ as above. An affine function on U is a function $u: U \rightarrow \mathbb{R}$ such that $u = l \circ \pi$ for an affine function l on V , with $\pi: U \rightarrow V$ the projection. We denote by $\text{Aff}(U)$ the set of affine functions on U . More generally, given an open subset W of $\bar{\Sigma}$, an affine function on W is a continuous function whose restrictions to basic open subsets $U \subset W$ is affine. They form the set $\text{Aff}(W)$, and together define the sheaf of affine functions on $\bar{\Sigma}$.

In this paper $\bar{\Sigma}$ will always refer to the canonical compactification of Σ .

2.1.3. Tropical varieties. Let X be a connected Hausdorff space. A polyhedral chart is a pair (W, ϕ) with $W \subset X$ a nonempty open set and $\phi: W \rightarrow U$ a homeomorphism onto a basic open neighborhood U of a point in the canonical compactification of a simplicial rational fan Σ , as in the previous section. We suppose without loss of generality that the linear span of Σ coincides with the ambient space in which it is embedded. We refer to the lattice of Σ as the ambient lattice of the chart (W, ϕ) .

Two charts $(W_i, \phi_i: W_i \rightarrow U_i)$ and $(W_j, \phi_j: W_j \rightarrow U_j)$, with ambient fans Σ_i and Σ_j and lattices N_i and N_j , are compatible if either $W_i \cap W_j = \emptyset$ or the transition map

$$\phi_i \circ \phi_j^{-1}: \phi_j(W_i \cap W_j) \rightarrow \phi_i(W_i \cap W_j)$$

is induced by a global affine $\mathbb{Z}$ -linear map (*i. e.* affine map with a $\mathbb{Z}$ -linear part) $\psi_{ij}: N_j \rightarrow N_i$ that restricts to a $\mathbb{Z}$ -linear isomorphism between the sedentarity fans of $\phi_j(W_i \cap W_j)$ and $\phi_i(W_i \cap W_j)$ in Σ_j and Σ_i , respectively.

An extended polyhedral space is a connected Hausdorff space endowed with an atlas of pairwise compatible polyhedral charts.

Let $(W_i, \phi_i: W_i \rightarrow U_i)$ be a chart. We say that two points $x, y \in W_i$ have the same sedentarity if the images $\phi_i(x)$ and $\phi_i(y)$ do in U_i . The transitive closure in X of the property of having the same sedentarity in a chart yields a stratification of X into sedentarity types, denoted $\text{Sed}(X)$. (The sedentarity types play the role of generic points of the strata, analogous to stratified algebraic varieties.) For $s \in \text{Sed}(X)$, denote by X_s the stratum of type s .

The set $\text{Sed}(X)$ carries a natural partial order: $s_1 < s_2$ if each point of type s_2 is a limit of points of type s_1 . Using the connectivity of X , it is easy to show that $\text{Sed}(X)$ has a minimum element (see [ACM25, Prop. 2.2]) that we denote by o , and refer to X_o as the open part of X .

A point $x \in X$ is regular if it has a neighborhood homeomorphic to an open subset of $\mathbb{R}^d$ for some d . We denote by X_{reg} the locus of regular points.

Let X be an extended polyhedral space. A tropical weight function is a map $\varpi: X_{\text{reg}} \rightarrow \mathbb{N}$ such that, on each chart (W_i, ϕ_i) with image U_i , the restriction $\varpi|_{W_{i,\text{reg}}}$ is the pullback of a tropical weight function on Σ_i (restricted to $U_{i,\text{reg}}$).

A tropical variety is a pair (X, ϖ) consisting of an extended polyhedral space and a tropical weight function. For each sedentarity type s of X , the stratum $\overline{X}_s$ inherits the structure of a tropical variety.

There is a natural notion of morphism between tropical varieties. A continuous map $\varphi: X \rightarrow X'$ between two tropical varieties X and X' is a tropical morphism if the image $\varphi(X_o)$ lies in a single sedentarity stratum X'_s of X' , and there exist compatible atlases $\{(W_i, \phi_i)\}_{i \in I}$ on X and $\{(W'_j, \phi'_j)\}_{j \in J}$ on $\overline{X}'_s$, with lattices N_i and N'_j such that, for each $i \in I$, there is some $j \in J$ satisfying $\varphi(W_i) \subset W'_j$, and $\phi'_j \circ \varphi \circ \phi_i^{-1}$ is induced by a $\mathbb{Z}$ -linear map of lattices $N_i \rightarrow N'_j$.

2.1.4. The sheaf of affine functions. Let W be an open subset of a tropical variety X . A function $u: W \rightarrow \mathbb{R}$ is called affine for every point $x \in W$, there exists a basic open neighborhood U of x (in a local polyhedral chart) such that the restriction $u|_U$ is affine. We denote by $\text{Aff}(W)$ the set of affine functions on W . They form a sheaf of abelian groups on X .

2.1.5. The sheaf of PL-functions. Let $U \subset X$ be an open subset. A function $f: U \rightarrow \mathbb{R}$ is called piecewise linear (PL) if $f \in C^0(U)$ and there exists a polyhedral structure $\overline{\Pi}$ for X such that the restriction of $f|_{U \cap P}$ is affine for each face P of Π . We denote by PL the sheaf of PL functions on X .

2.1.6. Polyhedral structures. We now discuss polyhedral structures on extended polyhedral spaces.

Let σ be a rational simplicial cone, and $Q \subseteq \sigma$ be a polyhedron. We say that Q is compatible with σ if the recession cone $\text{rec}(Q)$ is a face of σ . In this case, writing $Q = Q' + \tau$, for a polytope Q' and $\tau = \text{rec}(Q)$, the closure of Q in $\overline{\sigma}$ is $Q' + \overline{\tau} \subset \overline{\sigma}$. A (simplicial) rational polytope P in $\overline{\sigma}$ is by definition the closure of a compatible (simplicial) rational polytope Q in σ . A face of P is a nonempty intersection of P with a face of $\overline{\sigma}$ (for the face structure defined in § 2.1.2).

Let Σ be a rational simplicial fan in $N_{\mathbb{R}}$ with canonical compactification $\overline{\Sigma}$. A rational polytope in $\overline{\Sigma}$ is by definition a rational polytope in a face of $\overline{\Sigma}$ (that is a rational polytope in the canonical compactification $\overline{\sigma}_{\infty}^{\tau}$ of a face $\sigma_{\infty}^{\tau} \in \Sigma_{\infty}^{\tau}$, for a pair of elements $\tau < \sigma$ in Σ).

A rational polytope in an extended polyhedral space X is a closed set $P \subset X$ such that, for each chart (W, ϕ) with $P \subset W$, the image $\phi(P)$ is a rational polytope.

A (simplicial) rational polyhedral structure on X is a collection $\overline{\Pi}$ of (simplicial) rational polytopes satisfying:

- For each $P \in \overline{\Pi}$, there exists a chart (W, ϕ) with $P \subset W$. Moreover, for any such chart, if F is a face of $\phi(P)$, then $\phi^{-1}(F) \in \overline{\Pi}$.
- If $P_1, P_2 \in \overline{\Pi}$ and $(W_1, \phi_1), (W_2, \phi_2)$ are charts containing them, then $P_{12} = P_1 \cap P_2 \in \overline{\Pi}$, with $\phi_1(P_{12})$ a face of $\phi_1(P_1)$ and $\phi_2(P_{12})$ a face of $\phi_2(P_2)$.

We refer to $\overline{\Pi}$ as a (simplicial) rational polyhedral complex. A (simplicial) rational polyhedral structure $\overline{\Pi}$ on X induces a (simplicial) rational polyhedral structure on X_o denoted by Π .

All tropical varieties admit simplicial polyhedral structures. For simplicity, we assume that the tropical varieties studied in this paper admit a rational simplicial polyhedral structure. In particular, this includes tropical varieties coming from tropicalization of algebraic varieties over a discretely valued field. Throughout the remainder of this paper, unless stated otherwise, polyhedral complexes (*resp.* structures) are assumed to be rational and simplicial.

Given a polyhedral structure $\bar{\Pi}$ on X , with induced polyhedral structure Π on X_o , for each point $x \in Z$, we set

$$\bar{\Pi}^{(x)} := \bigcup_{x \in P, P \in \bar{\Pi}} \text{relint}(P).$$

Note that $\bar{\Pi}^{(x)} \subset X$ is open. When $x \in X_o$, we have $\Pi^{(x)} = \bar{\Pi}^{(x)} \subset X_o$. An affine function defined on a neighborhood of $x \in X$ can be extended facewise linearly to $\bar{\Pi}^{(x)}$.

For each $\sigma \in \Pi$, we denote by Π^σ the star fan of σ in Π , defined as follows. Let $N_{\mathbb{R}}$ be the linear span of all the faces $\sigma' \geq \sigma$, with underlying lattice N , and $N_{\sigma, \mathbb{R}} \subset N_{\mathbb{R}}$ be the linear span of σ with underlying lattice N_σ . The star fan Π^σ is the fan in $N_{\mathbb{R}}^\sigma = N_{\mathbb{R}}/N_{\sigma, \mathbb{R}}$ defined as the collection of all the cones generated by the projection of a face $\sigma' \geq \sigma$ into $N_{\mathbb{R}}^\sigma$. In local polyhedral chart (W, ϕ) that contains σ , let $\bar{\Sigma}$ be the canonically compactified fan with where the image of ϕ lands, i.e., $\phi(W) \subset \bar{\Sigma}$. Let η be the smallest face of Σ that contains $\phi(\sigma)$. Then, $\Pi^\sigma = \Sigma^\eta$. (Note that, more generally, for each sedentarity stratum X_s , we get an induced polyhedral structure Π_s from $\bar{\Pi}$, and for $\sigma \in \Pi_s$, we can define its star fan Π_s^σ in a similar way.)

Let $\bar{\Pi}$ be a polyhedral structure on an extended polyhedral space X . A polyhedral subcomplex $\bar{\Pi}'$ of $\bar{\Pi}$ is a collection of polytopes $P \in \bar{\Pi}$ such that $\bar{\Pi}'$ is a polyhedral structure. This means that the union Y of polytopes in $\bar{\Pi}'$ is itself an extended polyhedral space admitting $\bar{\Pi}'$ as a polyhedral structure. In this case we say that Y is a polyhedral subspace of X .

Let $\bar{\Pi}$ and $\bar{\Pi}'$ be two polyhedral structures on X . We say that $\bar{\Pi}'$ is a refinement of $\bar{\Pi}$ if each $P' \in \bar{\Pi}'$ is contained in an element $P \in \bar{\Pi}$.

2.1.7. Pushforward of weight functions. Suppose $\varphi: X \rightarrow X'$ is a proper tropical morphism, and let $n = \dim X$. The pushforward $\varphi_*(X)$ is defined as follows.

Replacing X' by the closure of the sedentarity stratum that contains $\varphi(X_o)$, we may assume $\varphi(X_o) \subset X'_o$.

Let Π be a rational polyhedral structure on X_o . After taking a refinement of Π if necessary, we may assume that $\varphi(P)$ is contained in a rational polytope of X'_o for each full-dimensional polytope $P \in \Pi$. Then the set Y_o underlying $\varphi_*(X)_o$ is defined as the union of all $\varphi(P)$ for $P \in \Pi$ such that $\varphi(P)$ is n -dimensional. Then, $\varphi_*(X)$ is the closure Y of Y_o in X' . We define a weight function on Y_o . Let $Z \subset Y_{\text{reg}}$ be the subset of points z such that $\varphi^{-1}(z)$ is a finite subset of X_{reg} (with Y_{reg} the set of points in Y that admit an open neighborhood homeomorphic to a ball in $\mathbb{R}^n$).

Define $\varpi_Y: Z \rightarrow \mathbb{N}$ by

$$(2.2) \quad \varpi_Y(z) = \sum_{w \in \varphi^{-1}(z)} |N_z/d\varphi(N_w)| \varpi_X(w),$$

where $d\varphi$ denotes the linear part of φ at w . By [MR, Prop. 6.2.1], the function ϖ_Y extends to a weight function on Y_{reg} such that (Y_o, ϖ_Y) is a tropical variety. The pushforward $\varphi_*(X)$ is then defined as the closure of Y in X' endowed with this weight function on Y_o . In general, this is not a tropical variety in the sense defined in this section, as it might have “singularities at infinity”. We view $\varphi_*(X)$ as a tropical variety with singularities at infinity.

2.1.8. *Projective tropical toric varieties.* Let Ξ be a simplicial rational fan on $N_{\mathbb{R}}$. We say the tropical toric variety $\mathbb{TP}_{\Xi}$ is projective if the corresponding toric variety $\mathbb{P}_{\Xi}$ is projective. That is to say, the rational fan Ξ satisfies the following two properties.

- (1) Ξ is complete.
- (2) There exists a strictly convex conewise linear function g on Ξ , *i.e.* for each $\delta \in \Xi$, there exists a linear function l which agrees with g on δ and satisfies $g > l$ on $E \setminus \delta$.

We recall the following basic result in toric geometry.

Theorem 2.1. [Theorem 6.1.18 and Theorem 11.1.9 in [CLS11]] *For any complete simplicial rational fan Ξ in $N_{\mathbb{R}}$, there exists a successive stellar subdivision Ξ' of Ξ such that Ξ' is simplicial and rational, and $\mathbb{TP}_{\Xi'}$ is projective.*

Let X be a tropical variety. We say that X is a tropical subvariety of $\mathbb{TP}_{\Xi}$ if X is a closed polyhedral subspace of $\mathbb{TP}_{\Xi}$ and $X \cap (\mathbb{TP}_{\Xi})_o$ is nonempty. Let Y be a tropical variety embedded in $N_{\mathbb{R}}$. Suppose that there exists a simplicial rational polyhedral structure Π on Y such that $\Sigma := \text{rec}(\Pi)$ is a subfan of Ξ . Then the closure of Y in $\mathbb{TP}_{\Xi}$ is isomorphic to the canonical compactification of Y with respect to Σ , and it is a tropical subvariety of $\mathbb{TP}_{\Xi}$.

We say that a tropical variety X is projective if it is isomorphic to a tropical subvariety of a projective tropical toric variety.

2.2. **Chow rings of fans.** Let Σ be a simplicial rational fan of dimension d . By definition, the Chow ring of Σ is the Chow ring of the corresponding toric variety $\mathbb{P}_{\Sigma}$. With real coefficients, it is the graded algebra defined by quotienting the polynomial ring in variables x_{ζ} , $\zeta \in \Sigma_1$,

$$A^{\bullet} := \mathbb{R}[x_{\zeta}]_{\zeta \in \Sigma_1} / (I + J), \quad \text{where}$$

- I is the homogeneous ideal generated by the products $x_{\zeta_1} \cdots x_{\zeta_k}$, $k \in \mathbb{N}$, for rays $\zeta_1, \dots, \zeta_k$ that do not span a cone in Σ , and
- J is the homogeneous ideal generated by the elements

$$\sum_{\zeta \in \Sigma_1} m(\mathbf{e}_{\zeta}) x_{\zeta}, \quad m \in M_{\mathbb{R}}.$$

For each face $\sigma \in \Sigma$, set $x_{\sigma} := \prod_{\substack{\zeta \in \Sigma_1 \\ \zeta < \sigma}} x_{\zeta}$. It can be shown that the homogeneous part A^k is spanned by elements of the form x_{σ} with $\sigma \in \Sigma_k$. In particular, $A^k(\Sigma) = 0$ unless $0 \leq k \leq d$.

Let now $(\Sigma, \varpi_{\Sigma})$ be a simplicial tropical fan. The tropical weight ϖ_{Σ} induces a map

$$\deg: A^d \rightarrow \mathbb{R}.$$

Explicitly, for each face σ of Σ_d , we have

$$\deg(x_{\sigma}) = \frac{1}{|\det(\mathbf{e}_{\rho_1} \dots \mathbf{e}_{\rho_d})|} \varpi_{\Sigma}(\sigma),$$

where $(\mathbf{e}_{\rho_1} \dots \mathbf{e}_{\rho_d})$ is the $d \times d$ matrix with columns the primitive vectors of the rays $\rho_1, \dots, \rho_d$ of σ , written in a basis of the lattice N_{σ} .

Throughout we use x, y, z for elements of the Chow rings.

2.3. Mixed differential forms relative to a polyhedral structure. We give a brief account of the formalism of mixed differential forms introduced in [APW26]. This yields tropical semipositive and Kähler forms.

Let X be a tropical variety of dimension d with tropical weight function ϖ . Let $\bar{\Pi}$ be a simplicial polyhedral structure on X , with Π the induced structure on X_o , and let $\Pi_f \subset \Pi$ be the subcomplex defined by compact faces (simplices).

The differential calculus is defined first relative to Π . It behaves well under refinements and yields a differential calculus depending only on the support of X .

For $\delta \in \Pi$, let $\iota_\delta: \delta \hookrightarrow X$ denote the inclusion.

2.3.1. Coefficients. For each $\tau \in \Pi$, the star fan Π^τ has dimension $d - d_\tau$, and is a simplicial tropical fan with the tropical weight function induced from ϖ . Denote by $A^\bullet(\tau)$ the Chow ring of the star fan Π^τ . This is a graded $\mathbb{R}$ -algebra

$$A^\bullet(\tau) = A^0(\tau) \oplus \cdots \oplus A^{d-d_\tau}(\tau).$$

In the presentation given in Section 2.2, the generator given by a ray η^τ of Π^τ (for $\eta \succ \tau$ in Π), is denoted by x_{η^τ} . For any cone σ^τ in Π^τ , we denote by x_{σ^τ} the corresponding element in $A^{d_\sigma-d_\tau}(\tau)$, defined as the product of x_{η^τ} for the rays η^τ of σ^τ .

The map $A^\bullet, \tau \mapsto A^\bullet(\tau)$, from Π to the category of finite-dimensional graded $\mathbb{R}$ -algebras, comes with the following additional structure:

- For each $\tau \in \Pi$, we have the degree map

$$\deg_\tau: A^{d-d_\tau}(\tau) \rightarrow \mathbb{R}.$$

- For elements $\tau < \sigma$ of Π , we have the restriction map

$$\text{Res}_{\tau < \sigma}: A^\bullet(\tau) \rightarrow A^\bullet(\sigma).$$

- For elements $\tau < \sigma$ of Π , we have the Gysin map

$$\text{Gys}_{\sigma > \tau}: A^\bullet(\sigma) \rightarrow A^{\bullet+d_\sigma-d_\tau}(\tau).$$

We refer to [APW26, §3.1] for a list of compatibility properties between these maps. Mixed differentials restrict to smooth forms on each τ taking values in the Chow ring $A^\bullet(\tau)$.

2.3.2. Tropical conjugation. We first recall the formalism of Lagerberg [Lag12]. Let δ be a face of Π with tangent space T_δ , and lattice $N_\delta \subset T_\delta$. Take a second copy $\bar{T}_\delta$ and $\bar{N}_\delta$ of T_δ and N_δ , respectively. Set

$$W_\delta := T_\delta \oplus \bar{T}_\delta.$$

For each positive integer k , we have a natural decomposition

$$\bigwedge^k W_\delta = \bigoplus_{p+q=k} \bigwedge^{p,q} W_\delta \quad \text{with} \quad \bigwedge^{p,q} W_\delta := \bigwedge^p T_\delta \otimes \bigwedge^q \bar{T}_\delta.$$

The isomorphism $T_\delta \simeq \bar{T}_\delta$ given by the identity map induces an involution $\mathfrak{a} \mapsto \bar{\mathfrak{a}}$ on W_δ

$$\begin{pmatrix} 0 & \text{id} \\ \text{id} & 0 \end{pmatrix}: T_\delta \oplus \bar{T}_\delta \rightarrow T_\delta \oplus \bar{T}_\delta$$

that we call the *tropical conjugation*. It extends naturally to $\bigwedge^k W_\delta$, for all integers k , giving the identification $\overline{\bigwedge^{p,q} W_\delta} = \bigwedge^{q,p} W_\delta$, for all integers p, q with $p + q = k$.

We denote by $W_\delta^* = T_\delta^* \oplus \overline{T_\delta^*}$ the dual of W_δ , and by $\bigwedge^k W_\delta^*$ and $\bigwedge^{p,q} W_\delta^*$ the duals of $\bigwedge^k W_\delta$ and of $\bigwedge^{p,q} W_\delta$, respectively. We have a decomposition

$$\bigwedge^k W_\delta^* = \bigoplus_{p+q=k} \bigwedge^{p,q} W_\delta^*,$$

and an induced tropical conjugation on $\bigwedge^k W_\delta$ that verifies $\overline{\bigwedge^{p,q} W_\delta^*} = \bigwedge^{q,p} W_\delta^*$.

If $\mathbf{v} \in \bigwedge^k W_\delta$ and $\alpha \in \bigwedge^k W_\delta^*$, we call k the *degree* of $\mathbf{v}$ and α , and write $\deg(\mathbf{v}) = \deg(\alpha) = k$.

In the following, if D is an endomorphism of a vector space W endowed with the tropical conjugation as above, we denote by $\overline{D}$ the *tropical conjugate* of D defined as follows:

$$\overline{D}(\alpha) := \overline{D(\overline{\alpha})} \quad \text{for any element } \alpha \in W.$$

2.3.3. Dolbeault forms on faces. For each face δ in Π , let C_δ^∞ be the sheaf of smooth functions on δ . The sheaf of *Dolbeault forms of bidegree* (r, s) on δ is the sheaf

$$C_\delta^\infty \otimes \bigwedge^{r,s} W_\delta^*.$$

For an inclusion of faces $\tau < \delta$, we have the restriction map of sheaves

$$\cdot|_\tau : C_\delta^\infty \otimes \bigwedge^{r,s} W_\delta^* \longrightarrow C_\tau^\infty \otimes \bigwedge^{r,s} W_\tau^*.$$

2.3.4. The sheaf $\Omega^{p,q}$ of mixed (p, q) -forms relative to Π . Let δ be an element of Π . Consider the sheaf

$$A^\bullet(\delta) \otimes C_\delta^\infty \otimes \bigwedge^{r,s} W_\delta^*$$

that we view as the sheaf of differential (r, s) -forms on δ taking values in the ring $A^\bullet(\delta)$. We will glue these sheaves and define the sheaf of mixed differential forms, relative to Π .

For a triple of nonnegative integers p, q, k , consider the sheaf

$$\Omega_\delta^{p,q,k} := A^k(\delta) \otimes C_\delta^\infty \otimes \bigwedge^{p-k,q-k} W_\delta^*.$$

Sections of $\Omega_\delta^{p,q,k}$ are $(p-k, q-k)$ -differential forms which take values in $A^k(\delta)$. Here, we view elements of the Chow ring A^k as contributing by bidegree (k, k) , so that the bidegree of the elements in the right hand side is (p, q) . In the triple (p, q, k) on the left hand side, (p, q) is the bidegree, and k is the part that comes from the Chow ring (cohomology) of the star fan.

The sheaf $\Omega_\delta^{p,q,0}$ coincides with the sheaf of Dolbeault forms $C_\delta^\infty \otimes \bigwedge^{p,q} W_\delta^*$. For $k > \min(p, q, d - d_\delta)$, we get $\Omega_\delta^{p,q,k} = 0$.

The sheaf $\Omega_\delta^{p,q}$ of *mixed (p, q) -forms on δ* is

$$\Omega_\delta^{p,q} := \bigoplus_{k \geq 0} \Omega_\delta^{p,q,k}.$$

By an abuse of notation, we simply write $\Omega_\delta^{p,q}$ and $\Omega_\delta^{p,q,k}$ for the pushout of these sheaves to X via the inclusion map $\iota_\delta : \delta \hookrightarrow X$.

For an inclusion of faces $\tau < \delta$ of Π , the two restriction maps

$$\text{Res}_{\tau < \delta} : A^k(\tau) \longrightarrow A^k(\delta) \quad \text{and} \quad \cdot|_\tau : C_\delta^\infty \otimes \bigwedge^{p-k,q-k} W_\delta^* \longrightarrow C_\tau^\infty \otimes \bigwedge^{p-k,q-k} W_\tau^*$$

yield the following maps

$$(2.3) \quad \begin{array}{ccc} \Omega_\delta^{p,q,k} = A^k(\delta) \otimes C_\delta^\infty \otimes \bigwedge^{p-k,q-k} W_\delta^* & \xrightarrow{\text{id} \otimes -|_\tau} & A^k(\delta) \otimes C_\tau^\infty \otimes \bigwedge^{p-k,q-k} W_\tau^* \\ & & \uparrow \text{Res}_{\tau < \delta} \otimes \text{id} \\ & & \Omega_\tau^{p,q,k} = A^k(\tau) \otimes C_\tau^\infty \otimes \bigwedge^{p-k,q-k} W_\tau^* \end{array}$$

We define the sheaf $\Omega^{p,q,k}$ on X as the gluing of the sheaves $\Omega_\delta^{p,q,k}$, $\delta \in \Pi$, via the diagrams in (2.3). More precisely, the sheaf $\Omega^{p,q,k}$ of $(p-k, q-k)$ -forms with values in A^k , relative to Π , is the kernel of the map of sheaves on X

$$(2.4) \quad \bigoplus_{\delta \in \Pi} \Omega_\delta^{p,q,k} \xrightarrow{\bigoplus_{\tau < \delta} \left((\text{id} \otimes -|_\tau) - (\text{Res}_{\tau < \delta} \otimes \text{id}) \right)} \bigoplus_{\substack{\tau, \delta \in \Pi \\ \tau < \delta}} A^k(\delta) \otimes C_\tau^\infty \otimes \bigwedge^{p-k,q-k} W_\tau^*.$$

Explicitly, a section of $\Omega^{p,q,k}$ is given as follows. On an open subset $U \subseteq X$, a section $\mathfrak{a}$ in $\Omega^{p,q,k}(U)$ is a collection of mixed forms

$$\mathfrak{a}_\delta \in A^k(\delta) \otimes C_\delta^\infty(U \cap \delta) \otimes \bigwedge^{p-k,q-k} W_\delta^*, \quad \delta \in \Pi,$$

with compatibility relations

$$\mathfrak{a}_\delta|_\tau = \text{Res}_{\tau < \delta}(\mathfrak{a}_\tau) \quad \text{in} \quad A^k(\delta) \otimes C_\tau^\infty(U \cap \tau) \otimes \bigwedge^{p-k,q-k} W_\tau^* \quad \text{for each } \tau < \delta.$$

We define the sheaf of mixed (p, q) -forms on X , relative to Π , as the direct sum

$$(2.5) \quad \Omega^{p,q} = \bigoplus_{k \geq 0} \Omega^{p,q,k}.$$

We naturally define a wedge product on the sheaves $\Omega^{\bullet,\bullet}$, see [APW26, § 3].

Note that $\Omega^{0,0}$ is the sheaf of continuous functions on X_o that are smooth on each face $\delta \in \Pi$. When necessary, we write Ω_Π to emphasize the dependency on Π .

2.3.5. Tropical conjugation on mixed forms. We extend the tropical conjugation to $\Omega_\delta^{\bullet,\bullet}$, $\delta \in \Pi$, and $\Omega^{\bullet,\bullet}$ by setting

$$\overline{z \otimes \alpha} := (-1)^k z \otimes \bar{\alpha}$$

for each $\delta \in \Pi$, $z \in A^k(\delta)$, and α a section of $C_\delta^\infty \otimes \bigwedge^{p-k,q-k} W_\delta^*$, for nonnegative integers k, r, s . Note that the tropical conjugation only acts on $\bigwedge^{\bullet,\bullet} W_\delta^*$ (elements of the Chow ring are real).

2.4. Mixed differential calculus. We explain the construction of the two operators ∂ and $\bar{\partial}$ of bidegrees $(1, 0)$ and $(0, 1)$, respectively, acting on mixed differential forms.

Let δ be a face of Π . We have the two operators ∂_δ and $\bar{\partial}_\delta$ acting on Dolbeault forms $C_\delta^\infty \otimes \bigwedge^{\bullet,\bullet} W_\delta^*$. Namely,

$$\partial_\delta: C_\delta^\infty \otimes \bigwedge^{p,0} W_\delta^* = C_\delta^\infty \otimes \bigwedge^p T_\delta^* \longrightarrow C_\delta^\infty \otimes \bigwedge^{p+1,0} W_\delta^* = C_\delta^\infty \otimes \bigwedge^{p+1} T_\delta^*$$

is the usual differential on smooth forms on δ . Tensoring by $\bigwedge^q \overline{T_\delta^*}$ on both sides, this yields

$$\partial_\delta: C_\delta^\infty \otimes \bigwedge^{p,q} W_\delta^* \longrightarrow C_\delta^\infty \otimes \bigwedge^{p+1,q} W_\delta^*.$$

We similarly define

$$\bar{\partial}_\delta: C_\delta^\infty \otimes \bigwedge^{p,q} W_\delta^* \longrightarrow C_\delta^\infty \otimes \bigwedge^{p,q+1} W_\delta^*.$$

Note that $\bar{\partial}_\delta$ is the tropical conjugation of ∂_δ , that is, we have

$$\bar{\partial}_\delta \alpha = \overline{(\partial_\delta \bar{\alpha})}.$$

If $\mathbf{v} \in \bigwedge^{p,q} W_\delta$ is a (p, q) -multivector, we denote by $\iota_{\mathbf{v}}$ the contraction of Dolbeault forms on δ by $\mathbf{v}$, that is,

$$\iota_{\mathbf{v}}: C_\delta^\infty \otimes \bigwedge^{\bullet, \bullet} W_\delta^* \rightarrow C_\delta^\infty \otimes \bigwedge^{\bullet-p, \bullet-q} W_\delta^*, \quad \alpha \mapsto \iota_{\mathbf{v}}(\alpha) = \alpha(\mathbf{v} \wedge \cdot).$$

By our convention on the tropical conjugation of endomorphisms, we have $\bar{\iota}_{\mathbf{v}} = \iota_{\bar{\mathbf{v}}}$.

We naturally extend the operators ∂_δ , $\bar{\partial}_\delta$, and $\iota_{\mathbf{v}}(\alpha)$, defined above, to the mixed sheaves $\Omega_\delta^{\bullet, \bullet}$ and $\Omega_\delta^{\bullet, \bullet}$. In this context, we have a Leibniz and a commutator rule, see [APW26, § 3.7].

2.4.1. *Definition of ∂ and $\bar{\partial}$.* We define the mixed differential operators

$$\partial: \Omega^{\bullet, \bullet} \rightarrow \Omega^{\bullet+1, \bullet} \quad \text{and} \quad \bar{\partial}: \Omega^{\bullet, \bullet} \rightarrow \Omega^{\bullet, \bullet+1}$$

by taking into account the contributions of $\partial_\bullet$ and $\bar{\partial}_\bullet$ on boundary faces, as follows.

The definition is local, so taking a chart, we can assume without loss of generality that Π is embedded in $\mathbb{R}^n$. For each $\delta \in \Pi$, choose an arbitrary complement $\mathcal{N}^\delta$ to N_δ in $\mathbb{Z}^n$ so that

$$\mathcal{N}^\delta \oplus N_\delta = \mathbb{Z}^n.$$

For each pair $\eta > \delta$, we pick $\mathbf{n}_{\eta/\delta}$ to be the primitive vector in $N_\eta \cap \mathcal{N}^\delta$ pointing to η .

By definition, $\Omega^{\bullet, \bullet}$ is a subsheaf of the sheaf $\bigoplus_{\delta \in \Pi} \Omega_\delta^{\bullet, \bullet}$ on X . We define first

$$\partial: \bigoplus_{\delta \in \Pi} \Omega_\delta^{\bullet, \bullet} \longrightarrow \bigoplus_{\delta \in \Pi} \Omega_\delta^{\bullet+1, \bullet}$$

as the sum

$$\partial := \partial_{\parallel} + \partial_{\perp}$$

of *tangential* $\partial_{\parallel}$ and *transversal* $\partial_{\perp}$ parts. The tangential part $\partial_{\parallel}$ is given simply by the ∂_δ

$$\partial_{\parallel} := \sum_{\delta \in \Pi} \partial_\delta: \bigoplus_{\delta \in \Pi} \Omega_\delta^{\bullet, \bullet} \longrightarrow \bigoplus_{\delta \in \Pi} \Omega_\delta^{\bullet+1, \bullet}.$$

The transversal part

$$\partial_{\perp}: \bigoplus_{\delta \in \Pi} \Omega_\delta^{\bullet, \bullet} \longrightarrow \bigoplus_{\delta \in \Pi} \Omega_\delta^{\bullet+1, \bullet}$$

absorbs the boundary contribution (in the distributional sense) of $\partial_{\parallel}$ at each face of codimension one by transferring the contraction along the corresponding primitive vector into the Chow ring via the Gysin map.

Formally, it is defined by sending a section $z \otimes \alpha$ of $\Omega_\delta^{\bullet, \bullet}$ to the section $\partial_{\perp}(z \otimes \alpha)$ of $\bigoplus_{\gamma < \delta} \Omega_\gamma^{\bullet+1, \bullet}$ given by

$$\partial_{\perp}(z \otimes \alpha) := \sum_{\gamma < \delta} \text{Gys}_{\delta > \gamma}(z) \otimes \bar{\iota}_{\mathbf{n}_{\delta/\gamma}}(\alpha)|_{\gamma}.$$

Note that for z of degree k , $\text{Gys}_{\delta > \gamma}(z)$ has degree $k+1$. Also, if α has bidegree $(p-k, q-k)$, then $\bar{\iota}_{\mathbf{n}_{\delta/\gamma}}(\alpha)|_{\gamma}$ has bidegree $(p-k, q-k-1)$. Therefore,

$$\partial_{\perp}: \bigoplus_{\delta \in \Pi} \Omega_\delta^{p,q,k} \longrightarrow \bigoplus_{\gamma \in \Pi} \Omega_\gamma^{p+1,q,k+1}.$$

We define $\bar{\partial}$ as the tropical conjugate of ∂ .

We proved in [APW26, Thm.3.4] that two operators preserve the subsheaves of mixed differentials, that is, they induce differential operators

$$\partial_{\parallel} : \Omega^{\bullet,\bullet} \rightarrow \Omega^{\bullet+1,\bullet}, \quad \partial_{\perp} : \Omega^{\bullet,\bullet} \rightarrow \Omega^{\bullet+1,\bullet} \quad \text{and} \quad \partial = \partial_{\parallel} + \partial_{\perp} : \Omega^{\bullet,\bullet} \rightarrow \Omega^{\bullet+1,\bullet}.$$

Moreover, these resulting operators are independent of the choices of the subspace complements $\mathcal{N}^{\delta}$ for $\delta \in \Pi$. We also proved in [APW26] that

$$\partial \circ \partial = 0, \quad \bar{\partial} \circ \bar{\partial} = 0, \quad \partial \circ \bar{\partial} = \bar{\partial} \circ \partial.$$

2.5. The degree map. We can define a *degree map* by integrating top-degree mixed forms. Namely, the integration map $\int$ on $\Omega^{d,d}$ is the map

$$\int : \bigoplus_{\delta \in \Pi} \Omega_{\delta}^{d,d} \rightarrow \mathbb{R} \cup \{\infty\}$$

defined for sections on bounded open sets, or globally for nonnegative (d,d) forms, as follows.

Let $U \subset X$ be an open set. For a face $\delta \in \Pi$ of dimension k , a mixed (d,d) form on $U \cap \delta$ is a linear combination of elements of the form $z \otimes \alpha$ for $z \in A^{d-k}(\delta)$ and α a section in $C_{\delta}^{\infty}(U \cap \delta) \otimes \bigwedge^{d-k,d-k} W_{\delta}^*$. This is because $\Omega_{\delta}^{d,d,j} = 0$ if $j \neq d_{\delta}$.

Then, we set

$$\int_U z \otimes \alpha := (-1)^{\binom{k+1}{2}} \deg_{\delta}(z) \cdot \int_{U \cap \delta} \bar{\iota}_{\nu_{\delta}}(\alpha),$$

and extend it by linearity to all of $\Omega_{\delta}^{d,d}$. Here, ν_{δ} refers to a canonical multivector of δ defined as $\mathbf{n}_1 \wedge \dots \wedge \mathbf{n}_k$ for a basis $\mathbf{n}_1, \dots, \mathbf{n}_k$ of the lattice N_{δ} . Note that there are two choices for ν_{δ} differing by a sign. This choice defines an orientation of δ , and the integration of the d -form $\bar{\iota}_{\nu_{\delta}}(\alpha)$ on δ is defined relative to this orientation. Therefore, the definition of $\int$ is independent of the choice of canonical multivector.

The chosen of sign in the above expression guarantees that the degree of the (d,d) form ω^d for a semipositive mixed form $\omega \in \Omega^{1,1}$ is nonnegative.

The degree map is by definition the integration of top-degree mixed forms on X (for those that are either semipositive or compactly supported). We use both the notation $\deg$ and $\int_X$ for the degree map, so $\deg(\omega^d) = \int_X \omega^d$.

2.6. The sheaf of mixed differential forms $\Omega_X^{\bullet,\bullet}$. For a refinement $\bar{\Pi}'$ of $\bar{\Pi}$, with Π' the corresponding refinement of Π , we get a natural injection

$$\Phi : \Omega_{\Pi}^{\bullet,\bullet} \hookrightarrow \Omega_{\Pi'}^{\bullet,\bullet},$$

where $\Omega_{\Pi}^{\bullet,\bullet}$ and $\Omega_{\Pi'}^{\bullet,\bullet}$ are mixed differential forms relative to Π and Π' , respectively.

The inclusion Φ commutes with the inclusions of $\Omega_{\Pi}^{0,0}$ and $\Omega_{\Pi'}^{0,0}$ into $C^0(X)$, with the tropical conjugation, with the wedge product, with the differentials ∂ and $\bar{\partial}$, and the degree map.

For each pair of nonnegative integers p, q , we define

$$\Omega_X^{p,q} := \varinjlim_{\Pi} \Omega_{\Pi}^{p,q}$$

as Π varies over all simplicial polyhedral structures on X_o induced by a simplicial polyhedral structures $\bar{\Pi}$ on X . We infer from the above discussion that the operations of tropical conjugation, wedge product, ∂ and $\bar{\partial}$, and the degree map all extend to $\Omega_X^{p,q}$.

The sheaf $\Omega_X^{0,0}$ coincides with the sheaf PC^∞ of piecewise smooth functions on X_o . Here, a continuous function $f: X_o \rightarrow \mathbb{R}$ is called piecewise smooth if there exists a polyhedral structure Π for X_o such that the restriction of f to each face P of Π is smooth.

As we showed in [APW26, §3], the sheaf of tropical Dolbeault forms on X , defined in [JSS19, GK17, CLD12], as well as δ -forms defined in [GK17, Mih24, BGGK25] inject into $\Omega_X^{\bullet,\bullet}$.

2.7. Integration-by-parts formula. For future use, we note the following consequence of Stokes' theorem. We say that a subset $S \subset X_o$ has polyhedral boundary if there exists a polyhedral structure Π on X_o such that the boundary of S is a polyhedral subcomplex of Π . We recall the integration-by-parts formula established in [APW26], which is used in establishing Bedford–Taylor theory.

Proposition 2.2 ([APW26, Cor. 3.19]). *Let $U \subset X_o$ be an open subset. Let $\zeta \in \Omega^{d-1,d-1}(U)$ be a mixed form such that $\partial\zeta = \bar{\partial}\zeta = 0$. Let $\varphi, \psi \in \text{PC}^\infty(U)$, such that $\text{Supp}(\varphi) \Subset U$. Then,*

$$\int_U (\varphi \cdot \partial\bar{\partial}\psi \wedge \zeta) = - \int_U (\partial\varphi \wedge \bar{\partial}\psi \wedge \zeta) = \int_U (\psi \cdot \partial\bar{\partial}\varphi \wedge \zeta).$$

In the following, $\partial\bar{\partial}$ plays a role analogous to the operator dd^c in complex geometry.

3. LOCALLY CONVEX FUNCTIONS

Throughout this section, we fix a tropical variety X and denote by X_o its open part. Let $U \subset X$ be an open subset. We begin by defining the class of locally convex functions on U , which will be the central object of study in this paper.

Definition 3.1. Let $u: U \rightarrow \mathbb{R}$.

- (1) We say that u admits a local affine support at a point $x \in U$ if there exist a neighborhood U' of x contained in U , and an affine function $l \in \text{Aff}(U')$ such that

$$u(x) = l(x) \quad \text{and} \quad u(y) \geq l(y) \quad \text{for all } y \in U'.$$

In this case, we say that l is an affine support function (of u at x).

- (2) We say that u is locally convex at x if u is continuous at x , and admits a local affine support at x . We say that u is locally convex on U if u is locally convex at each point $x \in U$. We denote by $\text{LConv}(U)$ the set of locally convex functions on U .

Remark 3.2. It is not hard to see that $\text{LConv}(U)$ is a convex cone: For $u_1, u_2 \in \text{LConv}(U)$ and $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 u_1 + \lambda_2 u_2 \in \text{LConv}(U)$. Moreover, LConv defines a sheaf on X . $\diamond$

The rest of this section is devoted to fundamental properties of locally convex functions.

3.1. Locally convex functions on the open part X_o . In this subsection, we assume that U is an open subset of X_o , and unless otherwise stated, $u \in \text{LConv}(U)$. Recall that for $x \in X_o$ and a polyhedral structure $\bar{\Pi}$ on X , with induced polyhedral structure Π on X_o , we denote $\Pi^{(x)} = \bigcup_{x \in \sigma, \sigma \in \Pi} \text{relint}(\sigma)$. Note that $\Pi^{(x)} \subset X_o$ is open, and an affine support function l of u at x can be extended facewise linearly to $\Pi^{(x)}$.

Let $x, y \in X_o$ such that there exists a polyhedral structure Π on X_o and $\sigma \in \Pi$ with $x, y \in \sigma$. Denote by $L_{x,y} := \{tx + (1-t)y \mid t \in [0, 1]\} \subset \sigma \subset X_o$ be the line segment connecting x and y . We remark that the definition of $L_{x,y}$ as a subset of X_o does not depend on the choices of Π and σ . We also remark that if $y \in \Pi^{(x)}$, then $L_{x,y} \subset \Pi^{(x)}$.

- Proposition 3.3.** (1) Suppose that Ω is a convex open subset in $\mathbb{R}^n$. Then u is locally convex on Ω if and only if u is convex in the usual sense.
- (2) Let $U \subset X_o$ be an open subset, and let Π be a polyhedral structure on X_o . Let $x \in U$ and l be an affine support function of u at x . Assume that the line segment $L_{x,y} \subset U \cap \Pi^{(x)}$ for each $y \in U \cap \Pi^{(x)}$. Then, we have $u \geq l$ on $U \cap \Pi^{(x)}$. In particular, $u \geq l$ on $\Pi^{(x)}$ when $\Pi^{(x)} \subset U$.

Proof. For statement (1), it is a standard result that u is convex implies u is locally convex. For the inverse direction, we claim that for any $g \in C^0([0, 1])$, if there exists an affine function l and $\epsilon > 0$ such that $g(0) = l(0)$ and $g \geq l$ on $[0, \epsilon]$, then $g \geq l$ on $[0, 1]$. Since a function is convex on Ω if and only if its restriction to every line segment in Ω is convex, it is not hard to see that this claim concludes statement (1).

Now we prove this claim. By adding an affine function to g , we may assume that $g(0) = 0$ and $l = 0$. For the sake of a contradiction, suppose there exists $t \in (0, 1]$ such that $g(t) < 0$. Let $M = \sup\{g(s) \mid 0 \leq s \leq t\}$. Since $g(0) = 0$, we have $M \geq 0$. Let $t_0 := \sup\{s \in [0, t] \mid g(s) = M\}$. Since $g|_{[0, \epsilon]} \geq 0$, we have $t_0 > 0$. Since g is continuous, we have $g(t_0) = M$, and hence $t_0 < t$. By assumption, there exists an affine function $l_0(s) := a(s - t_0) + M$ such that $l_0 \leq g$ for $s \in (t_0 - c, t_0 + c)$ for some $c > 0$. If $a < 0$, we have $l_0 > M \geq g$ on $(t_0 - c, t_0)$ yielding a contradiction. If $a \geq 0$, then $l_0 \geq M$ on $(t_0, t_0 + c)$. At the same time, $g < M$ on $(t_0, t_0 + c)$ by assumption. This is again a contradiction. Therefore, we infer that $g \geq l$ on $[0, 1]$, and statement (1) is proved.

For statement (2), let $y \in \Pi^{(x)}$. By assumption, the line segment $L_{x,y}$ connecting x and y are contained in U . Then, statement (2) is concluded by applying the claim to $u|_{L_{x,y}} \in C^0(L_{x,y})$. $\square$

Remark 3.4. In Definition 3.1 of locally convex functions, the assumption that u is continuous is necessary. Let $u: (-1, 1) \rightarrow \mathbb{R}$ be the function defined by $u(0) = 0$, and $u(x) = 1$ for $x \neq 0$. Then, u admits a local affine support at each point in $(-1, 1)$, but u is not a convex function in the usual sense. $\diamond$

In [BBS22], the authors introduced a class of convex functions on tropical varieties without infinity boundaries, that satisfy $X_o = X$. Their definition is based on the notion of *polyhedral convex combination*. We next show in Proposition 3.6 that their class of convex functions is equivalent to our notion of locally convex function in X_o .

Let $U \subset N_{\mathbb{R}}$ by a local polyhedral chart of X_o . A *convex combination* in U is a triple

$$\left(x, (x_i)_{i \in I}, (\nu_i)_{i \in I}\right)$$

consisting of points x and x_i in U for i in a finite set I , and nonnegative real numbers ν_i , $i \in I$, satisfying

$$\sum_{i \in I} \nu_i = 1 \quad \text{and} \quad \sum_{i \in I} \nu_i x_i = x \quad \text{in } N_{\mathbb{R}}.$$

A convex combination in U , as above, is called *polyhedral* if there exist a polyhedral structure Π on X_o and polyhedra $\tau, \sigma_i \in \Pi$ for $i \in I$ with $\sigma_i \geq \tau$ such that $x \in \tau, x_i \in \sigma_i$, $i \in I$.

We need the following lemma. For a convex subset $\Omega \subset \mathbb{R}^n$, denote by $\text{Conv}(\Omega)$ the set of all convex functions $u: \Omega \rightarrow \mathbb{R}$.

Lemma 3.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a convex domain, and let $f: \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ be an arbitrary function. Define the function $P(f): \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ by*

$$P(f)(x) := \inf \left\{ \sum_{i \in I} \nu_i f(x_i) \mid (x, (x_i)_{i \in I}, (\nu_i)_{i \in I}) \text{ is a convex combination in } \Omega \right\}, \quad \forall x \in \Omega.$$

Then, we have

$$P(f) = \sup \left\{ u \in \text{Conv}(\Omega) \mid u \leq f \right\}.$$

In particular, $P(f) \in \text{Conv}(\Omega)$.

Proof. It is clear that $P(f) \leq f$ by taking convex the combination $(x, (x), (1))$ (for I a singleton). For any $u \in \text{Conv}(\Omega)$ with $u \leq f$, since $u(x) \leq \sum_{i \in I} \nu_i u(x_i)$ for any convex combination $(x, (x_i)_{i \in I}, (\nu_i)_{i \in I})$ in Ω , we have $u \leq P(f)$. Therefore, it suffices to prove that $P(f) \in \text{Conv}(\Omega)$.

Let $y_1, y_2 \in \Omega$ and $0 \leq t \leq 1$. For any $\epsilon > 0$, by definition we may choose convex combinations $(y_1, (x_{1,i})_{i \in I}, (\nu_{1,i})_{i \in I})$ and $(y_2, (x_{2,i})_{i \in I}, (\nu_{2,i})_{i \in I})$, such that

$$\sum_{i \in I} \nu_{j,i} f(x_{j,i}) \leq P(f)(y_j) + \frac{\epsilon}{2}, \quad j \in \{1, 2\}$$

Then,

$$P(ty_1 + (1-t)y_2) \leq \sum_{i \in I} t\nu_{1,i} f(x_{1,i}) + \sum_{i \in I} (1-t)\nu_{2,i} f(x_{2,i}) \leq tP(f)(y_1) + (1-t)P(f)(y_2) + \epsilon$$

Here the first inequality follows from the observation that $(x, (x_{1,i}, x_{2,i})_{i \in I}, (t\nu_{1,i}, (1-t)\nu_{2,i})_{i \in I})$ is a convex combination. We conclude the proof of the lemma by letting $\epsilon \rightarrow 0$. $\square$

Proposition 3.6. *Let $U \subset N_{\mathbb{R}}$ be a local polyhedral chart of X_o , and let $u: U \rightarrow \mathbb{R}$. Suppose that there exists a convex open subset $\tilde{\Omega} \subset N_{\mathbb{R}}$ such that $U = \tilde{\Omega} \cap X_o$. Then $u \in \text{LConv}(U)$ if and only if, for any polyhedral convex combination $(x, (x_i)_{i \in I}, (\nu_i)_{i \in I})$ in U , we have*

$$(3.1) \quad u(x) \leq \sum_{i \in I} \nu_i u(x_i).$$

Proof. Suppose that $u \in \text{LConv}(U)$. Let $(x, (x_i)_{i \in I}, (\nu_i)_{i \in I})$ be a polyhedral convex combination for a polyhedral structure Π on X , with $x \in \tau$, $x_i \in \sigma_i$, and $\tau, \sigma_i \in \Pi$ for $i \in I$ with $\sigma_i \geq \tau$. Let l be a local affine support function of u at x . Note that $\sigma \cap U = \sigma \cap \tilde{\Omega}$ is a convex subset of $N_{\mathbb{R}}$ for every $\sigma \in \Pi$. Thus, for any $y \in \Pi^{(x)}$, the line segment $L_{x,y}$ connecting x and y is contained in $\sigma \cap U$, where $\sigma \in \Pi$ satisfies $x, y \in \sigma$. Then, Proposition 3.3 (2) implies that $u \geq l$ on $\Pi^{(x)}$. In particular, we have $u(x_i) \geq l(x_i)$ for each $i \in I$, and inequality (3.1) follows.

Conversely, assume that the inequality (3.1) holds for any polyhedral convex combination. We show that $u \in \text{LConv}(U)$. By [BBS22, Thm. 6.2], we have $u \in C^0(U)$. It remains to prove that u admits a local affine support at each point x . Since this is a local statement, we may reduce to the case where x is the origin of a tropical fanfold Z , and U is an open neighborhood of 0 in Z . We need to prove that u admits a local affine support at 0.

By restricting to a linear subspace, we may assume that $\text{span}_{\mathbb{R}}(U) = N_{\mathbb{R}}$. Take a convex domain Ω_0 in $N_{\mathbb{R}}$ such that $\Omega_0 \cap Z \subset U$. The balancing condition of Z implies that the convex hull Ω of $\Omega_0 \cap Z$ is an open neighborhood of 0. We extend u to a function $\tilde{u}$ on Ω by taking $+\infty$ on $\Omega \setminus U$. Take the convex envelop $P(\tilde{u}) \in \text{Conv}(\Omega)$ as in Lemma 3.5. By definition, $P(\tilde{u})(0) = u(0)$. Since u takes finite value on $\Omega_0 \cap Z$, we have $P(\tilde{u}) < +\infty$ on Ω . Therefore, we may take an affine support function $l \in \text{Aff}(\Omega)$ of $P(\tilde{u})$ at 0. Then, we have $l \leq P(\tilde{u}) \leq u$

on $\Omega \cap Z$, and $l(0) = P(\tilde{u})(0) = u(0)$. Hence, l is also an affine support function of u on $\Omega \cap Z$, as required. $\square$

We get the following consequences on pointwise limits and supremums in $\text{LConv}(U)$.

Proposition 3.7. *Let U be an open subset in X_o , and let $\{u_i\}_{i \geq 1}$ be a sequence of locally convex functions on U . Suppose that u_i converges pointwise to a function $u: U \rightarrow \mathbb{R}$. Then $u \in \text{LConv}(U)$.*

Proof. Since $u \in \text{LConv}(U)$ is a local condition, we may assume that U is a local polyhedral chart of X_o satisfying the condition in Proposition 3.6. It is easy to see that inequality (3.1) for u holds for any polyhedral convex combination on U . Therefore by Proposition 3.6, $u \in \text{LConv}(U)$. $\square$

Proposition 3.8. *Let U be an open subset in X_o , and let $\{u_i\}_{i \in I}$ be a collection of locally convex functions on U . Suppose that $\sup_{i \in I} u_i(x) < +\infty$ for every $x \in U$. Then $\sup_{i \in I} u_i$ is a locally convex function on U .*

Proof. Similar to the proof of Proposition 3.7. $\square$

Now, we discuss the Lipschitz property of locally convex functions. Let (Z, d) be a metric space and $f: Z \rightarrow \mathbb{R}$ a function on it. Following the convention in [BBS22], we define the *Lipschitz constant* of f on Z with respect to d as

$$\text{Lip}_{(Z,d)}(f) = \sup_{\substack{x,y \in Z \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x,y)} \in \mathbb{R}_{\geq 0} \cup \{+\infty\},$$

with $\text{Lip}_{(Z,d)}(f) = 0$ if Z consists of a single point. A function f is called *Lipschitz continuous* on Z with respect to d if $\text{Lip}_{(Z,d)}(f) < +\infty$.

Combining Proposition 3.6 with [BBS22, Thm. 6.18]⁷, we get the following.

Proposition 3.9. *Let $U \subset N_{\mathbb{R}}$ be a local polyhedral chart of X_o . Let d the restriction to U of an Euclidean metric on $N_{\mathbb{R}}$. Let K be a compact subset of U . Then, there exists a constant $C > 0$ depending only on U and K such that, for any $u \in \text{LConv}(U)$, we have $\text{Lip}_{(K,d)}(u) \leq C \|u\|_{C^0(U)}$. In particular, for any $u \in \text{LConv}(U)$, u is Lipschitz on K .*

3.2. Locally convex functions around the boundary. In this subsection, we assume that U is an open subset of X , which may intersect the boundary strata of X . The main result of this subsection is the following.

Proposition 3.10. *Let U be an open subset of X .*

- (1) *Let $u_o: U \cap X_o \rightarrow \mathbb{R}$ be a bounded locally convex function. Then, u_o can be extended uniquely to a continuous function $u \in C^0(U)$. Furthermore, we have $u \in \text{LConv}(U)$.*
- (2) *Suppose that $u_o: U \cap X_o \rightarrow \mathbb{R}$ is a locally convex function which is locally bounded from above. Then, u_o can be extended uniquely to a continuous map $u: U \rightarrow \mathbb{R} \cup \{-\infty\}$.*

Here, $\mathbb{R} \cup \{-\infty\}$ is given the usual extended topology. We need the following lemma.

Lemma 3.11. *Let k be a positive integer. Suppose that $u: \mathbb{R}_+^k \rightarrow \mathbb{R}$ is a convex function bounded from above. Then,*

⁷The initial idea of the proof of this theorem comes from [BFJ14].

- (1) If x and y are two points in $\mathbb{R}_+^k$ satisfying $x \geq y$, then $u(x) \leq u(y)$.
(2) There exists a constant $c \in \mathbb{R} \cup \{-\infty\}$ such that for every sequence of points $\{x_i\}_{i=1}^\infty$ in $\mathbb{R}_+^k$ converging to $\underline{\infty} = (\infty, \dots, \infty) \in \mathbb{T}_+^k = (0, \infty]^k$, we have

$$\lim_{i \rightarrow +\infty} u(x_i) \rightarrow c$$

Proof. We treat first the case $k = 1$. We claim that $u(x) \leq u(y)$ whenever $x > y$. In fact, otherwise, the convexity of u would imply that $u(t) \geq \frac{(t-y)}{(x-y)}(u(x) - u(y))$ for any $t \geq x$, in which case, we would obtain $u(t) \rightarrow +\infty$ as $t \rightarrow \infty$. This would contradict the assumption that u is bounded from above. Therefore, the claim is proved, and we have

$$\lim_{t \rightarrow +\infty} u(t) = \inf_{x \in \mathbb{R}_{\geq 0}} u(x),$$

Thus, both statements in (1) and (2) hold when $k = 1$.

Now assume that $k \geq 2$. Let x and $y \in \mathbb{R}_+^k$ be distinct points satisfying $x \geq y$. The ray $\rho := \{y + t(x - y) \mid t \geq 0\}$ is contained in $\mathbb{R}_+^k$. The restriction $u|_\rho$ yields a convex function $\varphi: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, which is bounded from above. We thus obtain $u(x) = \varphi(1) \leq u(y) = \varphi(0)$, proving part (1) of the lemma.

It remains to prove part (2). Set $c := \inf_{x \in \mathbb{R}_+^k} u(x)$. Let $\{x_i\}_{i \geq 1} \subset \mathbb{R}_+^k$ be a sequence of points converging to $\underline{\infty} = (\infty, \dots, \infty)$. Suppose instead that $\lim_{i \rightarrow +\infty} u(x_i) = c' > c$. Then, there exists $y \in \mathbb{R}_{\geq 0}^k$ such that $u(y) < c'$. Observe that $x_i \geq y$ for large enough i . By part (1), however, this implies $u(x_i) \leq u(y)$ for large enough i , and so $c' = \lim_i u(x_i) \leq u(y)$, a contradiction. Therefore, the lemma is proved. $\square$

Proof of Proposition 3.10. (1) We claim that for any $x \in U$, there exists $c_x \in \mathbb{R}$ such that for each sequence $\{x_i\}_{i=1}^\infty$ in $U \cap X_o$ converging to x , we have $u_o(x_i) \rightarrow c_x$. Granting this claim, we define $u(x) := c_x$ for any $x \in U$. Since U is second countable and $U \cap X_o$ is dense in U , it follows that $u \in C^0(U)$.

To prove the claim, we may assume that U is a basic open set. By the definition of basic open sets, we have an isomorphism $U \simeq \mathbb{T}_+^k \times V \subset \mathbb{T}_+^k \times |\Sigma|$ induced by an affine map for an open subset V in the support of a tropical fan Σ , and that $x = (\underline{\infty}, z) \in \mathbb{T}_+^k \times V$. For each $y \in \mathbb{R}_+^k$, we define $u_y: V \rightarrow \mathbb{R}$ by $u_y(w) := u_o(y, w)$.

Let $\{x_i = (y_i, w_i)\}_{i=1}^\infty$ be a sequence of points in $\mathbb{R}_+^k \times V$ such that $x_i \rightarrow x$, equivalently, $y_i \rightarrow \underline{\infty}$ and $w_i \rightarrow z$, as i tends to $+\infty$. Passing to a subsequence, we can assume that $y_1 \leq y_2 \leq \dots$, that is, y_i form an increasing sequence. Lemma 3.11 implies that $\{u_{y_i}\}_{i \geq 1}$ is a decreasing function sequence converging pointwise to some function $u_{\underline{\infty}}: V \rightarrow \mathbb{R}$. Therefore, Proposition 3.7 implies that $u_{\underline{\infty}} \in \text{LConv}(V)$. In particular, $u_{\underline{\infty}} \in C^0(V)$. Hence, by Dini's theorem, $\{u_{y_i}\}_{i \geq 1}$ converges uniformly to $u_{\underline{\infty}}$. This implies that $\lim_{i \rightarrow +\infty} u(y_i, w_i) = u_{\underline{\infty}}(z) \in \mathbb{R}$.

Now suppose that $\{x'_i = (y'_i, w'_i)\}_{i=1}^\infty$ is another sequence of points in $\mathbb{R}_+^k \times V$ converging to x . Then, Lemma 3.11 implies that $\{u_{y'_i}\}_{i \geq 1}$ also converges pointwise to $u_{\underline{\infty}}$, and hence converges uniformly to $u_{\underline{\infty}}$ after the same argument as in the previous paragraph. Therefore, we have $\lim_{i \rightarrow +\infty} u(y_i, w_i) = \lim_{i \rightarrow +\infty} u(y'_i, w'_i) = u_{\underline{\infty}}(z)$. Hence, the claim is proven.

It remains to show that u admits a local affine support at x . Let l_0 be an affine support function of $u_{\underline{\infty}}$ at a point $z \in V$ defined on a neighborhood $V' \subset V$ of z . We define $l := l_0 \circ \pi$, where $\pi: \mathbb{T}_+^k \times V \rightarrow V$ is the projection map. Since for any $y \in \mathbb{R}_+^k$, $u_y \geq u_{\underline{\infty}}$, we have $u(y, w) \geq u(\underline{\infty}, w) \geq l_0(w)$ for any $w \in V'$. Therefore, $u \geq l$ on $U' = \mathbb{T}_+^k \times V'$, and hence l is an affine support function of u at $x = (\underline{\infty}, z)$.

(2) We keep the notation as in the proof of (1). Let $U \cong \mathbb{T}_+^k \times V$ be a basic open neighborhood of $x = (\underline{\infty}, z)$. Without loss of generality, we may assume that u_o is bounded from above on $U \cap X_o$. We may assume further that V satisfies the assumptions of Proposition 3.6, $V = \Pi^{(z)} = \bigcup_{z \in \sigma, \sigma \in \Pi} \text{relint}(\sigma)$ for some polyhedral structure Π on V .

Let $y_1 \leq y_2 \leq \dots$ be an increasing sequence in $\mathbb{R}_+^k$, and $u_{\underline{y}}: V \rightarrow \mathbb{R} \cup \{-\infty\}$ be the pointwise limit of $\{u_{y_i}\}_{i \geq 1}$. Then, by assumption, we have $\{u_{y_i}\}_{i \geq 1}$ is uniformly bounded from above on V . By the same arguments as in (1), it suffices to show that $\{u_{y_i}\}_{i \geq 1}$ converges uniformly to $u_{\underline{y}}$ on an open neighborhood $V' \subset V$ of z . We argue separately in each of the following two cases.

- **Case A**, $u_{\underline{y}}(z) > -\infty$. Take $z_1 \in V$. Then, the balancing condition of Π implies that there exist a finite subset $\{z_j\}_{2 \leq j \leq k} \subset V$ and positive real numbers $\{\nu_j\}_{1 \leq j \leq k}$ such that $(z, (z_j)_{1 \leq j \leq k}, (\nu_j)_{1 \leq j \leq k})$ is a convex combination. Proposition 3.6 implies that

$$u_{y_i}(z_1) \geq \frac{1}{\nu_1} \left(\sum_{2 \leq j \leq k} \nu_j u_{y_i}(z_j) - u_{y_i}(z) \right)$$

for each $i \geq 1$. On the other hand, the monotonicity of $\{u_{y_i}\}_{i \geq 1}$ implies that $u_{y_i}(z_j) \leq u_{y_1}(z_j)$ and $u_{y_i}(z) \geq u_{\underline{y}}(z)$ for each $i \geq 1$ and $2 \leq j \leq k$. Therefore, $\{u_{y_i}(z_1)\}_{i \geq 1}$ is uniformly bounded from above, and $u_{\underline{y}}(z_1) = \lim_{i \rightarrow +\infty} u_{y_i}(z_1) > -\infty$. This shows that $u_{\underline{y}}$ is a finite-valued function on V , and hence $u_{\underline{y}} \in \text{LConv}(V)$ by Proposition 3.7. In particular, $u_{\underline{y}} \in C^0(V)$. Thus, Dini's theorem implies that $\{u_{y_i}\}_{i \geq 1}$ converges uniformly to $u_{\underline{y}}$ on any compact open neighborhood $V' \Subset V$.

- **Case B**, $u_{\underline{y}}(z) = -\infty$. Let $V' \Subset V$ be a compact open neighborhood in V . We claim that $\{u_{y_i}\}_{i \geq 1}$ converges uniformly to $-\infty$ in this case. Proceeding by contradiction, suppose that there exist a constant $C_0 \in \mathbb{R}$ and a sequence of points $\{w_i\}_{i=1}^\infty \subset V'$ such that $u_{y_i}(w_i) \geq C_0$. After passing to a subsequence, we may assume that $w_i \rightarrow w$ with $w \in V$. Moreover, we may assume that there exists $\sigma \in \Sigma$ such that $w_i \in \sigma$ and $2w_i - w \in V \cap \sigma$ for each $i \geq 1$. Then, the convexity of $(u_{y_i})|_\sigma$ implies that $u_{y_i}(2w_i - w) \geq (2C_0 - u_{y_i}(w)) \rightarrow +\infty$. This contradicts the assumption that $\{u_{y_i}\}_{i \geq 1}$ is uniformly bounded from above on V . Therefore, the claim is proved. $\square$

The following result is the generalization of Proposition 3.3 to the boundary.

Corollary 3.12. *Let U be any open subset of X with $x \in U$, and let $u \in \text{LConv}(U)$. Suppose that l is a local affine support function of u at x , and that $\Pi^{(x)} \subset U$. Then $u \geq l$ on $\Pi^{(x)}$.*

Proof. We keep the notation in the proof of Proposition 3.10 (1). Recall that $U \cong \mathbb{T}_+^k \times V \subset \mathbb{T}_+^k \times |\Sigma|$, and $x = (\underline{\infty}, z) \in \{\underline{\infty}\} \times V$. We may assume that $\Pi|_U \cong \mathbb{T}_+^k \times \Pi'$, where Π' is a polyhedral structure on V . Hence, $\Pi^{(x)} = \mathbb{T}_+^k \times \Pi'^{(z)}$. By assumption, $l = l' \circ \pi$, where l' is a local affine support function of $u|_{\{\underline{\infty}\} \times V}$ at z . Therefore, by Proposition 3.3, we have $u|_{\{\underline{\infty}\} \times V} \geq l'$ on $\Pi'^{(z)}$. The proof of Proposition 3.10 (1) implies that $u(y) \geq u((\underline{\infty}, \pi(y)))$ for any $y \in U$. Therefore, $u(y) \geq l'((\underline{\infty}, \pi(y))) = l'(y)$ for any $y \in \Pi^{(x)}$. $\square$

Motivated by Proposition 3.11 (2), we introduce the following definition.

Definition 3.13. Let U be any open subset of X .

- (1) An *extended-real-valued function* on U is a map $u: U \rightarrow \mathbb{R} \cup \{-\infty\}$. Such a function u is called *continuous* if it is continuous when we equip $\mathbb{R} \cup \{-\infty\}$ with the usual

extended topology. We use $C^{0,\text{ext}}(U)$ to denote the set of continuous extended-real-valued functions on U .

- (2) A *plurisubharmonic (psh) function* on U is a function $u: U \rightarrow \mathbb{R} \cup \{-\infty\}$ satisfying $u \in C^{0,\text{ext}}(U)$ and $u|_{U \cap X_o} \in \text{LConv}(U \cap X_o)$. We use $\text{PSH}(U)$ to denote the set of psh functions on U .

Remark 3.14. (1) Let $u \in \text{PSH}(U)$. Note that $u(x) \in \mathbb{R}$ for all $x \in U \cap X_o$, since a locally convex function is always real-valued by definition. In particular, $\{U = -\infty\} \subset U \cap \partial X$, where $\partial X = X \setminus X_o$.

- (2) If $u \in \text{PSH}(U)$ satisfies $u(x) \in \mathbb{R}$ for all $x \in U$, then $u \in \text{LConv}(U)$. To see this, fix $x \in U$ and an open neighborhood V of x with $V \subseteq U$. It suffices to show that $u|_V \in \text{LConv}(V)$. Since $\bar{V}$ is compact and $u: U \rightarrow \mathbb{R} \cup \{-\infty\}$ is continuous, $u|_{V \cap X_o}$ is a bounded locally convex function. Therefore, by Proposition 3.10 (1), there exists $\tilde{u} \in \text{LConv}(V)$ such that $\tilde{u}|_{V \cap X_o} = u|_{V \cap X_o}$. Since both $\tilde{u}, u|_V \in C^0(V)$, and $V \cap X_o$ is dense in V , we conclude that $u|_V = \tilde{u} \in \text{LConv}(V)$. $\diamond$

Recall that $(\cdot)^*$ denotes the upper semicontinuous (usc) envelope of a function.

Proposition 3.15. *Let U be any open subset of X .*

- (1) *Let $\{u_i\}_{i \geq 1} \subset \text{PSH}(U)$. Suppose that u_i converge pointwise to a function $u: U \rightarrow \mathbb{R} \cup \{-\infty\}$ with $u|_{X_o} > -\infty$. Then, $u^* \in \text{PSH}(U)$.*
- (2) *Let $\{u_i\}_{i \in I}$ be a class of psh functions on U . Suppose that $\sup_{i \in I} u_i(x) < +\infty$ for any $x \in U$. Then, $(\sup_{i \in I} u_i)^* \in \text{PSH}(U)$. If in addition $u_j \in \text{LConv}(U)$ for some $j \geq 1$, then $(\sup_{i \in I} u_i)^* \in \text{LConv}(U)$.*
- (3) *Let $\{u_i\}_{i \in I} \subset \text{LConv}(U)$ be a finite set of locally convex functions on U . Then, $\sup_{i \in I} u_i \in \text{LConv}(U)$.*

Proof. For (1), we claim that $u|_{U \cap X_o}$ is bounded from above. Fix $x \in U$, and take a basic open neighborhood W of x satisfying $W \subseteq U$. By definition, $W \cong \mathbb{T}_+^k \times V \subset \mathbb{T}_+^k \times |\Sigma|$, where Σ is a tropical fan. For each $y \in \mathbb{T}_+^k$, we define $u_{i,y}: V \rightarrow \mathbb{R}$ (resp. u_y) by $u_{i,y}(z) := u_i(y, z)$ (resp. $u_y(z) := u(y, z)$). Set $y_0 := (1, \dots, 1) \in \mathbb{R}_+^k$, and $W' := (1, \infty]^k \times V$. As $u_{y_0} \in \text{LConv}(\bar{V})$ and $\bar{V}$ is compact, there exists $M \in \mathbb{R}$ such that $u_{y_0} \leq M$. Moreover, $u_{i,y_0} \geq u_{i,y}$ for each $i \geq 1$ and each $y \geq y_0$. Consequently, $u \leq M$ on $W' \cap X_o = (1, \infty)^k \times V$, and the claim follows.

By Proposition 3.7, we have $u|_{U \cap X_o} \in \text{LConv}(U \cap X_o)$. Therefore, Proposition 3.10 (2) implies that there exists $\tilde{u} \in \text{PSH}(U)$ such that $\tilde{u}|_{U \cap X_o} = u|_{U \cap X_o}$. To complete the proof of (1), it remains to show that $\tilde{u} = u^*$. Since $\tilde{u}$ is continuous and $U \cap X_o$ is dense in U , it is enough to show that $u \leq \tilde{u}$ on U . Take W' as in the previous paragraph, and fix $(y', z') \in W' = (1, \infty]^k \times V$. Choose an increasing sequence $\{y_j\}_{j \geq 1} \subset (1, \infty)^k$ converging to y' . For any $i \geq 1$ and $j \geq 1$, $u_i(y_j, z') \geq u_i(y', z')$. Therefore, $u(y_j, z') = \lim_{i \rightarrow +\infty} u_i(y_j, z') \geq \lim_{i \rightarrow +\infty} u_i(y', z') = u(y', z')$ for all $j \geq 1$. On the other hand, $\tilde{u}(y', z') = \lim_{j \rightarrow +\infty} u(y_j, z')$ since $\tilde{u}$ is continuous. We conclude that $u(y', z') \leq \tilde{u}(y', z')$. Hence, statement (1) is proven.

For (2), the proof of the first part of the statement is similar to the proof of Statement (1), which we omit. Now suppose that $u_j \in \text{LConv}(U)$ for some $j \geq 1$. Then, $(\sup_{i \in I} u_i)^*(x) \geq u_j(x) > -\infty$ for all $x \in U$. Hence, $(\sup_{i \in I} u_i)^* \in \text{LConv}(U)$ by Remark 3.14 (2).

For (3), since the supremum of finitely many continuous functions is continuous, we have $(\sup_{i \in I} u_i) = (\sup_{i \in I} u_i)^* \in \text{LConv}(U)$. $\square$

4. TROPICAL ω -PSH FUNCTIONS

In this section, we always assume that X is a compact tropical variety.

4.1. Semipositive forms and ω -psh functions. We recall the following definitions from [APW26].

- Definition 4.1.** (1) A *real closed simplicial $(1,1)$ -form* on X is a mixed $(1,1)$ -form ω on X_o satisfying the following. For any $x \in X$, there exist a neighborhood U of x and $h \in \text{PL}(U)$ such that $\omega = \partial\bar{\partial}h$ on $U \cap X_o$. When there is no ambiguity, we simply call such ω a *simplicial form* on X .
- (2) A *real closed semipositive $(1,1)$ -form* on X is a mixed $(1,1)$ -form ω on X_o satisfying the following. For any $x \in X$, there exist a neighborhood U of x and $h \in \text{LConv}(U)$ such that $h|_{U \cap X_o} \in \text{PC}^\infty(U \cap X_o)$, and $\omega = \partial\bar{\partial}h$ in $U \cap X_o$. If this is satisfied, h is called a *local potential* of ω on U . When there is no ambiguity, we simply call such ω a *semipositive form* on X .
- (3) We use $\mathcal{K}^{\geq 0}(X)$ to denote the set of semipositive forms on X , and use $\mathcal{K}_{\text{PL}}^{\geq 0}(X)$ for its subset consisting of semipositive simplicial forms on X .

- Remark 4.2.** (1) Let ω be a semipositive form on X . By definition, a local potential h on an open subset $U \subset X$ of ω is *real-valued* and *continuous* (since a locally convex function is by definition real-valued and continuous). However, we require h to be piecewise smooth only on $U \cap X_o$.
- (2) To give an example, let $X = \mathbb{TP}^1$ and $\omega = 0$. Then, the function h defined by $h(t) := t$ cannot be extended to a local potential of ω on an open neighborhood of $+\infty$, since it admits no real-valued continuous extension to such an open neighborhood. $\diamond$

We give the definition of ω -psh functions.

Definition 4.3. Let ω be a semipositive form on X . We say that a function $\varphi: X \rightarrow \mathbb{R} \cup \{-\infty\}$ is an ω -psh function if

- (1) $\varphi \in C^{0,\text{ext}}(X)$. (See Definition 3.13.)
- (2) For any $x \in X$, there exists a neighborhood U of x and a local potential $h \in \text{LConv}(U)$ of ω on U such that $h + \varphi \in \text{PSH}(U)$.

We use $\text{PSH}_\omega(X)$ to denote the set of ω -psh functions on X . Furthermore, we use $\text{CPSH}_\omega(X)$ to denote the set of bounded ω -psh functions on X . $\diamond$

Remark 4.4. Let $\varphi \in C^{0,\text{ext}}(X)$. Let $U \subset X$ be an open subset and let $h \in \text{LConv}(U)$ be a local potential of ω . Assume that $h + \varphi \in \text{PSH}(U)$, then [APW26, Proposition 3.10] implies that $h' + \varphi \in \text{PSH}(U)$ for any local potential $h' \in \text{LConv}(U)$ on U . $\diamond$

- Remark 4.5.** (1) By definition, each $\varphi \in \text{CPSH}_\omega(X)$ is a continuous real-valued function, i.e. $\text{CPSH}_\omega(X) \subset C^0(X)$.
- (2) For each $\varphi \in \text{CPSH}_\omega(X)$ satisfying $\varphi|_{X_o} \in \text{PC}^\infty(X_o)$, $\omega_\varphi := \omega + \partial\bar{\partial}\varphi$ defines a semipositive form on X .
- (3) Observe that by Proposition 3.15 (3), for any $\varphi \in \text{PSH}_\omega(X)$ and any constant $\lambda \in \mathbb{R}$, $\sup\{\varphi, \lambda\} \in \text{CPSH}_\omega(X)$.
- (4) Since $\text{LConv}(U)$ is a convex space for any open subset U , $\text{PSH}_\omega(X)$ is also a convex space. $\diamond$

We now introduce the definition of numerical class for semipositive forms.

Definition 4.6. Let $\omega, \omega' \in \mathcal{K}^{\geq 0}(X)$. We say that ω and ω' are in the *same numerical class* if there exists $\varphi \in C^0(X)$ satisfying $\varphi|_{X_o} \in \text{PC}^\infty(X_o)$ such that $\omega' := \omega + \partial\bar{\partial}\varphi$. We use $[\omega]$ to denote the numerical class of ω .

Remark 4.7. If $\omega, \omega' \in \mathcal{K}^{\geq 0}(X)$ are in the same numerical class, then the space $\text{PSH}_\omega(X)$ is canonically identified with $\text{PSH}_{\omega'}(X)$. $\diamond$

The following theorem is the tropical analogue of Theorem I.5.26 in [Dem].

Theorem 4.8. Let $\varphi_o: X_o \rightarrow \mathbb{R}$ be a function bounded from above which satisfies Property (2) in Definition 4.3. Then, φ_o can be extended uniquely to an ω -psh function φ on X .

Proof. Take a finite open cover $X = \bigcup_{i \in I} W_i$ by tropical charts $(W_i, \phi_i, W_i \rightarrow U_i)$. Let $h_i \in \text{LConv}(W_i)$ such that $\omega = \partial\bar{\partial}h_i$ on W_i . By Proposition 3.10, $\varphi_o|_{W_i \cap X_o}$ can be extended to φ_i over W_i such that $\varphi_i + h_i \in \text{PSH}(W_i)$. Since φ_i is continuous and X_o is dense in X , we obtain that $\varphi_i|_{W_i \cap W_j} = \varphi_j|_{W_i \cap W_j}$ for all i, j . Therefore, the theorem is concluded. $\square$

Example 4.9. Let $\mathbb{TP}_\Xi$ be a projective tropical toric variety, and $X \subset \mathbb{TP}_\Xi$ be a tropical subvariety. Suppose that g be a convex conewise linear function on Ξ , and $h := g|_\Xi$. In [APW26, Example 4.5], we showed that $\omega := \partial\bar{\partial}h$ extends to a semipositive simplicial form on X . Therefore, by Theorem 4.8, there are bijective maps

$$\begin{aligned} \text{PSH}_\omega(X) &\rightarrow \{u \in \text{LConv}(X_o) \mid u - h \leq O(1)\}, & \varphi &\mapsto \varphi + h, \text{ and} \\ \text{CPSH}_\omega(X) &\rightarrow \{u \in \text{LConv}(X_o) \mid |u - h| = O(1)\}, & \varphi &\mapsto \varphi + h. \end{aligned}$$

Moreover, for any $h' \in \text{LConv} \cap \text{PC}^\infty(X_o)$ satisfying $|h' - h| = O(1)$, $\omega' := \partial\bar{\partial}h'$ extends to a semipositive form on X . $\diamond$

Proposition 4.10. (1) Let $\{\varphi_i\}_{i \geq 1}$ be a sequence of ω -psh functions on X . Suppose that φ_i converge pointwise to a function $\varphi: X \rightarrow \mathbb{R} \cup \{-\infty\}$ with $\varphi|_{X_o} > -\infty$. Then, $\varphi^* \in \text{PSH}_\omega(X)$, where $(\cdot)^*$ denotes the usc envelope.
(2) Let $\{\varphi_i\}_{i \in I}$ be a class of ω -psh functions on X . Suppose that for each $x \in X$, we have $\sup_{i \in I} \varphi_i(x) < +\infty$. Then, $(\sup_{i \in I} \varphi_i)^* \in \text{PSH}_\omega(X)$.
(3) Let $\{\varphi_i\}_{i \in I} \subset \text{CPSH}_\omega(X)$ be a finite set of bounded ω -psh functions. Then, $\sup_{i \in I} \varphi_i \in \text{CPSH}_\omega(X)$.

Proof. These are immediate corollaries of Proposition 3.15. $\square$

Corollary 4.11. Let $\{\varphi_i\}_{i \geq 1}$ be a decreasing sequence of ω -psh functions such that $\lim_{i \rightarrow +\infty} \varphi_i(x) > -\infty$ for any $x \in X_o$. Then, the pointwise limit φ of $\{\varphi_i\}_{i \geq 1}$ is an ω -psh function on X . Furthermore, suppose that there exists C such that $\varphi_i \geq C$ on X for any i . Then φ_i converges uniformly to $\varphi \in \text{CPSH}_\omega(X)$.

Proof. Since $\{\varphi_i\}_{i \geq 1}$ is a decreasing sequence of continuous function, φ is usc. Therefore, by Proposition 4.10 (1), $\varphi = \varphi^* \in \text{PSH}_\omega(X)$. The second part of the corollary follows directly from Dini's theorem. $\square$

We will apply the following construction several times in our paper.

Construction 4.12. Let ω be a semipositive form on X , and $\varphi \in \text{CPSH}_\omega(X)$. Let $U \subset X$ is an open subset, and let $u \in \text{LConv}(U)$. Let $h \in \text{LConv}(U)$ satisfy $\omega = \partial\bar{\partial}h$ on U . Suppose that $W := \{\varphi + h < u\} \subseteq U$. We define a function denoted $\sup\{\varphi, u - h\}: X \rightarrow \mathbb{R}$ by setting

$$(4.1) \quad \sup\{\varphi, u - h\}(x) := \begin{cases} \sup\{\varphi(x), u(x) - h(x)\}, & x \in U, \\ \varphi(x), & x \in X \setminus U. \end{cases}$$

Lemma 4.13. *In the setting of Construction 4.12, we have $\sup\{\varphi, u - h\} \in \text{CPSH}_\omega(X)$.*

Proof. Since $W \subseteq U$, $\{X \setminus \overline{W}, U\}$ forms an open cover of X . Denote $\varphi' := \sup\{\varphi, u - h\}$. Remark that $\varphi'|_{X \setminus \overline{W}} = \varphi|_{X \setminus \overline{W}} \in C^0(X)$, and $\varphi'|_U = \sup\{\varphi|_U, u - h\} \in C^0(U)$. Hence, $\varphi' \in C^0(X)$. It remains to verify Property (2) in Definition 4.3. When $x \in X \setminus \overline{W}$, this is satisfied by the assumption that $\varphi \in \text{CPSH}_\omega(X)$. When $x \in U$, by Proposition 3.8, $\varphi' + h = \sup\{\varphi|_U + h, u\} \in \text{LConv}(U)$. This completes the proof. $\square$

4.2. Uniform Lipschitz property for ω -psh functions. Let ω be a semipositive form on X . Fix a finite open cover $X_o = \bigcup_{i \in I} W_i$ by tropical charts $(W_i, \phi_i, W_i \rightarrow U_i)$. For each $i \in I$, fix an Euclidean metric on U_i , and let d_i denote its pullback to W_i via ϕ_i .

Let $S \subset X_o$ and $\varphi: S \rightarrow \mathbb{R}$. We say that φ is C -Lipschitz on S , for $C > 0$, if for any $i \in I$, $\varphi|_{S \cap W_i}$ is C -Lipschitz with respect to d_i .

We have the following theorem.

Theorem 4.14. *Let K be a compact subset of X_o . There exists a positive constant C_K such that for any $\varphi \in \text{PSH}_\omega(X)$, $\varphi|_K$ is C_K -Lipschitz.*

Proof. This is Theorem A.1 proved in Appendix A. $\square$

4.3. Envelope property of ω -psh functions.

Definition 4.15 (Envelope). Let $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function bounded from below. The *envelope* of f denoted $P_\omega(f)$ is defined to be $-\infty$ if there is no ω -psh function $\varphi \leq f$, otherwise,

$$P_\omega(f) := \left(\sup \left\{ \varphi \mid \varphi \in \text{PSH}_\omega(X) \text{ and } \varphi \leq f \right\} \right)^*.$$

Here, as before, $(\cdot)^*$ denotes the usc envelope. $\diamond$

Proposition 4.16. *Let $x \in X_o$. There exists $C > 0$ such that for any $\varphi \in \text{PSH}_\omega(X)$ satisfying $\varphi(x) \leq 0$, we have $\varphi \leq C$ on X .*

Proof. Fix a polyhedral structure $\overline{\Pi}$ on X such that $x \in \Pi_f$. Since Π_f is a compact subset of X_o , Theorem 4.14 implies that there exists $C > 0$ such that $|\varphi| \leq C$ on Π_f for any $\varphi \in \text{PSH}_\omega(X)$ satisfying $\varphi(x) \leq 0$. Let φ be such a function. We now show that $\varphi \leq C$ on X . Since $\varphi \in C^{0, \text{ext}}(X)$ by definition, we have $\sup_{x \in X_o} \varphi(x) = \sup_{x \in X} \varphi(x)$, and φ is bounded from above on X . Let $\sigma \in \Pi$ be an unbounded polyhedron in X_o . We can write $\sigma = \tau + \delta$ for a polytope $\tau \in \Pi_f$ and a polyhedral cone δ (the recession cone of σ). Let $y = y_1 + y_2 \in \sigma$ with $y_1 \in \tau$ and $y_2 \in \delta$. Then, Lemma 3.11 implies that $\varphi(y) \leq \varphi(y_1) \leq C$, as required. $\square$

Corollary 4.17. *Suppose that $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a function such that f is bounded from below, and there exists $x \in X_o$ with $f(x) < +\infty$. Then, $P_\omega(f) \in \text{CPSH}_\omega(X)$.*

Proof. Consider the family of ω -psh functions $\mathcal{F} := \{\varphi \mid \varphi \in \text{PSH}_\omega(X) \text{ and } \varphi \leq f\}$. That f is bounded from below implies that $-M \in \mathcal{F}$ for $M \gg 0$. Hence, $\mathcal{F}$ is a nonempty subset of $\text{PSH}_\omega(X)$. Proposition 4.16 implies that there exists $C \in \mathbb{R}$ such that $\varphi < C$ on X for each $\varphi \in \mathcal{F}$. Therefore, $P_\omega(f) \in \text{PSH}_\omega(X)$ by Proposition 4.10 (2). On the other hand, $P_\omega(f) \geq -C$, so $P_\omega(f) \in \text{CPSH}_\omega(X)$. $\square$

Remark 4.18. It follows from the definition that, as an operator, the psh envelope is monotone and concave. $\diamond$

Example 4.19. We call $\mathcal{S} = \{(x_i, \lambda_i) \in X_o \times \mathbb{R}\}_{1 \leq i \leq k}$ a *finite collection of pairs* if x_i are pairwise distinct points in X_o . We use S to denote the finite subset $\{x_i\}_{1 \leq i \leq k}$ of X_o . We associate to $\mathcal{S}$ a lower semicontinuous (lsc) function f_S by setting

$$f_S(x) = \begin{cases} \lambda_i & \text{if } x = x_i \text{ for some } x_i \in S, \\ +\infty & \text{if } x \in X \setminus S. \end{cases}$$

Then, by Corollary 4.17, we have $P_\omega(f_S) \in \text{CPSH}_\omega(X)$. $\diamond$

We finish this section with the following Lemma.

Definition 4.20 (ω -PL functions on X). Let ω be a semipositive form on X , and let φ be a function on X .

- (1) We say that φ is ω -PL if for any open subset $U \subset X$ and for any local potential $h \in \text{LConv}(U)$ of ω on U , we have $h + \varphi \in \text{PL}(U)$. We use $\text{PL}_\omega(X)$ to denote the space of ω -PL functions on X . In particular, when ω is simplicial, then $\text{PL}_\omega(X) = \text{PL}(X)$.
- (2) Suppose that $\overline{\Pi}$ is a polyhedral structure on X . We say that ψ is ω -facewise affine on $\overline{\Pi}$ if for any open subset $U \subset X$ and for any local potential $h \in \text{LConv}(U)$ of ω on U , $h + \varphi$ is facewise affine on $\overline{\Pi} \cap U$. We say that ω is facewise affine on $\overline{\Pi}$ if 0 is an ω -facewise affine function on $\overline{\Pi}$.

Lemma 4.21. Let ω be a semipositive form on X . Let $\overline{\Pi}$ be a polyhedral structure on X with induced polyhedral structure Π on X_o . Let ψ be an ω -facewise affine function on $\overline{\Pi}$. Then, for any $\varphi \in \text{PSH}_\omega(X)$, $\varphi \leq \psi$ on X iff $\varphi \leq \psi$ on Π_0 , the set of vertices of Π . Equivalently, we have $\psi = P_\omega(f_S)$ for the finite collection of pairs $\mathcal{S} = \{(x, \psi(x)) \mid x \in \Pi_0\}$.

Proof. Fix $\varphi \in \text{PSH}_\omega(X)$ and $\sigma \in \Pi$. Observe that $(\varphi - \psi)|_\sigma$ is convex. Write $\sigma = \tau + \delta$, where τ is a bounded polytope and δ is a polyhedral cone. By Lemma 3.11 applied to each cone $x + \delta$, $x \in \tau$, we have $\sup_\sigma(\varphi - \psi) = \sup_\tau(\varphi - \psi)$. Furthermore, since the set of vertices σ_0 of σ coincides with the set of vertices τ_0 of τ , we have $\sup_\tau(\varphi - \psi) = \sup_{\tau_0}(\varphi - \psi) = \sup_{\sigma_0}(\varphi - \psi)$. Hence, $\sup_X(\varphi - \psi) = \sup_{\Pi_0}(\varphi - \psi)$. $\square$

4.4. Topology on $\text{PSH}_\omega(X)$. We endow the space $\text{PSH}_\omega(X)$ with the topology of pointwise convergence on X_o .

For each $x \in X_o$, let

$$\text{PSH}_{\omega,x}(X) := \{\varphi \in \text{PSH}_\omega(X) \mid \varphi(x) = 0\} \subset \text{PSH}_\omega(X).$$

Proposition 4.22. For each point $x \in X_o$, the subspace $\text{PSH}_{\omega,x}(X)$ of $\text{PSH}_\omega(X)$ is compact.

Proof. We follow the proof of Theorem 5.11 in [BJ22]. Pick a sequence $\{\varphi_i\}_{i \geq 1}$ in $\text{PSH}_{\omega,x}(X)$. By Theorem 4.14, $\{\varphi_i(y)\}_{i \geq 1}$ is a bounded sequence in $\mathbb{R}$ for each $y \in X_o$. By Tychonoff's

theorem, after passing to a subsequence, we may thus assume that $\{\varphi_i\}_{i \geq 1}$ converges pointwise on X_o to a function $\varphi_o: X_o \rightarrow \mathbb{R}$.

By assumption, $\varphi_i + h \in \text{LConv}(W)$ for any local tropical chart W of X and for any local potential h of ω on W . Hence, Proposition 3.7 implies that $\varphi_o + h \in \text{LConv}(W \cap X_o)$. On the other hand, Proposition 4.16 implies that there exists $C \in \mathbb{R}$ such that $\varphi_i < C$ on X for each $i \geq 1$, which implies further that $\varphi_o < C$ on X_o . Therefore, by Theorem 4.8, φ_o extends to a function φ in $\text{PSH}_{\omega, x}(X)$. $\square$

5. REGULARIZATION THEOREM ON COMPACT PREKÄHLER TROPICAL VARIETIES

In this section we prove our regularization theorem on prekähler tropical varieties.

5.1. Kähler forms. We first introduce Kähler forms and prekähler tropical varieties.

Definition 5.1. Let U be an open subset of X and $u: U \rightarrow \mathbb{R}$.

- (1) We say that u *admits a strict local affine support at x* if there exist an open neighborhood U' of x and $l \in \text{Aff}(U')$ such that $u(x) = l(x)$, and $u(y) > l(y)$ for any $y \in U' \setminus \{x\}$.
- (2) We say that u is *strictly locally convex at x* if u is continuous at x , and admits a strict local affine support at x . We say that u is *strictly locally convex on U* if u is strictly locally convex at x for each $x \in U$.

Definition 5.2 (Kähler forms). Let ω be a semipositive form on X .

- (1) Suppose $\varphi \in \text{CPSH}_{\omega}(X)$. We say that φ is *strictly ω -psh* if for any $x \in X$, there exist an open neighborhood W of x and a local potential h of ω on W such that $\varphi + h$ is strictly locally convex at x . We use $\text{CPSH}_{\omega}^{+}(X)$ to denote the set of strictly ω -psh functions on X .
- (2) We say that ω is a *Kähler form on X* if $0 \in \text{CPSH}_{\omega}^{+}(X)$. We use $\mathcal{K}^{>0}(X)$ to denote the set of Kähler forms on X .
- (3) A numerical class $[\eta]$ is called a *Kähler class* if $[\eta]$ can be represented by a Kähler form.

Remark 5.3. Similar to [APW26, Rem. 4.3], the definition of strictly ω -psh functions and Kähler forms does not depend on the choice of the local potentials. This follows from [APW26, Prop. 3.10] which implies that if $\partial\bar{\partial}h = \partial\bar{\partial}h'$ for two potentials on an open set, then $h - h'$ is affine. $\diamond$

Remark 5.4. For a semipositive form ω , $\text{CPSH}_{\omega}^{+}(X)$ is nonempty if $[\omega]$ is a Kähler class. $\diamond$

Definition 5.5 (Compact prekähler tropical varieties). A compact tropical variety X is called *prekähler* if X admits a Kähler form θ .

Remark 5.6. In tropical geometry, a tropical variety is called Kähler if each of its star fans satisfies the Kähler package (see [AP20a, AP20b, AP26]), and it admits a “simplicial Kähler form” in the sense of [AP20a]. (It is shown in *loc.cit.* that the variety itself then verifies the Kähler package.) Therefore, we choose the terminology “prekähler tropical varieties” in our paper, as we do not impose any cohomological requirement on the variety. Also, we note that our definition of Kähler forms is different from the simplicial Kähler forms in [AP20b]. In the setting of *loc. cit.*, a simplicial Kähler form is a semipositive form whose local potential induces a strictly convex conewise linear function on each star fan (in the sense of Section 2.1.8). For a comparison between these two definitions, see the discussion below. $\diamond$

Let Ξ be a complete simplicial fan, and let g be a strictly convex conewise linear function on Ξ (see Section 2.1.8). Notice that the semipositive form $\partial\bar{\partial}g$ is not a Kähler form. This is because g is linear on $\text{relint}(\sigma)$ for each $\sigma \in \Xi$, which implies that g is strictly locally convex at x iff $x = 0$. However, we have the following result.

Proposition 5.7. *Any projective tropical toric variety $\mathbb{TP}_\Xi$, for a complete simplicial fan Ξ , is a compact prekähler tropical variety. More precisely, suppose that g is a strictly convex conewise linear function on Ξ . Then $[\partial\bar{\partial}g]$ is a Kähler class on $\mathbb{TP}_\Xi$.*

To prove this proposition, we need the following lemma.

Lemma 5.8. *Let $E = \mathbb{R}^n$ be an Euclidean space. Consider the Gaussian function*

$$\rho(x) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{\|x\|^2}{2}\right), x \in E.$$

Let $g: E \rightarrow \mathbb{R}$ be a convex function. Then the convolution

$$(g \star \rho)(x) = \int_E g(x - x') \rho(x') dx'.$$

is also a convex function on E . Moreover, let $x_1, x_2 \in E$ and $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$ with $\lambda_1 + \lambda_2 = 1$. We have

$$(g \star \rho)(\lambda_1 x_1 + \lambda_2 x_2) < \lambda_1 \cdot (g \star \rho)(x_1) + \lambda_2 \cdot (g \star \rho)(x_2)$$

if there exists $x_0 \in E$ such that

$$g(\lambda_1 x_1 + \lambda_2 x_2 - x_0) < \lambda_1 \cdot g(x_1 - x_0) + \lambda_2 \cdot g(x_2 - x_0)$$

Proof. The first part of the statement is proven in [BPS14, Lem. 2.7.7]. The second part comes from the fact that the function $x' \mapsto \lambda_1 \cdot g(x_1 - x') + \lambda_2 \cdot g(x_2 - x') - g(\lambda_1 x_1 + \lambda_2 x_2 - x')$ is a nonnegative continuous function on E which takes positive value at x_0 . $\square$

Proof of Proposition 5.7. We use E to denote the support of the fan Ξ . Set $\tilde{g} := (g \star \rho)$. Then, Lemma 5.8 implies that $\tilde{g}$ is convex on E . That g is Lipschitz on E implies that $|\tilde{g} - g| = O(1)$. Therefore, by Example 4.9, $\tilde{\omega} := \partial\bar{\partial}(\tilde{g})$ extends to a semipositive form on $\mathbb{TP}_\Xi$ with $[\tilde{\omega}] = [\partial\bar{\partial}g]$. Our goal is to show that $\tilde{\omega}$ is a Kähler form on $\mathbb{TP}_\Xi$.

Since g is strictly convex on Ξ , for any $x_1, x_2 \in E$ and $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$ with $\lambda_1 + \lambda_2 = 1$, there exists $x' \in E$ such that $g(\lambda_1 x_1 + \lambda_2 x_2 - x') < \lambda_1 \cdot g(x_1 - x') + \lambda_2 \cdot g(x_2 - x')$. Otherwise, g would be linear on the line $x'' + \mathbb{R}[x_2 - x_1]$ for any $x'' \in E$, which would contradict with the assumption that g is strictly convex. Hence, by Lemma 5.8, $\tilde{g}$ is a strictly locally convex function on $E = (\mathbb{TP}_\Xi)_o \subset \mathbb{TP}_\Xi$.

Now, fix $x \in \mathbb{TP}_\Xi$ with $\text{sed}(x) = \delta$. By definition, there exists a linear function l on E such that $g|_\delta = l|_\delta$, and $g > l$ on $\eta \setminus \delta$ for each $\eta > \delta$. Since g is convex on E , this implies that $g > l$ over $E \setminus \delta$. Consider the tangent space T_δ , and let $p: E \rightarrow E^\delta := E/T_\delta \simeq \mathbb{O}(\delta)$ be the quotient map. Set $g_\delta := g - l$, and $\tilde{g}_\delta := (g_\delta \star \rho)$. Since g_δ vanishes on δ , it induces a strictly convex linear function on the star fan Ξ^δ that we denote by g^δ .

We identify E with $T_\delta \times E^\delta$, and identify δ with $\mathbb{R}_{\geq 0}^k \subset T_\delta$ for $k = \dim \delta$. We identify further x with (∞, z) for $z \in E^\delta$. Then, $\mathbb{T}_+^k \times E^\delta$ is a basic open neighborhood of x in E . Notice that for any bounded open subset $V \subset E^\delta$, g_δ is bounded above on $\mathbb{T}_+^k \times V$. And since $|g_\delta - \tilde{g}_\delta| = O(1)$ on E , $\tilde{g}_\delta$ is also bounded above on $\mathbb{T}_+^k \times V$. Therefore, Proposition 3.10 implies that both g_δ and $\tilde{g}_\delta$ can be extended to locally convex functions on $\mathbb{T}_+^k \times V$. Slightly

abusing of notation, we use the same symbol to denote the extensions. Notice that $\tilde{g}_\delta - \tilde{g} = l$ on E . Hence, $\tilde{g}_\delta$ is a local potential of $\tilde{\omega}$ on $\mathbb{T}_+^k \times E^\delta$.

It is not hard to see $(g_\delta)|_{\{\underline{\infty}\} \times E^\delta} = g^\delta$, the induced conewise linear function on Ξ^δ by g^δ . Now we seek to determine $\tilde{g}_\delta|_{\{\underline{\infty}\} \times E^\delta}$. Let $\{y_i\}_{i=1}^\infty \subset \mathbb{T}_+^k$ be a sequence of points such that $y_i \rightarrow \underline{\infty} \in \mathbb{T}_+^k$, and let $z \in E^\delta$. Observe that for any $x' \in E$, $\{g_\delta((y_i, z) - x')\}_{i \geq 1}$ is a decreasing sequence with $\lim_{i \rightarrow +\infty} g_\delta((y_i, z) - x') = g^\delta(z - p(x'))$. Therefore, the monotone convergence theorem implies that

$$\tilde{g}_\delta(\underline{\infty}, z) = \lim_{i \rightarrow +\infty} \tilde{g}_\delta((y_i, z)) = \lim_{i \rightarrow +\infty} \int_E g^\delta(z - p(x')) \rho(x') dx' = ((g^\delta \circ p) \star \rho)((0, z)).$$

Since g^δ is strictly convex on Ξ^δ , for any $z_1, z_2 \in E^\delta$ and $\lambda_1, \lambda_2 \in \mathbb{R}_{>0}$ with $\lambda_1 + \lambda_2 = 1$, there exists $z' \in E^\delta$ such that $g^\delta(\lambda_1 z_1 + \lambda_2 z_2 - z') < \lambda_1 \cdot g^\delta(z_1 - z') + \lambda_2 \cdot g^\delta(z_2 - z')$. Therefore, by Lemma 5.8, $((g^\delta \circ p) \star \rho)$ is strictly convex on $\{0\} \times E^\delta$, which implies further that $\tilde{g}_\delta|_{\{\underline{\infty}\} \times E^\delta}$ is strictly convex on $\{\underline{\infty}\} \times E^\delta$.

Let $l \in \text{Aff}(E^\delta)$ such that l is a strict affine support of $\tilde{g}_\delta|_{\{\underline{\infty}\} \times E^\delta}$ at z . We aim to show that $l \circ p$ is a strict affine support function of $\tilde{g}_\delta$ on $\mathbb{T}_+^k \times E^\delta$, i.e. $\tilde{g}_\delta(y', z') > l \circ p(y', z')$ for any $(y', z') \in \mathbb{T}_+^k \times E^\delta$ with $(y', z') \neq (\underline{\infty}, z')$. By assumption, this inequality holds when $y' = \underline{\infty}$, so we may henceforth assume that $y' \neq \underline{\infty}$. Take $\{y_i\}_{i \geq 1} \subset \mathbb{T}_+^k$ such that $y_1 = y'$, $y_{i+1} \geq y_i$ for any $i \geq 1$, and $\lim_{i \rightarrow \infty} y_i = \underline{\infty}$. Since $\tilde{g}_\delta$ is strictly convex on E by Lemma 5.8 and $\tilde{g}_\delta$ is bounded above on $\mathbb{T}_+^k \times E^\delta$, Lemma 3.11 implies that $\tilde{g}_\delta(y_{i+1}, z') > \tilde{g}_\delta(y_i, z')$ for any $i \geq 1$. Therefore, we have $\tilde{g}_\delta(y', z') > \tilde{g}_\delta(\underline{\infty}, z') \geq l(z') = l(p((y', z')))$. This proves that $\tilde{g}_\delta$ is locally strictly convex at $(\underline{\infty}, z)$, and hence $\tilde{\omega}$ is a Kähler form. $\square$

The next corollary shows that one can always compactify an open tropical variety in $\mathbb{R}^n$ to a compact prekähler tropical variety.

Corollary 5.9. *Let Y be a tropical variety embedded in $E = \mathbb{R}^n$. Then,*

- (1) *There exists a complete simplicial rational fan Ξ on E , and a simplicial rational polyhedral structure Π on Y such that*
 - TP_Ξ is projective.
 - $\text{rec}(\Pi)$ is a subfan of Ξ .
 - Let $\bar{Y}$ denote the closure of Y in TP_Ξ . Then, $\bar{Y}$ is a tropical subvariety of TP_Ξ . In particular, $\bar{Y}$ is a prekähler tropical variety.
- (2) *Furthermore, there exists a convex conewise linear function g on Ξ with $g > 0$ on $E \setminus \{0\}$ such that $[\partial\bar{\partial}h]$ is a Kähler class on $\bar{Y}$, where $h := g|_Y$.*
- (3) *If Y is a fanfold, then Π can be taken as a fan structure on Y , and $h \in \text{LConv} \cap \text{PL}_{\text{hom}}(Y)$.*

Proof. We first prove statement (1). This is a folklore result, for example, there is a variant in [KKMS06, EI06]. For the sake of completeness, we give the argument. Let Π' be any simplicial rational polyhedral structure on Y . Denote by $\Sigma' := \text{rec}(\Pi')$ the recession cone of Π' . By Theorem 2.1, there exist a complete simplicial rational fan Ξ supported on E and a subfan Σ of Ξ such that TP_Ξ is projective and $|\Sigma| = |\Sigma'|$.

Recall that any simplicial polyhedron σ can be written uniquely as a Minkowski sum of a simplex τ and a simplicial cone δ such that their tangent spaces satisfy $T_\tau \cap T_\delta = \{0\}$. Any face of σ is also a simplicial polyhedron which can be written as $\tau' + \delta'$ for $\tau' < \tau$ and

$\delta' < \delta$. Moreover, for any subsets $T, T' \subset \tau$ and $D, D' \subset \delta$, we have $(T \cap T') + (D \cap D') = (T + D) \cap (T' + D')$.

Now, we define

$$\Pi := \{\tau + \delta \mid \tau \in \Pi'_f, \delta \in \Sigma, \text{ such that there exists } \delta' \in \Sigma' \text{ satisfying } \delta \subset \delta', \tau + \delta' \in \Pi'\}$$

Any $\tau + \delta \in \Pi$ is a simplicial polyhedron because $T_\tau \cap T_\delta \subset T_\tau \cap T_{\delta'} = \{0\}$. Let $(\tau_1 + \delta_1), (\tau_2 + \delta_2) \in \Pi$, and let $\delta'_1, \delta'_2 \in \Sigma'$ such that $\delta_i \subset \delta'_i$ and $(\tau_i + \delta'_i) \in \Pi'$ for each $i \in \{1, 2\}$. By assumption, $(\tau_1 + \delta'_1) \cap (\tau_2 + \delta'_2)$ is a face of $\tau_i + \delta'_i$. Hence, $(\tau_1 + \delta'_1) \cap (\tau_2 + \delta'_2) = \tau + \delta' \in \Pi'$, where $\tau = (\tau_1 \cap \tau_2) \in \Pi$ and $\delta' = (\delta'_1 \cap \delta'_2)$. Therefore,

$$\begin{aligned} (\tau_1 + \delta_1) \cap (\tau_2 + \delta_2) &= ((\tau_1 + \delta_1) \cap (\tau + \delta')) \cap ((\tau_2 + \delta_2) \cap (\tau + \delta')) \\ &= (\tau + (\delta_1 \cap \delta')) \cap (\tau + (\delta_2 \cap \delta')) = \tau + (\delta_1 \cap \delta_2) \in \Pi \end{aligned}$$

This implies that Π is a simplicial polyhedral complex. Furthermore, it is a refinement of Π' and satisfies $\text{rec}(\Pi) = \Sigma$. Therefore, $\overline{Y}$ coincides with the canonical compactification of Y with respect to Π , and hence $\overline{Y}$ is a tropical subvariety of TP_Ξ .

For statement (2), choose a strictly convex conewise linear function g' on Ξ . By definition, there exists an affine function l on E such that $g' > l$ on $E \setminus \{0\}$. Then, $g := g' - l$ satisfies $g > 0$ on $E \setminus \{0\}$, and Proposition 5.7 implies $[\partial\bar{\partial}h]$ is a Kähler class on $\overline{Y}$. Statement (3) is clear from the proof of Statement (1). $\square$

5.2. Regularization Theorem. The main theorem of this section is Theorem 5.13. We begin with a result that allows us to construct an ω -psh function ψ on X satisfying a prescribed local property near a fixed point $x \in X$, together with a C^0 -estimate relative to a given strictly ω -psh function φ .

Proposition 5.10. *Let ω be a semipositive form. Suppose that $[\omega]$ is a Kähler class, and let $\varphi \in \text{CPSH}_\omega^+(X)$. Then, for every open set $W \subset X$, every $f \in \text{LConv}(W)$, every $x \in W$, every local potential h of ω on W , and every $\epsilon > 0$, there exist $C > 0$, $l \in \text{Aff}(V)$, $\psi \in \text{CPSH}_\omega(X)$ and an open neighborhood $V \Subset W$ of x such that*

$$\psi + h = Cf + l \text{ on } V, \quad \psi = \varphi \text{ on } X \setminus V, \quad \text{and } \varphi \leq \psi \leq \varphi + \epsilon \text{ on } X.$$

We recall the following lemma.

Lemma 5.11. *Let T be a compact topological space, and $x \in T$. Suppose that $K_1 \supset K_2 \supset \dots$ be a nested sequence of closed subset of T satisfying $\bigcap_{i=1}^{+\infty} K_i = \{x\}$. Then for any open neighborhood U of x in T , there exists $j \geq 1$ such that $K_i \subset U$ for any $i \geq j$.*

Proof. For each i , consider the open set $U_i := T \setminus K_i$ of T . Observe that $\{U_i \mid i \geq 1\}$ is an open cover of T . Since T is compact, and $U_i \subset U_{i'}$ for any $i \leq i'$, there exists $j \geq 1$ such that $T = U_j \cup U$. Therefore, for any $i \geq j$, $K_i \subset K_j = T \setminus U_j \subset U$. $\square$

Proof of Proposition 5.10. Fix $W \subset X$, $x \in W$, a local potential h on W and $\epsilon > 0$. We may assume without loss of generality that $\overline{W}$ is compact. Denote $g := \varphi + h \in \text{LConv}(W)$. By assumption, g is strictly locally convex on W . Let l_g be a strict local affine support function of g at x , and l_f be a local affine support function of f at x .

For any $C, \delta > 0$, define the open subset

$$V(C, \delta) := \{y \in \overline{W} \mid g(y) - l_g(y) < C(f(y) - l_f(y)) + \delta\} \subset \overline{W}$$

Let $\{(C_i, \delta_i)\}_{i \geq 1}$ be a decreasing sequence of pairs of positive numbers converging to $(0, 0)$. Then, by Dini's theorem, $\{C_i(f - l_f) + \delta_i\}_{i \geq 1}$ converges uniformly to 0 on $\overline{W}$. Therefore, by definition,

$$\bigcap_i \overline{V(C_i, \delta_i)} = \{y \in \overline{W} \mid f(y) - l_f(y) \leq 0\} = \{x\}.$$

Here $\overline{V(C_i, \delta_i)}$ denotes the closure of $V(C_i, \delta_i)$ in $\overline{W}$. Now, take an open neighborhood V of x such that $V \subseteq W$. Then, applying Lemma 5.11 to $T = \overline{W}$ and $K_i = \overline{V(C_i, \delta_i)} \cap \overline{W}$ and open neighborhood V of x , there exists $k \geq 1$ such that $V(C_k, \delta_k) \subseteq V$, and $C_k(f - l_f) + \delta_k \leq \epsilon$ on W . Set $u := C_k(f - l_f) + \delta_k + l_g \in \text{LConv}(V)$, and set

$$\psi(y) := \begin{cases} \sup\{\varphi(y), u(y) - h(y)\}, & y \in V \\ \varphi(y), & y \in X \setminus V \end{cases}$$

Remark that $\varphi(y) + h(y) < u(y)$ iff $y \in V(C_k, \delta_k)$. Therefore, $\psi + h = C_k f + (\delta_k + l_g - C_k l_f)$ on $V(C_k, \delta_k)$, and $\psi = \varphi$ on $X \setminus V(C_k, \delta_k)$. Furthermore,

$$0 \leq \psi - \varphi = C_k(f - l_f) + \delta_k + l_g - g \leq C_k(f - l_f) + \delta_k \leq \epsilon \text{ on } V(C_k, \delta_k).$$

On the other hand, applying Lemma 4.13 to $\varphi \in \text{CPSH}_\omega(X)$ and $u \in \text{LConv}(V)$, we have $\psi \in \text{CPSH}_\omega(X)$. Therefore, ψ satisfies all the conditions in the statement, and the proposition is proved. $\square$

Now, we prove the following simplicial version of Proposition 5.10.

Proposition 5.12. *Let ω be a semipositive form. Suppose that $[\omega]$ is a Kähler class, and let $\varphi \in \text{CPSH}_\omega^+(X)$. Then, for every $\epsilon > 0$, there exists $\psi \in \text{CPSH}_\omega \cap \text{PL}_\omega(X)$ such that $\varphi \leq \psi \leq \varphi + \epsilon$ on X . Moreover, for every open set $W \subset X$, every $f \in \text{LConv} \cap \text{PL}(W)$, every $x \in W$, every local potential h of ω on W , and every $\epsilon > 0$, there exist $C > 0$, $l \in \text{Aff}(V)$, $\psi \in \text{CPSH}_\omega \cap \text{PL}_\omega(X)$ and an open neighborhood $V \subseteq W$ of x such that*

$$\psi + h = Cf + l \text{ on } V, \quad \text{and } \varphi \leq \psi \leq \varphi + \epsilon \text{ on } X.$$

Proof. Fix $W \subset X$ and $x \in W$. Fix $f \in \text{LConv} \cap \text{PL}(W)$ and a local potential h of ω on W . Fix $\epsilon > 0$. Applying Proposition 5.10 to $(\varphi, W, x, f, h, \epsilon)$, we obtain an open neighborhood $V_x \subseteq W$ of x , and $\psi_x \in \text{CPSH}_\omega(X)$ such that $\varphi \leq \psi_x \leq \varphi + \epsilon$ on X , and $\psi_x + h = C_x f + l_x$ on V_x for some constant $C > 0$ and $l \in \text{Aff}(V_x)$. Take an open neighborhood V of x such that $V \subseteq V_x$.

For any $y \in X \setminus \overline{V}$, we apply Proposition 5.10 to $(\varphi, X \setminus \overline{V}, y, 0, h, \epsilon)$, where h is any local potential of ω around y . Then, we obtain an open neighborhood $V_y \subseteq X \setminus \overline{V}$ of y , $\psi_y \in \text{CPSH}_\omega(X)$, and $l_y \in \text{Aff}(V_y)$, such that $\psi_y + h = l_y$ on V_y , $\psi_y = \varphi$ on $V \subset X \setminus V_y$, and $\varphi \leq \psi_y \leq \varphi + \epsilon$ on X .

Notice that X is compact, and $\{V_x\} \cup \{V_y \mid y \in X \setminus \overline{V}\}$ is an open cover of X . Then, there exists a finite subset $\{y_j\}_{j \in J} \subset X \setminus \overline{V}$ such that $X = V_x \cup (\bigcup_{j \in J} V_{y_j})$, where J is a finite index set. We define $\psi := \sup\{\psi_x, \psi_{y_j} \mid j \in J\}$. Then by construction, we have $\varphi \leq \psi \leq \varphi + \epsilon$ on X , and we have $\psi + h = \sup\{Cf + l, \varphi + h\} = Cf + l$ on V . Furthermore, by Proposition 4.10 (3), $\psi \in \text{CPSH}_\omega(X)$.

To finish the proof of the proposition, it remains to show that $\psi \in \text{PL}_\omega(X)$. Let $X = \bigcup_{i \in I} W_i$, where I is a finite index set. Let $(W_i, \phi_i, W_i \rightarrow U_i)$ be a tropical chart of X for any $i \in I$. Let h_i be a local potential of ω on W_i . We now show that $\psi + h_i \in \text{PL}(W_i)$ for any $i \in I$. To simplify the notation, we denote $V_0 := V_x$, $V_j := V_{y_j}$ for each $j \in J$, and $J' := \{0\} \cup J$. Let

$f_0 \in \text{PL}(W_i)$ be an extension of $Cf + l \in \text{PL}(V_0 \cap W_i)$, and $f_j \in \text{PL}(W_i)$ be an extension of $l_{y_j} \in \text{PL}(V_j \cap W_i)$ for each $j \in J$.

For any $z \in W_i$, by construction, $(\psi + h_i)(z) = \sup\{(h_i + \varphi)(z), f_j(z) \mid j \in J', z \in V_j\}$. Since $h_i + \varphi \leq f_j$ on V_j for each $j \in J'$, we have $(\psi + h_i)(z) = \sup\{f_j(z) \mid j \in J', z \in V_j\} \in \{f_j(z)\}_{j \in J'}$. Since $(\psi + h_i) \in C^0(W_i)$ and $\{f_j\}_{j \in J'}$ is a finite set of PL function on W_i , by [AT07, Def. 7.10 & Prop. 7.13], $(\psi + h_i) \in \text{PL}(W_i)$. The proof is complete. $\square$

We are now ready to state and prove the following regularization theorem for ω -psh functions.

Theorem 5.13. *Let X be a compact prekähler tropical variety. Let ω be any semipositive form on X and θ be any Kähler form on X .*

- (1) *For any $\varphi \in \text{CPSH}_\theta(X)$, there exists a sequence of functions $\{\varphi_i\}_{i \geq 1} \subset \text{PSH}_\theta \cap \text{PL}_\theta(X)$ such that φ_i converges uniformly to φ .*
- (2) *For each $\varphi \in \text{PSH}_\omega(X)$, there exist a decreasing sequence of positive numbers $\{\alpha_i\}_{i \geq 1}$ with $\alpha_i \rightarrow 0$, and a decreasing sequence of functions $\{\varphi_i\}_{i \geq 1}$ with $\varphi_i \in \text{PSH}_{\omega + \alpha_i \theta} \cap \text{PL}_{\omega + \alpha_i \theta}(X)$, for each $i \geq 1$, such that $\{\varphi_i\}_{i \geq 1}$ converges pointwisely to φ .*

Remark 5.14. When $\varphi \in \text{CPSH}_\omega(X)$, the sequence of functions φ_i in (2) converges uniformly to φ by Dini's theorem. $\diamond$

Proof of Theorem 5.13. Set $\theta_i := \omega + \alpha_i \theta$. We first prove statement (1). When $\varphi \in \text{CPSH}_\theta^+(X)$, this is implied by Proposition 5.12. In general, since $0 \in \text{CPSH}_\theta^+(X)$, we have $(1 - \delta)\varphi \in \text{CPSH}_\theta^+(X)$ for any $\varphi \in \text{CPSH}_\theta(X)$ and $\delta > 0$. Since $(1 - \delta)\varphi$ converges uniformly to φ when $\delta \rightarrow 0+$, this concludes statement (1).

Now, we prove statement (2). Fix $\varphi \in \text{PSH}_\omega(X)$. Let $\{\alpha_i\}_{i \geq 1}$ be an arbitrary decreasing sequence of positive numbers such that $\alpha_i \rightarrow 0$. Take a sequence of positive numbers $\{C_i\}_{i \geq 1}$ such that $C_i \rightarrow +\infty$. Set $\psi'_i := \sup\{\varphi, -C_i\} \in \text{CPSH}_\omega(X)$. Remark that $\{\psi'_i\}_{i \geq 1}$ is a decreasing sequence converging pointwise to φ . Observe that for any $i \geq 1$, $\theta_i = \omega + \alpha_i \theta$ is a Kähler form, and $\psi'_i \in \text{CPSH}_{\theta_i}(X)$. Therefore, statement (1) implies that for each i , one can take $\psi_i \in \text{PSH}_{\theta_i} \cap \text{PL}_{\theta_i}(X)$ that $0 \leq \psi_i - \psi'_i \leq 1/2^i$ on X . Set $\varphi_i := \psi_i + 1/2^i$. Then, $\{\varphi_i\}_{i \geq 1}$ is also a decreasing sequence converging pointwise to φ . Hence, statement (2) is proved. $\square$

We end this section with another application of Proposition 5.12.

Corollary 5.15. *Let θ be a Kähler form. For any $f \in \text{PL}(X)$, there exist $C > 0$ and $\psi_1, \psi_2 \in \text{PSH}_\theta \cap \text{PL}_\theta(X)$ such that $f = C(\psi_1 - \psi_2)$.*

Proof. For any $x \in X$, there exist an open neighborhood W_x of x and $g_x \in \text{LConv} \cap \text{PL}(W_x)$ such that $f + g_x \in \text{LConv}(W_x)$. Let $h_x \in \text{LConv}(W_x)$ be a local potential of θ . Remark that $0 \in \text{CPSH}_\theta^+(X)$. Applying Proposition 5.12 to $(\varphi = 0, W_x, x, h_x, g_x, \epsilon = 1)$, there exist $C_x > 0$, $\psi_x \in \text{PSH}_\theta \cap \text{PL}_\theta(X)$, and an open neighborhood V_x of x such that $C_x(\psi_x + h_x) = g_x + l_x$ on V_x for some constant $C_x > 0$ and $l_x \in \text{Aff}(V_x)$. Therefore, we have $C_x(\psi_x + h_x) + f \in \text{LConv}(V_x)$.

Since $X = \bigcup_{x \in X} V_x$ and X is compact, there exists a finite subset $\{x_j\}_{j \in J} \subset X$ such that $X = \bigcup_{j \in J} V_{x_j}$, where J is a finite index set. Now, we set

$$C := \sum_{j \in J} C_{x_j}, \quad \text{and} \quad \psi_2 := \frac{1}{C} \sum_{j \in J} C_{x_j} \psi_{x_j} \in \text{PSH}_\theta \cap \text{PL}_\theta(X).$$

Then, for each $j \in J$, we have

$$C(\psi_2 + h_{x_j}) + f = C_{x_j}(\psi_{x_j} + h_{x_j}) + f + \sum_{k \in J, k \neq j} C_{x_k}(\psi_{x_k} + h_{x_k}) \in \text{LConv} \cap \text{PL}(V_{x_j}).$$

Therefore $\psi_1 := \psi_2 + \frac{1}{C}f \in \text{PSH}_\theta \cap \text{PL}_\theta(X)$. $\square$

5.3. Strongly semipositive forms.

Definition 5.16. Let ω be a semipositive form. We say that ω is *strongly semipositive* if it satisfies the following conditions. For every open set $W \subset X_o$, every $f \in \text{LConv}(W)$, every $x \in W$, and every local potential h of ω on W , there exist $C > 0$, $l \in \text{Aff}(V)$, $\varphi \in \text{CPSH}_\omega(X)$ and an open neighborhood $V \subseteq W$ of x such that $\varphi + h = Cf + l$ on V . Moreover, if $f \in \text{PL}(W)$, then φ can be chosen in $\text{PSH}_\omega(X) \cap \text{PL}_\omega(X)$.

Remark 5.17. Remark 5.4 together with Propositions 5.10 and 5.12 imply that any Kähler form is strongly semipositive. We prove in Proposition 5.19 below that the same holds for simplicial Kähler forms. We remind the reader that in the statement of Proposition 5.10 and Proposition 5.12, we allow W to be any open subset of X ; while in the definition of strongly semipositive forms, we require W to be an open subset of X_o . $\diamond$

Remark 5.18. The terminology “strongly semipositive” is justified by the following. For any $x \in X_o$, there exist $\varphi \in \text{CPSH}_\omega(X)$ and an open neighborhood $V \subset X_o$ of x such that $\varphi + h$ is strictly locally convex on V , where h is a local potential of ω on V . This is achieved by taking a strictly convex function f on an open neighborhood W of x , and then applying the conditions of Definition 5.16 to W and f . $\diamond$

The next proposition provides examples of strongly semipositive forms that are not Kähler.

Proposition 5.19. Let $X \subset \mathbb{TP}_\Xi$ be a projective tropical variety, and g be a convex conewise linear function on Ξ such that $g > 0$ on $E \setminus \{0\}$, where $E = (\mathbb{TP}_\Xi)_o = \mathbb{R}^n$. We use $\omega := \partial\bar{\partial}g$ to denote the semipositive simplicial form on X defined as in Example 4.9. Then, ω is strongly semipositive.

More precisely, let $U \subset X_o$ be an open subset. Then, for any $x \in U$ and any $f \in \text{LConv}(U)$, there exist an open neighborhood $V \subseteq U$ of x , a constant $C > 0$, and $\varphi \in \text{CPSH}_\omega(X)$ such that $\varphi + g = Cf$ on V , and $\varphi + g \geq Cf$ on U . Furthermore, if $f \in \text{PL}(U)$, φ can be chosen in $\text{CPSH}_\omega \cap \text{PL}_\omega(X)$.

We recall the following basic result before we go to the proof.

Lemma 5.20. Let $E = \mathbb{R}^n$ with its Euclidean norm $\|\cdot\|$, and let $g \in \text{PL}_{\text{hom}}(E)$ satisfy $g > 0$ on $E \setminus \{0\}$. Then, there exists a constant $C > 0$ such that $g(y) \geq C\|y\|$ for each $y \in E$.

Proof of Proposition 5.19. Let $f \in \text{LConv}(U)$. Take a compact neighborhood $U' \subseteq U$ of x . We may assume $0 \leq f \leq M$ on U' for some $M > 0$. Define $g_x(y) := g(y - x)$. Set $h = g|_{X_o}$ and $h_x = g_x|_{X_o}$, where $X_o = X \cap E$. Then, by Example 4.9, $h_x - h \in \text{PSH}_\omega \cap \text{PL}(X)$.

By Lemma 5.20, there exists $C \gg 0$ such that $\{y \in E \mid Ch_x(y) - 1 \leq M\} \subset U'$. Therefore, the open subset $V := \{y \in U \mid Ch_x(y) - 1 < f\}$ is contained in U' , and hence $V \subseteq U$. Since $Ch_x(x) - 1 = -1$ and $f(x) \geq 0$, we have $x \in V$.

Let $\varphi := \frac{1}{C} \sup\{C(h_x - h) - 1, f - Ch\}$ as defined in Construction 4.12. Then, Lemma 4.13 implies that $\varphi \in \text{CPSH}_\omega(X)$. By definition, $\varphi + h = \frac{1}{C}f$ on V and $\varphi + h \geq \frac{1}{C}f$ on U . Furthermore, it's not hard to see that $\varphi \in \text{PSH}_\omega \cap \text{PL}(X)$ when $f \in \text{LConv} \cap \text{PL}(U)$. Hence, the proposition is concluded. $\square$

We end this section with an application of Proposition 5.19.

Let W be any open subset of X_o . We say that a locally convex function u on W is *approximable*, and write $u \in \text{LConv}^{\text{app}}(W)$ if for any $x \in W$, there exist an open neighborhood V of x in W and a function sequence $\{u_i\}_{i \geq 1} \subset \text{LConv} \cap \text{PL}(V)$ converging uniformly to u on V .

Proposition 5.21. *Let W be any open subset of X_o . Then $\text{LConv}^{\text{app}}(W) = \text{LConv}(W)$, that is, any locally convex function on W is approximable.*

Proof. The inclusion $\text{LConv}^{\text{app}}(W) \subset \text{LConv}(W)$ follows directly from Proposition 3.7. We only need to show that $\text{LConv}(W) \subset \text{LConv}^{\text{app}}(W)$.

Let $u \in \text{LConv}(W)$ and $x \in W$. Since this is a local statement, one can assume that W is a bounded open subset of a tropical fanfold $|\Sigma|$, where Σ is embedded in an Euclidean space E . By Corollary 5.9, replacing Σ with another fan with the same support, we can suppose without loss of generality that the canonical compactification $\bar{\Sigma}$ is a projective tropical variety. This gives $h \in \text{LConv} \cap \text{PL}_{\text{hom}}(\Sigma)$ such that $h > 0$ on $\Sigma \setminus \{0\}$. By Proposition 5.7, $[\theta]$ is a Kähler class for $\theta := \partial\bar{\partial}h$.

Applying Proposition 5.19 to $\bar{\Sigma}$, h , and $u \in \text{LConv}(W)$, we obtain an open neighborhood V of x in W , a constant $C > 0$, and $\varphi \in \text{CPSH}_{\theta}(\bar{\Sigma})$ such that $\varphi + h = Cu$ on V .

By Theorem 5.13 (using the equality $\text{PL}_{\theta}(\bar{\Sigma}) = \text{PL}(\bar{\Sigma})$), there exists a sequence $\{\varphi_i\}_{i \geq 1} \subset \text{PSH}_{\theta} \cap \text{PL}(\bar{\Sigma})$ such that φ_i converges uniformly to φ . Setting $u_i := \frac{1}{C}(\varphi_i + h)$, we have $u_i \in \text{LConv} \cap \text{PL}(V)$, and $\{u_i\}_{i \geq 1}$ converges uniformly to u on V . This completes the proof. $\square$

6. MONGE–AMPÈRE MEASURES

6.1. Bedford–Taylor theory. In this section, we assume that X is a compact prekähler tropical variety of dimension d , and ω is a semipositive form on X . We fix a Kähler form θ on X . Denote by $\text{Radon}_{\geq 0}(X)$ the set of nonnegative Radon measures on X .

Recall the following definition from [APW26, § 5].

Definition 6.1. Let $\omega_1, \dots, \omega_d$ be semipositive forms on X , and let $\varphi_i \in \text{PL}_{\omega_i}(X)$. The *mixed Monge–Ampère measure for PL functions* $\text{MA}_{\omega_1, \dots, \omega_d}(\varphi_1, \dots, \varphi_d)$ is the Radon measure on X associated to $\omega_{\varphi_1} \wedge \dots \wedge \omega_{\varphi_d}$. Here, $\omega_{\varphi_i} := \omega_i + \partial\bar{\partial}\varphi_i$. When no confusion arises, we also write $\omega_{\varphi_1} \wedge \dots \wedge \omega_{\varphi_d}$ for this measure on X .

The Radon measure associated to $\omega_{\varphi_1} \wedge \dots \wedge \omega_{\varphi_d}$ is given by the integration of compactly supported top-degree mixed differential forms on X , see Section 2.5 and [APW26, § 5].

Theorem 6.2. *For each $(\omega_1, \dots, \omega_n) \in (\mathcal{K}^{\geq 0}(X))^d$, there is a unique operator*

$$\begin{aligned} \text{MA}_{\omega_1, \dots, \omega_d} : \text{CPSH}_{\omega_1}(X) \times \dots \times \text{CPSH}_{\omega_d}(X) &\longrightarrow \text{Radon}_{\geq 0}(X) \\ (\varphi_1, \dots, \varphi_d) &\longmapsto \text{MA}_{\omega_1, \dots, \omega_d}(\varphi_1, \dots, \varphi_d) \end{aligned}$$

that extends the Monge–Ampère operator on $(\text{PSH}_{\omega_1} \cap \text{PL}_{\omega_1}(X)) \times \dots \times (\text{PSH}_{\omega_d} \cap \text{PL}_{\omega_d}(X))$ defined in Definition 6.1, and satisfies the following properties:

- (1) *Let $\{\epsilon_i^k\}_{k \geq 1}$ be a sequence of positive real numbers such that $\epsilon_i^k \rightarrow 0$ for each $1 \leq i \leq d$. Let $\{\varphi_i^k\}_{k \geq 0} \subset C^0(X)$ be a function sequence for any $1 \leq i \leq d$ such that $\{\varphi_i^k\}_{k \geq 0}$ converges uniformly to φ_i , and $\varphi_i^k \in \text{CPSH}_{\theta_i^k}(X)$ for each $k \geq 0$ with $\theta_i^k := \omega_i + \epsilon_i^k \theta$. Then,*

$$\text{MA}_{\theta_1^k, \dots, \theta_d^k}(\varphi_1^k, \dots, \varphi_d^k) \rightarrow \text{MA}_{\omega_1, \dots, \omega_d}(\varphi_1, \dots, \varphi_d), \quad (k \rightarrow +\infty)$$

In particular, the operator $\text{MA}_{\omega_1, \dots, \omega_d}$ is continuous.

- (2) Suppose that $(W, \phi : W \rightarrow U)$ is a tropical chart of X_o , and for any $1 \leq i \leq d$, $\omega_i = \partial\bar{\partial}h_i$ for some $h_i \in \text{LConv}(W)$ on W . If $\varphi_i + h_i \in \text{PC}^\infty \cap \text{LConv}(W)$ for each $1 \leq i \leq d$, then

$$\text{MA}_{\omega_1, \dots, \omega_d}(\varphi_1, \dots, \varphi_d) = \partial\bar{\partial}(h_1 + \varphi_1) \wedge \dots \wedge \partial\bar{\partial}(h_d + \varphi_d)$$

- (3) $\text{MA}_{\omega_1, \dots, \omega_d}$ is multilinear.

Proof. The proof is exactly the same as the proof of Theorem 5.13 in [APW26] for the Bedford–Taylor theory of the projective case, except that here we use Theorem 5.13 which implies that $\text{CPSH}_\omega(X) = \text{CPSH}_\omega^{\text{app}}(X)$, and Lemma 5.14 in *loc.cit.* needs to be replaced by Corollary 5.15 in this paper. $\square$

In the remaining part of this subsection, we assume that $\omega_1, \dots, \omega_{d-1}$ are semipositive forms on X , and $(\varphi_1, \dots, \varphi_{d-1}) \in \text{CPSH}_{\omega_1}(X) \times \dots \times \text{CPSH}_{\omega_{d-1}}(X)$. We denote

$$T := (\omega_1 + \partial\bar{\partial}\varphi_1) \wedge \dots \wedge (\omega_{d-1} + \partial\bar{\partial}\varphi_{d-1}).$$

Definition 6.3. We say that $f \in \text{dCPSH}(X)$ if there exist $\omega \in \mathcal{K}^{\geq 0}(X)$ and $\psi_1, \psi_2 \in \text{CPSH}_\omega(X)$ such that $f = \psi_1 - \psi_2$. We define $\partial\bar{\partial}\varphi \wedge T := (\omega + \partial\bar{\partial}\psi_1) \wedge T - (\omega + \partial\bar{\partial}\psi_2) \wedge T$.

Corollary 6.4 (Cauchy–Schwartz inequality). *For any $f, g \in \text{dCPSH}(X)$, we have*

$$\int_X f \partial\bar{\partial}f \wedge T \leq 0$$

Furthermore,

$$\left| \int_X f \partial\bar{\partial}g \wedge T \right| \leq \left(\int_X -f \partial\bar{\partial}f \wedge T \right)^{\frac{1}{2}} \left(\int_X -g \partial\bar{\partial}g \wedge T \right)^{\frac{1}{2}}$$

Proof. The proof is exactly the same as the proof of Corollary 5.16 in [APW26]. $\square$

Corollary 6.5. (1) *For any $\varphi, \psi \in \text{dCPSH}(X)$, we have*

$$\int_X \psi \partial\bar{\partial}\varphi \wedge T = \int_X \varphi \partial\bar{\partial}\psi \wedge T$$

- (2) *If $(\varphi_1, \dots, \varphi_d) \in \text{CPSH}_{\omega_1}(X) \times \dots \times \text{CPSH}_{\omega_d}(X)$, then, we have*

$$\text{MA}_{\omega_1, \dots, \omega_d}(\varphi_1, \dots, \varphi_d) = \deg(\omega_1 \wedge \dots \wedge \omega_d).$$

Proof. The proof is exactly the same as the proof of Corollary 5.18 in [APW26]. $\square$

Theorem 6.6. *Let $\varphi \in \text{CPSH}_\omega(X)$. Then $\text{MA}_\omega(\varphi)(\partial X) = 0$.*

Remark 6.7. This result was proven for projective tropical varieties in [APW26, Thm. 5.19]. The proof presented below for arbitrary compact prekähler tropical varieties reduces to the projective case. $\diamond$

Proof of Theorem 6.6. Let $x \in \partial X$. We take a basic open neighborhood U of x such that there exists an isomorphism $\phi : U \rightarrow W := \mathbb{T}_+^k \times V \subset Z_1 \times \mathbb{T}_+^k$, and $\phi(x) = (0, \infty) \in V \times \mathbb{T}_+^k$, where Z_1 is a tropical fanfold embedded in $E_1 = \mathbb{R}^n$, and V is a bounded open subset of Z_1 . Set $E_2 = \mathbb{R}^k$ and $E = E_1 \times E_2$. Set $Z = Z_1 \times E_2$.

We claim that there exists a fan structure Σ on Z , $\eta \in \mathcal{K}_{\text{PL}}^{\geq 0}(\bar{\Sigma})$ (for $\bar{\Sigma}$ the canonical compactification of Σ), and $\psi \in \text{CPSH}_\eta(\bar{\Sigma})$ such that $\bar{\Sigma}$ is prekähler and $(\text{MA}_\omega(\varphi)) = \phi^*(\text{MA}_\eta(\psi))$

on an open neighborhood $U' \subset U$ of x . Granted this claim, since $\text{MA}_\eta(\psi)(\partial\bar{\Sigma}) = 0$ by [APW26, Thm. 5.19], and $\partial X \cap U' = \phi^{-1}(\partial\bar{\Sigma} \cap \phi(U'))$, the theorem follows.

It remains to prove the claim. We first give the construction of the fan structure Σ . By Corollary 5.9, there exists a fan structure Σ_1 on Z_1 such that the canonical compactification $\bar{\Sigma}_1$ is projective. Furthermore, there exists a convex conewise linear function g_1 on Σ_1 such that $g_1 > 0$ on $\Sigma_1 \setminus \{0\}$, and the numerical class $[\partial\bar{\partial}g_1]$ is a Kähler class. On the other hand, let Σ_2 be the standard complete fan on E_2 of the projective space, and let $g_2 := \max\{0, -x_1, -x_2, \dots, -x_k\}$. Then, g_2 is a strictly convex conewise linear function on Σ_2 , and hence $[\partial\bar{\partial}g_2]$ is a Kähler class on $\bar{\Sigma}_2$. Let $\Sigma = \Sigma_1 \times \Sigma_2$, it gives a fan structure on Z . Notice that $\pi_1^*[\partial\bar{\partial}g_1] + \pi_2^*[\partial\bar{\partial}g_2]$ is a Kähler class on $\bar{\Sigma}$, where π_1 and π_2 are projection maps to Σ_1 and Σ_2 respectively. Therefore, $\bar{\Sigma}$ is prekähler.

We proceed to construct η and $\psi \in \text{CPSH}_\eta(\bar{\Sigma})$. By Theorem 5.13, there exists a sequence of positive numbers $\{\alpha_i\}_{i \geq 1}$ satisfying $\alpha_i \rightarrow 0$, and there exists a sequence of functions $\{\varphi_i\}_{i \geq 1}$ such that $\varphi_i \in \text{PSH}_{\theta_i} \cap \text{PL}_{\theta_i}(X)$ for all $i \geq 1$, and $\varphi_i \rightrightarrows \varphi$, where $\theta_i := \omega + \alpha_i \theta$. Let $h \in \text{LConv}(U)$ and $h_i \in \text{LConv}(U)$ be local potentials of ω of θ_i on U respectively such that $h_i \rightrightarrows h$ on U . Set $u_i := \varphi_i + h_i$ and $u := \varphi + h$. Then, we have $u_i \in \text{LConv} \cap \text{PL}(U)$, $u \in \text{LConv}(U)$, and $u_i \rightrightarrows u$ on U . Notice that $\bar{U}$ is compact, and u_i are continuous. So after adding constants to h_i , we may assume that $0 \leq u_i \leq M$ for all $i \geq 1$, where $M > 0$ is a constant.

Let V_0 be an open neighborhood of 0 in $Z_1 = |\Sigma_1|$ such that $V_0 \Subset V$. Define $W_0 := V_0 \times (1, +\infty]^k \Subset W$. Let $g'_2 := \max\{0, 2 - x_1, \dots, 2 - x_k\}$. Fix $C \gg 0$. Observe that we have $Cg'_2 - 1 > M$ on $[1, +\infty]^k \setminus (1, +\infty]^k$. On the other hand, Lemma 5.20 implies that $Cg_1 - 1 > M$ on ∂V_0 when $C \gg 0$. We denote $g' := C(g_1 \circ \pi_1) + C(g'_2 \circ \pi_2) - 2$. Therefore, $g' > M$ on ∂W_0 , and hence $\{g' < u_i \circ \phi^{-1}\}, \{g' < u \circ \phi^{-1}\} \Subset W_0$.

Now, we define $g := C(g_1 \circ \pi_1 + g_2 \circ \pi_2)$, and $\eta := \partial\bar{\partial}g = C(\pi_1^* \partial\bar{\partial}g_1 + \pi_2^* \partial\bar{\partial}g_2) \in \mathcal{K}_{\text{PL}}^{\geq 0}(\bar{\Sigma})$. Since $|g' - g| = O(1)$ on Σ , Example 4.9 implies that $g' - g$ extends to $\text{CPSH}_\eta(\bar{\Sigma})$. Then,

$\psi_i := \sup\{g' - g, u_i \circ \phi^{-1}\} \in \text{PSH}_\eta \cap \text{PL}(\bar{\Sigma})$, $i = 1, 2$ and $\psi := \sup\{g' - g, u \circ \phi^{-1}\} \in \text{CPSH}_\eta(\bar{\Sigma})$, by Lemma 4.13. Set

$$v_i := \sup\{g', u_i \circ \phi^{-1}\} \in \text{LConv} \cap \text{PL}(W_0), \text{ and } v := \sup\{g', u \circ \phi^{-1}\} \in \text{LConv}(W_0).$$

Then, $(\psi_i + g)|_{W_0} = v_i$, and $(\psi + g)|_{W_0} = v$.

Set $W' := (\{g' < 0\} \cap W) \subset W$, and $U' := \phi^{-1}(W')$. It remains to prove that $\text{MA}_\omega(\varphi) = \phi^*(\text{MA}_\eta(\psi))$ on U' . Since $u_i \circ \phi^{-1} \geq 0$ and $u \circ \phi^{-1} \geq 0$ on W by assumption, we have $v_i = u_i \circ \phi^{-1}$, $v = u \circ \phi^{-1}$ on W' . By definition (see Definition 6.1), for any $i \geq 1$, we have $\text{MA}_\omega(\varphi_i) = (\partial\bar{\partial}(u_i))^{\wedge n}$ on $U' \cap X_o$, $\text{MA}_\eta(\psi_i) = (\partial\bar{\partial}(v_i))^{\wedge n}$ on $W' \cap \Sigma$, and $\text{MA}_\omega(\varphi_i)(U' \setminus X_o) = \text{MA}_\eta(\psi_i)(W' \setminus \Sigma)$. Therefore, we conclude that $\text{MA}_\omega(\varphi_i) = \phi^*(\text{MA}_\eta(\psi_i))$ on U' . Now, Theorem 6.2 (1) implies that $\text{MA}_\omega(\varphi) \rightarrow \text{MA}_\omega(\varphi)$, and $(\text{MA}_\eta(\psi)) \rightarrow (\text{MA}_\eta(\psi))$. This proves the claim, and hence finishes the proof of the theorem. $\square$

6.2. Comparison Principle. We end this section with the following.

Theorem 6.8 (Comparison Principle). *Let X be a compact prekähler tropical variety, and let ω be a semipositive form on X . Let $\varphi, \psi \in \text{CPSH}_\omega(X)$.*

$$\int_{\{\varphi < \psi\}} \text{MA}_\omega(\psi) \leq \int_{\{\varphi < \psi\}} \text{MA}_\omega(\varphi).$$

Proof. By Proposition 4.10 (3), for any $\epsilon > 0$, $\max\{\varphi, \psi - \epsilon\} \in \text{CPSH}_\omega(X)$. Then, for $d = \dim X$,

$$\begin{aligned} \deg(\omega^d) &= \int_X \text{MA}_\omega(\max\{\varphi, \psi - \epsilon\}) \\ &\geq \int_{\{\varphi < \psi - \epsilon\}} \text{MA}_\omega(\max\{\varphi, \psi - \epsilon\}) + \int_{\{\varphi > \psi - \epsilon\}} \text{MA}_\omega(\max\{\varphi, \psi - \epsilon\}) \\ &= \int_{\{\varphi < \psi - \epsilon\}} \text{MA}_\omega(\psi) + \int_{\{\varphi > \psi - \epsilon\}} \text{MA}_\omega(\varphi) = \int_{\{\varphi < \psi - \epsilon\}} \text{MA}_\omega(\psi) + \deg(\omega^d) - \int_{\{\varphi \leq \psi - \epsilon\}} \text{MA}_\omega(\varphi) \end{aligned}$$

where the first equality comes from Corollary 6.5 (2). Therefore, we have $\int_{\{\varphi < \psi - \epsilon\}} \text{MA}_\omega(\psi) \leq \int_{\{\varphi \leq \psi - \epsilon\}} \text{MA}_\omega(\varphi)$. Letting $\epsilon \rightarrow 0$, we conclude. $\square$

7. ENERGY FUNCTIONAL

Let X be a compact prekähler tropical variety of dimension d , and ω a semipositive form on X . We define a tropical analogue of the energy functional on $\text{PSH}_\omega(X)$, following [BFJ15, § 6].

7.1. Tropical energy functional E_ω on $\text{CPSH}_\omega(X)$. For $\varphi \in \text{CPSH}_\omega(X)$, we define the value $E_\omega(\varphi)$ of the *tropical energy functional* E_ω as

$$E_\omega(\varphi) := \frac{1}{d+1} \sum_{j=0}^d \int_X \varphi (\omega + \partial\bar{\partial}\varphi)^j \wedge \omega^{d-j}.$$

Proposition 7.1. E_ω is non-decreasing, concave on $\text{CPSH}_\omega(X)$, and satisfies $E_\omega(\varphi + \lambda) = E_\omega(\varphi) + \lambda \deg(\omega^d)$ for any $\varphi \in \text{CPSH}_\omega(X)$ and $\lambda \in \mathbb{R}$.

Proof. Let $\varphi, \psi \in \text{CPSH}_\omega(X)$. It follows formally from an integration-by-parts argument (see e.g. [BJ25, § 1.6] or [Tia12, Lem. 6.2]) that if φ, ψ are two functions in $\text{CPSH}_\omega(X)$, then

$$(6.2) \quad E_\omega(\psi) - E_\omega(\varphi) = \frac{1}{d+1} \sum_{j=0}^d \int (\psi - \varphi) (\omega + \partial\bar{\partial}\varphi)^j \wedge (\omega + \partial\bar{\partial}\psi)^{d-j}.$$

That E_ω is non-decreasing then follows from the non-negativity of $(\omega + \partial\bar{\partial}\varphi)^j \wedge (\omega + \partial\bar{\partial}\psi)^{d-j}$ (Theorem 6.2). And $E_\omega(\varphi + \lambda) = E_\omega(\varphi) + \lambda \deg(\omega^d)$ follows from Corollary 6.5 (2).

Set $\varphi_t = (1-t)\varphi + t\psi$ for $t \in [0, 1]$. Expanding $E_\omega(\varphi_t) - E_\omega(\varphi)$ in t yields the following formulas for first and second derivatives of E_ω :

$$(6.3) \quad \left. \frac{d}{dt} \right|_{t=0+} E_\omega(\varphi_t) = \int (\psi - \varphi) \text{MA}_\omega(\varphi), \text{ and}$$

$$(6.4) \quad \left. \frac{d^2}{dt^2} \right|_{t=0+} E_\omega(\varphi_t) = d \int (\psi - \varphi) \partial\bar{\partial}(\psi - \varphi) \wedge (\omega + \partial\bar{\partial}\varphi)^{d-1}.$$

The Cauchy–Schwartz inequality (Corollary 6.4) implies that $\left. \frac{d^2}{dt^2} \right|_{t=0+} E_\omega(\varphi_t) \leq 0$. Therefore, the functional E_ω is concave. $\square$

Corollary 7.2. *For each pair $\psi, \varphi \in \text{CPSH}_\omega(X)$, we have*

$$\int_X (\psi - \varphi) \text{MA}_\omega(\psi) \leq E_\omega(\psi) - E_\omega(\varphi) \leq \int_X (\psi - \varphi) \text{MA}_\omega(\varphi)$$

Proof. For each $t \in [0, 1]$, set $\varphi_t := (1 - t)\varphi + t\psi$ and $\psi_t := (1 - t)\psi + t\varphi$. The concavity of the energy functional implies that

$$\left. \frac{d}{dt} \right|_{t=1-} E_\omega(\varphi_t) \leq E_\omega(\varphi_1) - E_\omega(\varphi_0) \leq \left. \frac{d}{dt} \right|_{t=0+} E_\omega(\varphi_t).$$

Applying Equation 6.3, we get

$$\begin{aligned} \int_X (\psi - \varphi) \text{MA}_\omega(\psi) &= - \left. \frac{d}{dt} \right|_{t=0+} E_\omega(\psi_t) = \left. \frac{d}{dt} \right|_{t=1-} E_\omega(\varphi_t) \\ &\leq E_\omega(\psi) - E_\omega(\varphi) \leq \left. \frac{d}{dt} \right|_{t=0+} E_\omega(\varphi) = \int_X (\psi - \varphi) \text{MA}_\omega(\varphi). \quad \square \end{aligned}$$

7.2. Energy functional for ω -psh functions. For $\varphi \in \text{PSH}_\omega(X)$ and $C \in \mathbb{R}$, we denote $\varphi_C := \max\{\varphi, C\} \in \text{CPSH}_\omega(X)$. We extend the tropical energy functional $E_\omega: \text{PSH}_\omega(X) \rightarrow \mathbb{R} \cup \{-\infty\}$ by

$$E_\omega(\varphi) := \lim_{C \rightarrow -\infty} E_\omega(\varphi_C).$$

Proposition 7.3. *The energy functional E_ω is well-defined on $\text{PSH}_\omega(X)$. It is non-decreasing, concave, and satisfies $E_\omega(\varphi + \lambda) = E_\omega(\varphi) + \lambda \deg(\omega^d)$ for any $\lambda \in \mathbb{R}$. It is also upper semi-continuous, and continuous along decreasing sequences on $\text{PSH}_\omega(X)$.*

Proof. Proposition 7.1 implies that $E_\omega(\varphi_C)$ is decreasing in C . Therefore, $\lim_{C \rightarrow -\infty} E_\omega(\varphi_C)$ exists.

That E_ω is non-decreasing, concave, and $E_\omega(\varphi + \lambda) = E_\omega(\varphi) + \lambda \deg(\omega^d)$ then follows formally from Proposition 7.1. That E_ω is continuous along decreasing sequences follows from the upper semicontinuity and monotonicity of E_ω . It remains to prove the upper semicontinuity for E_ω .

Let $\varphi \in \text{PSH}_\omega(X)$ and $M \in \mathbb{R}$ such that $E_\omega(\varphi) < M$. We need to find an open neighborhood U of φ inside $\text{PSH}_\omega(X)$ such that $E_\omega(\varphi') < M$ for any $\varphi' \in U$. By definition, we may choose $C \in \mathbb{R}$ and $\lambda > 0$ such that $E_\omega(\varphi_C + \lambda) < M$. Since $\text{PL}_\omega(X)$ is dense in $C^0(X)$, we may further choose $\psi \in \text{PL}_\omega(X)$ such that $\varphi_C < \psi < \varphi_C + \lambda$.

Consider the functional on $\text{PSH}_\omega(X)$ given by $\varphi' \mapsto \sup_X(\varphi' - \psi)$. We choose a polyhedral structure $\bar{\Pi}$ of X with induced polyhedral structure on X_o such that ψ is ω -facewise affine on $\bar{\Pi}$. By Lemma 4.21, $\sup_X(\varphi' - \psi) = \sup(\varphi' - \psi)|_{\Pi_0}$. Since Π_0 is a finite subset of X_o , the functional $\varphi' \mapsto \sup_X(\varphi' - \psi)$ is continuous on PSH_ω .

Suppose $\sup_X(\varphi - \psi) = -\epsilon$ with $\epsilon > 0$. Then,

$$U := \left\{ \varphi' \in \text{PSH}_\omega(X) \mid -2\epsilon < \sup_X(\varphi' - \psi) < 0 \right\}$$

is an open neighborhood of φ . For any $\varphi' \in U$, we have $\varphi' \leq \varphi_C + \lambda$. Hence, the monotonicity of E_ω implies that $E_\omega(\varphi') \leq E_\omega(\varphi_C + \lambda) < M$. This concludes the upper semicontinuity of E_ω . $\square$

7.3. The Lagrangian functional. Given a nonnegative Radon measure μ with compact support in X_o with $\text{vol}(\mu) = \deg(\omega^d)$, define the Lagrangian functional

$$\begin{aligned} F_\mu : \text{PSH}_\omega(X) &\longrightarrow \mathbb{R} \cup \{-\infty\} \\ \varphi &\longmapsto E_\omega(\varphi) - \int_X \varphi \, d\mu. \end{aligned}$$

Corollary 6.5(2) and Proposition 7.3 imply that $F_\mu(\varphi + \lambda) = F_\mu(\varphi)$ for any constant $\lambda \in \mathbb{R}$. Recall that $\text{PSH}_{\omega,x}(X) := \{\varphi \in \text{PSH}_\omega(X) \mid \varphi(x) = 0\}$, and $\text{CPSH}_{\omega,x}(X) := \{\varphi \in \text{PSH}_\omega(X) \mid \varphi(x) = 0\}$.

Proposition 7.4. *The functional F_μ is upper semicontinuous on $\text{PSH}_{\omega,x}(X)$, and it admits a maximizer $\varphi_\mu \in \text{CPSH}_{\omega,x}(X)$. In particular, $E_\omega(\varphi_\mu)$ is finite.*

Proof. Let K be a compact subset of X_o containing the support of μ .

We first prove the upper semicontinuity of F_μ . The upper semicontinuity of the functional E_ω is already established in Proposition 7.3. It remains to prove that the functional $\varphi \mapsto \int_X \varphi \, d\mu$ is continuous. Let $\{\varphi_i\}_{i \geq 1} \subset \text{PSH}_{\omega,x}(X)$ be a sequence of functions converging to φ in $\text{PSH}_{\omega,x}(X)$, that is, $\{\varphi_i\}_{i \geq 1}$ converge pointwise to φ on X_o . Then by Theorem 4.14, φ_i are C_K -Lipschitz on K for a uniform constant $C_K > 0$. Therefore, $\{\varphi_i\}_i$ converges uniformly to φ on K , and the continuity of the above functional follows.

Since $\text{PSH}_{\omega,x}(X)$ is compact (Proposition 4.22), F_μ admits a maximizer ψ on $\text{PSH}_{\omega,x}(X)$. Note that $\psi|_K$ is continuous. Hence, there exists $C \in \mathbb{R}$ such that $\psi \geq C$ on K . Set $\varphi_\mu = \sup\{\psi, C\}$. By monotonicity property of E_ω (Proposition 7.3), $E_\omega(\varphi_\mu) \geq E_\omega(\psi)$. At the same time, we have $\psi|_K = \varphi_\mu|_K$. Therefore, $\int_X \varphi_\mu \, d\mu = \int_X \psi \, d\mu$. Hence, φ_μ is a maximizer of F_μ satisfying $\varphi_\mu \in \text{CPSH}_{\omega,x}(X)$. $\square$

Proposition 7.5. *Let μ be a nonnegative Radon measure with compact support in X_o . Assume that $\varphi \in \text{CPSH}_{\omega,x}(X)$ satisfies $\text{MA}_\omega(\varphi) = \mu$. Then, φ is a maximizer of F_μ .*

Proof. For any $\psi \in \text{PSH}_{\omega,x}(X)$, on the segment joining φ and ψ in $\text{PSH}_{\omega,x}(X)$, we have

$$\left. \frac{d}{dt} \right|_{t=0+} F_\mu(t\varphi + (1-t)\psi) = \int_X (\psi - \varphi)(\text{MA}_\omega(\varphi) - \mu) = 0.$$

Since F_μ is concave, this implies that $F_\mu(\varphi) \geq F_\mu(\psi)$, and the proposition follows. $\square$

8. TROPICAL MONGE–AMPÈRE EQUATIONS

Let X be a compact prekähler tropical variety of dimension d , and let ω be a strongly semipositive form on X . We say that the pair (X, ω) verifies the tropical Calabi–Yau theorem if for any nonnegative Radon measure μ compactly supported in X_o of total volume $\deg(\omega^d)$, there exists a unique $\varphi \in \text{CPSH}_\omega(X)$ up to a constant to the tropical Monge–Ampère equation:

$$(8.1) \quad \text{MA}_\omega(\varphi) = \mu.$$

In this section, we prove the uniqueness part of the tropical Calabi–Yau statement.

Theorem 8.1. *Let X be a compact prekähler tropical variety of dimension d , and let ω be a strongly semipositive form on X . Suppose that $\varphi, \psi \in \text{CPSH}_\omega(X)$ satisfy $\text{MA}_\omega(\varphi) = \text{MA}_\omega(\psi)$. Then, $\varphi - \psi$ is constant on X .*

Remark 8.2. Note that $\deg(\omega^d) > 0$ when ω is strongly semipositive. To see this, take a simplicial polyhedral structure $\bar{\Pi}$ on X with induced polyhedral structure Π on X_o . Take $\tau \in \Pi_0$ with $\dim \tau = \dim X$, and take $x \in \text{relint}(\tau)$. Let $(t_1, \dots, t_d)$ be affine coordinates on the d -dimensional simplicial polyhedron τ , and set $u := t_1^2 + \dots + t_d^2 \in \text{LConv}(\text{relint}(\tau))$. Then, by definition (Definition 5.16), there exists an open neighborhood $U \subseteq \text{relint}(\tau)$ of x , $C > 0$, $l \in \text{Aff}(U)$, a potential h of ω on V and $\varphi \in \text{CPSH}_\omega(X)$ such that $\varphi + h = Cu + l$. Then, we have $\text{MA}_\omega(\varphi)(U) = C^d \cdot \varpi(\tau) \cdot \text{MA}_\mathbb{R}(u)(U) > 0$. Thus, by Corollary 6.5 (2), we obtain that $\deg(\omega^d) = \text{MA}_\omega(\varphi)(X) \geq \text{MA}_\omega(\varphi)(U) > 0$. $\diamond$

Remark 8.3. One can formulate the tropical Calabi–Yau theorem in exactly the same way for any semipositive form ω with $\deg(\omega^d) > 0$. We are however only able to prove the uniqueness part assuming strong semipositivity, and do not know whether this holds in general for any semipositive form. Note also that a semipositive form ω with $\deg(\omega^d) > 0$ is not necessarily strongly semipositive (see Remark 11.9). $\diamond$

In preparation for the proof, we have the following two lemmas.

Lemma 8.4. *Under the hypotheses of Theorem 8.1, for any $f \in \text{PL}(X)$, and for any $\tilde{\varphi} \in \text{PSH}_\omega \cap \text{PL}_\omega(X)$, we have*

$$\int_X (\psi - \varphi) \partial \bar{\partial}(f) \wedge (\omega + \partial \bar{\partial} \tilde{\varphi})^{d-1} = 0.$$

Proof. The proof proceeds as in the corresponding step in the proof of [BFJ15, Prop. 8.2]. The only inputs are the Cauchy–Schwartz inequality and the integration-by-parts formula, whose tropical versions are proven in Corollary 6.4 and Corollary 6.5 (1), respectively. $\square$

Lemma 8.5. *For any tropical fan Σ in $E = \mathbb{R}^n$, there exists $g \in \text{LConv} \cap \text{PL}_{\text{hom}}(|\Sigma|)$ such that $\text{MA}(g)(\{0\}) > 0$.*

Here MA denotes the local tropical Monge–Ampère operator on Σ , see [APW26, § 5].

Proof of Lemma 8.5. Take $h \in \text{Conv} \cap \text{PL}_{\text{hom}}(E)$ such that $h(0) = 0$, and $h > 0$ on $E \setminus \{0\}$. Let $g := h|_\Sigma$. It is not hard to show that $\text{MA}(g)$ is supported at $\{0\}$. We prove that $\text{MA}(g)(\{0\}) > 0$. Let σ be a facet of Σ , and let $\{x_i\}_i \subset \text{relint}(\sigma)$ such that $x_i \rightarrow 0$. We define $g_i \in \text{LConv}(|\Sigma|)$ by $g_i(x) := h(x - x_i)$ for any $x \in |\Sigma|$. Then, g_i converge uniformly to g in a neighborhood U of 0, so $\text{MA}(g_i) \rightarrow \text{MA}(g)$ on U . On the other hand, there exists a positive constant c depending only on σ such that $\text{MA}(g_i)(\{x_i\}) = c \text{MA}_\mathbb{R}(g_i|_\sigma)(\{x_i\}) = c \text{MA}_\mathbb{R}(h|_{T_\sigma})(\{0\})$, where T_σ is the tangent space to σ (the linear span of σ), and $\text{MA}_\mathbb{R}$ refers to the real Monge–Ampère operator on σ with respect to the lattice structure on σ . Notice that $h > 0$ on $T_\sigma \setminus \{0\}$, so $\text{MA}_\mathbb{R}(h|_{T_\sigma})(\{0\}) > 0$. Therefore, $\text{MA}(g)(\{0\}) \geq c \text{MA}_\mathbb{R}(h|_{T_\sigma})(\{0\}) > 0$. $\square$

We are ready to prove Theorem 8.1.

Proof of Theorem 8.1. We show that for any simplicial polyhedral structure $\bar{\Pi}$ on X with induced polyhedral structure Π on X_o , the restriction of $\varphi - \psi$ on the set of vertices Π_0 is constant. Taking refinements of the polyhedral structure and using the continuity of φ and ψ , we infer that $\varphi - \psi$ is constant on X .

Consider the set $\Pi_{f,1}$ of finite edges in Π . Let Γ be the metric graph with vertex set Π_0 , edge set $\Pi_{f,1}$, with the primitive vector of each edge having length one. So, Γ has the same support as the one-skeleton $\Pi_{f,(1)} = \Pi_0 \cup \Pi_{f,1}$, and the length of each edge is given in terms

of the primitive vector of that edge. By connectivity of X , Γ is connected. We will construct below a semipositive simplicial form ω' defined on some simplicial refinement $\bar{\Pi}'$ of $\bar{\Pi}$ such that Γ is included in the support of $(\omega')^{d-1}$, which is a nonnegative Minkowski weight of dimension one on Π' and belongs to $\text{MW}_1^{\geq 0}(\Pi')$.

Granted the existence of such ω' , we explain how to finish the proof. The weight on each edge $e' \in \Pi'_{f,1}$ is given by $\deg_{e'}((\omega')^{d-1})$. Denote by Γ' the support of $(\omega')^{d-1}$. Endowed with the weight function $\deg_{e'}((\omega')^{d-1}) > 0$ for each e' in the support, this is a tropical subvariety of dimension one in X_o . We associate edge lengths to finite edges in Π'_1 as in the definition of edge lengths in Γ above, by declaring the primitive vector of the edge to be of norm one. For each edge e' in $\Pi'_{f,1}$, denote by $l_{e'}$ its length. By the assumption, Γ' contains Γ .

Let now f be a facewise affine function on Π' such that $f(v) = \varphi(v) - \psi(v)$ for any $v \in \Pi'_0$. Applying Lemma 8.4, combined with Proposition 4.12 and Lemma 5.10 in [APW26], yields

$$-\sum_{e'=uv \in \Pi'_{f,1}} l_{e'}^{-1} (f(u) - f(v))^2 = \int_{\Gamma'} (\varphi - \psi) \partial \bar{\partial} f = \int_X (\varphi - \psi) \partial \bar{\partial} f \wedge (\omega')^{d-1} = 0.$$

Since $l_{e'} \in \mathbb{R}_+$, this implies that $f(u) = f(v)$ for any finite edge $e' = uv$ in $\Pi'_{f,1}$. Since Γ is included in Γ' and connected, f is constant on the set of vertices of Γ , and hence $\varphi - \psi$ is constant on Π_0 , as desired.

It remains to construct such ω' . We may assume that for any $e \in \Pi_{f,1}$, there exists a tropical chart $(W, \phi: W \rightarrow U)$ with $U \subset |\Sigma| \subset E = \mathbb{R}^n$ for Σ a tropical fan of dimension d such that W contains e , and such that, viewed in U , e is a segment on a ray ρ of Σ with one extremity the origin of E . Slightly abusing of notation, we identify U and W . We use L_e to denote the 1-dimensional subspace of E generated by e . Let $\pi: E \rightarrow E/L_e$ be the projection map and $\Sigma' := \pi(\Sigma)$ be the image of Σ with its structure of tropical fan, see Section 2.1. By Lemma 8.5, we can take $g' \in \text{LConv} \cap \text{PL}(\Sigma')$ such that $(\partial \bar{\partial} g')^{d-1}(\{0\}) > 0$. Set $g := g' \circ \pi$.

Let $\text{div}(g)$ denote the real Minkowski weight of dimension $d-1$ associated to g . The support of $\text{div}(g)$ coincides with the support of $\partial \bar{\partial} g$, and the weight on each $\tau \in \Pi_{f,d-1}$ in the support is given by $\deg_{\tau}(\partial \bar{\partial} g)$. Since Σ' is isomorphic to the star fan Σ^{ρ} , we have $(\text{div}(g))^{d-1}(\rho) = (\partial \bar{\partial} g')^{d-1}(\{0\}) > 0$.

By definition of the strongly semipositive forms (Definition 5.16), for any $x \in e$, there exist $\varphi_x \in \text{PSH}_{\omega} \cap \text{PL}_{\omega}(X)$, an open neighborhood U_x of x , a constant $C > 0$, local affine function l on U_x , and $h \in \text{LConv} \cap \text{PL}_{\omega}(U_x)$ verifying $\omega = \partial \bar{\partial} h$ such that $\varphi_x + h = Cg + l$ on U_x . Let Π_x be any refinement of Π such that φ_x is facewise affine on Π_x . Therefore, for any $e' \in (\Pi_x)_{f,1}$ such that $e' \subseteq e$ and $U_x \cap e' \neq \emptyset$, we have $\deg_{e'}(\omega_{\varphi_x}^{d-1}) > 0$. Since e is compact, there exists a finite collection of points $x_i \in e$, $i \in I_e$ such that $e \subset \bigcup_{i \in I_e} U_{x_i}$. Let I be the disjoint union of I_e , for $e \in \Pi_{f,1}$.

Set

$$\omega' := \omega + \frac{1}{|I|} \sum_{e \in \Pi_{f,1}} \sum_{i \in I_e} \partial \bar{\partial} \varphi_{x_i},$$

and let $\bar{\Pi}'$ be a simplicial refinement of $\bar{\Pi}$ such that ω' is simplicial on the induced polyhedral structure Π' on X_o . Then, for any $e' \in \Pi'_{f,1}$ such that $e' \subseteq e$, for $e \in \Pi_{f,1}$, we have $\deg_{e'}((\omega')^{d-1}) \geq \frac{1}{|I|} \deg_{e'}(\omega_{\varphi_{x_i}}^{d-1})(e') > 0$, where $i \in I_e$ satisfies $e' \cap U_{x_i} \neq \emptyset$. This implies that each edge $e \in \Pi_{f,1}$ is in the support of $(\omega')^{d-1}$, as desired. $\square$

9. THE ORTHOGONALITY PROPERTY AND THE EXISTENCE OF SOLUTIONS

In this section, X is a compact prekähler tropical variety and ω is a strongly semipositive $(1,1)$ -form on X . The main result of this section is Theorem 9.2 which relates the validity of the tropical Calabi–Yau theorem and the following orthogonality property.

Definition 9.1 (Orthogonality property). Suppose that $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ is an lsc function which is not constantly $+\infty$ in X_o . We say that f verifies the orthogonality property if we have

$$\int_X (f - P_\omega(f)) \text{MA}_\omega(P_\omega(f)) = 0. \quad \diamond$$

Theorem 9.2. Let (X, ω) be a compact prekähler tropical variety equipped with a strongly semipositive $(1,1)$ -form. The followings are equivalent:

- (1) For each finite collection of pairs $\mathcal{S} = \{(x_j, \lambda_j) \in X_o \times \mathbb{R}\}_{1 \leq j \leq k}$, we have

$$\text{Supp}(\text{MA}_\omega(P_\omega(f_{\mathcal{S}}))) \subset S.$$

Here, $S := \{x_j\}_{1 \leq j \leq k} \subset X_o$, and

$$f_{\mathcal{S}}(x) := \begin{cases} \lambda_i & \text{if } x = x_i \text{ for some } x_i \in S, \\ +\infty & \text{if } x \in X \setminus S. \end{cases}$$

is the lsc function on X as defined in Example 4.19.

- (2) Each lsc function $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ which is not constantly $+\infty$ on X_o verifies the orthogonality property.
 (3) Each function $f \in C^0(X)$ verifies the orthogonality property.
 (4) (X, ω) verifies the tropical Calabi–Yau theorem.

The rest of this section is devoted to the proof of this theorem.

9.1. Orthogonality implies derivability. We start by proving that the orthogonality property implies derivability of the energy functional.

Theorem 9.3. Let $f \in \text{CPSH}_\omega(X)$ and $g \in C^0(X)$. Let $\epsilon > 0$. Suppose that for all $t \in (-\epsilon, \epsilon)$, $f + tg$ verifies the orthogonality property. Then,

$$(9.1) \quad \left. \frac{d}{dt} \right|_{t=0} E_\omega(P_\omega(f + tg)) = \int_X g \text{MA}_\omega(f).$$

Proof. Let $t \in (-\epsilon, \epsilon)$. We apply Corollary 7.2 to $P_\omega(f)$ and $P_\omega(f + tg)$:

$$\begin{aligned} \int_X (P_\omega(f + tg) - P_\omega(f)) \text{MA}_\omega(P_\omega(f + tg)) &\leq E_\omega(P_\omega(f + tg)) - E_\omega(P_\omega(f)) \\ &\leq \int_X (P_\omega(f + tg) - P_\omega(f)) \text{MA}_\omega(P_\omega(f)). \end{aligned}$$

Using the orthogonality property for $f + tg$ and the inequality $P_\omega(f) \leq f$, we obtain

$$t \int_X g \text{MA}_\omega(P_\omega(f + tg)) \leq \int_X (P_\omega(f + tg) - P_\omega(f)) \text{MA}_\omega(P_\omega(f + tg)).$$

Similarly, we have

$$\int_X (P_\omega(f + tg) - P_\omega(f)) \text{MA}_\omega(P_\omega(f)) \leq t \int_X g \text{MA}_\omega(P_\omega(f)).$$

On the other hand, the Bedford–Taylor theory (Theorem 6.2) implies that

$$\lim_{t \rightarrow 0} \int_X g \, \text{MA}_\omega(P_\omega(f + tg)) = \int_X g \, \text{MA}_\omega(P_\omega(f)).$$

Combining the above results, Equation (9.1) follows. $\square$

9.2. Proof of Theorem 9.2. (1) $\implies$ (2). Let $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ be an lsc function which is not constantly $+\infty$ on X_o . Since X is a second countable space, one can take a countable dense subset A_c of $\{x \in X_o \mid f(x) \leq c\}$ for each $c \in \mathbb{Q}$. Set

$$A = \bigcup_{c \in \mathbb{Q}} A_c.$$

Take a nested sequence of finite subsets $S_1 \subset S_2 \subset S_3 \subset \dots$ of A such that $A = \bigcup_0^\infty S_i$. For each i , take the finite collection of pairs $\mathcal{S}_i = \{(x, f(x))\}_{x \in S_i}$ and set $f_i := f_{\mathcal{S}_i}$. Note that $\{f_i\}_{i \geq 1}$ is a decreasing sequence of functions converging pointwise to f . By Remark 4.18, $\{P_\omega(f_i)\}_{i \geq 1}$ is a decreasing sequence of ω -psh functions. Since X is compact, there exists $C \in \mathbb{R}$ such that $f \geq C$ on X . This implies that for each $i \geq 1$, we have $f_i \geq C$, and hence $P_\omega(f_i) \geq C$. Thus, by Corollary 4.11, $P_\omega(f_i)$ converges uniformly to some function $\varphi \in \text{CPSH}_\omega(X)$.

We claim that $\varphi = P_\omega(f)$. In fact, for each i , we have $P_\omega(f_i) \geq P_\omega(f)$, hence $\varphi \geq P_\omega(f)$. On the other hand, $\varphi \leq f$ on S_i for each $i \geq 1$. This implies that $\varphi \leq f$ on A . So $\varphi \leq c$ on $A_c \subset \{f \leq c\}$ for any $c \in \mathbb{Q}$. Since A_c is dense in $\{f \leq c\}$ and $\varphi \in C^0(X)$, it follows that $\varphi \leq c$ on $\{f \leq c\}$. Therefore, $\varphi \leq f$ on X and hence $\varphi \leq P_\omega(f)$, proving the claim.

Let $i \geq 1$. Define the finite subset $S'_i := \{y \in S_i \mid P_\omega(f_i)(y) = f(y)\} \subset X$, and the finite collection of pairs $\mathcal{S}'_i := \{(y, f(y)) \mid y \in S'_i\}$. By assumption, we have $P_\omega(f_{\mathcal{S}'_i}) = P_\omega(f_i)$, and $\text{Supp}(\text{MA}_\omega(P_\omega(f_{\mathcal{S}'_i}))) \subset S'_i$. Therefore,

$$\int_X (f - P_\omega(f_i)) \, \text{MA}_\omega(P_\omega(f_i)) = \int_{S'_i} (f - P_\omega(f_{\mathcal{S}'_i})) \, \text{MA}_\omega(P_\omega(f_{\mathcal{S}'_i})) = 0.$$

Then, we have

$$\begin{aligned} 0 &\leq \int_X (f - P_\omega(f)) \, \text{MA}_\omega(P_\omega(f)) \\ &= \lim_{i \rightarrow +\infty} \left(\int_X (f - P_\omega(f)) \, \text{MA}_\omega(P_\omega(f)) - \int_X (f - P_\omega(f_i)) \, \text{MA}_\omega(P_\omega(f_i)) \right) \\ &= \lim_{i \rightarrow +\infty} (R_{1,i} + R_{2,i}), \end{aligned}$$

with

$$\begin{aligned} R_{1,i} &:= \int_X (P_\omega(f_i) - P_\omega(f)) \, \text{MA}_\omega(P_\omega(f_i)), \text{ and} \\ R_{2,i} &:= \int_X (f - P_\omega(f)) (\text{MA}_\omega(P_\omega(f)) - \text{MA}_\omega(P_\omega(f_i))). \end{aligned}$$

On the one hand,

$$\lim_{i \rightarrow +\infty} R_{1,i} \leq \deg(\omega^d) \cdot \lim_{i \rightarrow +\infty} \|P_\omega(f) - P_\omega(f_i)\|_{C^0(X)} = 0$$

On the other hand, Theorem 6.2 implies that $\text{MA}_\omega(P_\omega(f_i)) \rightarrow \text{MA}_\omega(P_\omega(f))$. Since $(f - P_\omega(f))$ is a nonnegative lsc function, Portmanteau theorem [Bog18, Cor. 2.2.6] implies that

$$\lim_{i \rightarrow +\infty} R_{2,i} \leq \liminf_{i \rightarrow +\infty} \left(\int_X (f - P_\omega(f)) \text{MA}_\omega(P_\omega(f)) - \int_X (f - P_\omega(f)) \text{MA}_\omega(P_\omega(f_i)) \right) \leq 0.$$

Combining the above statements, we conclude that f verifies the orthogonality property.

(2) $\implies$ (3). This is obvious.

(3) $\implies$ (4). Let μ be a nonnegative Radon measure with compact support in X_o of total mass $\deg(\omega^d)$. Let $\varphi \in \text{CPSH}_\omega(X)$ be a maximizer of F_μ on $\text{PSH}_\omega(X)$ as in Proposition 7.4. We claim that the equality $\text{MA}_\omega(\varphi) = \mu$ holds.

Let g be a continuous function on X . Consider the function

$$G(t) := E_\omega(P_\omega(\varphi + tg)) - \int_X (\varphi + tg) d\mu.$$

Observe that

$$G(t) = F_\mu(P_\omega(\varphi + tg)) + \int_X (P_\omega(\varphi + tg) - (\varphi + tg)) d\mu.$$

For any $t \neq 0$, we have $G(t) \leq G(0)$, since $\int_X (P_\omega(\varphi + tg) - (\varphi + tg)) d\mu \leq 0$ and $F_\mu(P_\omega(\varphi + tg)) \leq F_\mu(\varphi)$.

Applying Theorem 9.3, we get

$$\left. \frac{d}{dt} \right|_{t=0} G(t) = \int_X g \text{MA}_\omega(\varphi) - \int_X g d\mu = 0.$$

Since this holds for all continuous functions g , we get $\text{MA}_\omega(\varphi) = \mu$.

(4) $\implies$ (1). We proceed by contradiction. Suppose that there exists a finite collection of pairs $\mathcal{S} = \{(x_j, \lambda_j) \in X_o \times \mathbb{R}\}_{1 \leq j \leq k}$ such that $\text{Supp}(\text{MA}_\omega(P_\omega(f_{\mathcal{S}}))) \not\subseteq S$ where $S = \{x_j\}_{1 \leq j \leq k}$. Consequently, $\text{MA}_\omega(P_\omega(f_{\mathcal{S}}))(S) < \deg(\omega^d)$. So we may choose a measure μ on X_o such that $\text{Supp}(\mu) = S$, $\text{Vol}(\mu) = \deg(\omega^d)$, and $\mu(\{x_j\}) > P_\omega(f_{\mathcal{S}})(\{x_j\})$ for each $j \in S$.

By assumption, (X, ω) satisfies the tropical Calabi–Yau theorem. Therefore, there exists $\varphi \in \text{PSH}_\omega(X)$ such that $\text{MA}_\omega(\varphi) = \mu$. Adding a constant to φ , we may assume that $\max_{x \in S} \{\varphi(x) - P_\omega(f_{\mathcal{S}})(x)\} = 0$.

Let $x \in S$ such that $\varphi(x) = P_\omega(f_{\mathcal{S}})(x)$. Choose a neighborhood U of x such that $\omega = \partial\bar{\partial}h$ for some $h \in \text{LConv}(U)$. Hence, $h + \varphi, h + P_\omega(f_{\mathcal{S}}) \in \text{LConv}(U)$. By the definition of the envelope, we have $\varphi \leq P_\omega(f_{\mathcal{S}})$ on X , which implies that $h + \varphi \leq h + P_\omega(f_{\mathcal{S}})$ on U . At the same time, $(h + \varphi)(x) = (h + P_\omega(f_{\mathcal{S}}))(x)$, and $\text{MA}((h + \varphi))(\{x\}) > \text{MA}((h + P_\omega(f_{\mathcal{S}})))(\{x\})$ by constructions. But these contradict the following lemma.

Lemma 9.4. *Let U be an open subset in X_o , and let $x \in U$. Suppose that $u, v \in \text{LConv}(U)$ such that $u(x) = v(x)$ and $u \leq v$ on U . Then we have $\text{MA}(u)(\{x\}) \leq \text{MA}(v)(\{x\})$.*

Proof. This is a local statement, so we may assume that $(X_o, x) = (|\Sigma|, 0)$, where Σ is a tropical fan embedded in a linear space $E = N_{\mathbb{R}}$, and 0 is the origin in E . After adding a local affine support function of u at 0, we assume further that $u(0) = v(0) = 0$, and $0 \leq u \leq v$.

By Corollary 5.9, possibly refining Σ if necessary, we can suppose that the canonical compactification $\bar{\Sigma}$ is prekähler. Furthermore, there exists $h \in \text{LConv} \cap \text{PL}_{\text{hom}}(\Sigma)$ such that $h > 0$ on $|\Sigma| \setminus \{0\}$, and $\omega := \partial\bar{\partial}h$ is a semipositive simplicial form on $\bar{\Sigma}$.

For $\epsilon > 0$, we consider $v_\epsilon = v + \epsilon h \in \text{LConv}(U)$. By local Bedford–Taylor theory [APW26, Prop. 5.6], $\text{MA}(v_\epsilon) \rightarrow \text{MA}(v)$ when $\epsilon \rightarrow 0$. Moreover, $\text{MA}(v_\epsilon) \geq \text{MA}(v) \geq 0$ for any $\epsilon > 0$, and $\text{MA}(v_\epsilon)(U)$ are uniformly bounded. Therefore, $\text{MA}(v_\epsilon)\{0\} \rightarrow \text{MA}(v)\{0\}$. It is thus enough to prove the lemma for v_ϵ for any $\epsilon > 0$.

From now on, we fix $\epsilon > 0$, and a constant $C \gg 1$. We choose an open neighborhood V of 0 such that $V \subseteq U$. By Proposition 3.9, u and v_ϵ are Lipschitz on V . Therefore, there exists $A \gg 0$ such that $V_1 := \{(Ah - C) < u\} \subseteq V$, and $V_2 := \{Ah - (C - 1) < v_\epsilon\} \subseteq V$. We define functions $\varphi, \psi: \bar{\Sigma} \rightarrow \mathbb{R}$ by

$$\varphi(y) = \begin{cases} u - Ah & \text{if } y \in V_1 \\ -C & \text{if } y \in \bar{\Sigma} \setminus V_1 \end{cases}, \quad \text{and} \quad \psi(y) = \begin{cases} v_\epsilon - Ah & \text{if } y \in V_2 \\ -(C - 1) & \text{if } y \in \bar{\Sigma} \setminus V_2 \end{cases}.$$

By Lemma 4.13, we have $\varphi, \psi \in \text{CPSH}_{A\omega}(\bar{\Sigma})$.

For $\delta > 0$, we define $W_\delta = \{\epsilon h - \delta < 0\} \subset \Sigma$. Since h is homogeneous, and $h > 0$ on $\Sigma \setminus \{0\}$, we have $W_\delta = \delta \cdot W_1$, and W_δ is a bounded subset of E . Therefore, when $\delta \ll 1$, $W_\delta \subset V_1 \cap V_2$. We also define the subset $U_\delta := \{\psi - \delta < \varphi\} \subset \bar{\Sigma}$.

Claim 9.5. *When $0 < \delta \ll 1$, we have $U_\delta \subset W_\delta$.*

Proof of the claim. Take $y \in U_\delta$. By definition, $\psi(y) - \varphi(y) < \delta$. We need to verify that $\epsilon h(y) < \delta$, which proves $y \in W_\delta$.

If $y \in V_1 \cap V_2$, $\psi(y) - \varphi(y) = v(y) + \epsilon h(y) - u(y) < \delta$. By assumption $v(y) - u(y) \geq 0$. Thus, $\epsilon h(y) < \delta$, as required.

Proceeding by a case by case analysis, we show that $y \in V_1 \cap V_2$ is the only possible case.

- (1) If $y \in \bar{\Sigma} \setminus (V_1 \cup V_2)$, then $\psi(y) - \varphi(y) = 1 > \delta$, which is impossible.
- (2) If $y \in V_1 \setminus V_2$, then by assumption,

$$v_\epsilon(y) \leq Ah(y) - C + 1, \quad \text{and}$$

$$\psi(y) - \varphi(y) = Ah(y) - C + 1 - u(y) < \delta.$$

This implies that $v(y) + \epsilon h(y) - u(y) \leq Ah(y) - C + 1 - u(y) < \delta$. Since $v(y) - u(y) \geq 0$, it follows that $\epsilon h(y) < \delta$. Therefore, $y \in W_\delta \subset V_1 \cap V_2$, which is again impossible by our assumption on y .

- (3) In the remaining case, $y \in V_2 \setminus V_1$. Then by assumption,

$$Ah(y) - C + 1 \leq v_\epsilon(y), \quad \text{and}$$

$$\psi(y) - \varphi(y) = v_\epsilon(y) - Ah(y) + C.$$

This implies that $\psi(y) - \varphi(y) \geq 1$, which is impossible.

Therefore, the claim is proved. $\square$

For any $0 < \delta \ll 1$, Claim 9.5 implies that $U_\delta \subset V_1 \cap V_2$. Therefore, by constructions, $\text{MA}_{A\omega}(\psi)(U_\delta) = \text{MA}(v_\epsilon)(U_\delta)$, and $\text{MA}_{A\omega}(\varphi)(U_\delta) = \text{MA}(u)(U_\delta)$. On the other hand, the comparison principle (Theorem 6.8) implies that $\text{MA}_{A\omega}(\psi)(U_\delta) \geq \text{MA}_{A\omega}(\varphi)(U_\delta)$. Therefore, $\text{MA}(v_\epsilon)(U_\delta) \geq \text{MA}(u)(U_\delta)$.

Claim 9.5 also implies that $\bigcap_{\delta>0} U_\delta \subset \bigcap_{\delta>0} W_\delta = \{0\}$. So when $\delta \rightarrow 0+$, we have $\text{MA}(v_\epsilon)(U_\delta) \rightarrow \text{MA}(v_\epsilon)(\{0\})$, and $\text{MA}(u)(U_\delta) \rightarrow \text{MA}(u)(\{0\})$. This finishes the proof of the lemma. $\square$

At this point, the proof of Theorem 9.2 is complete.

10. ORTHOGONALITY PROPERTY IN CODIMENSION ONE

Let X be a compact prekähler tropical variety of dimension d equipped with a strongly semipositive $(1,1)$ -form ω .

Recall that a point $x \in X$ is regular if it admits a tropical chart $\phi: W \rightarrow U \subset \mathbb{R}^d$, $x \in W$. In particular, this implies that $x \in X_o$. We introduce the following definition.

Definition 10.1. We say that $x \in X$ is *regular in codimension one* if there exists a tropical chart $U \rightarrow W \subset \mathcal{L} \times \mathbb{R}^{d-1}$, where $\mathcal{L}$ is a generalized tropical line (see Section 2.1) and W is an open subset in $\mathcal{L} \times \mathbb{R}^{d-1}$. Denote by $\mathcal{R}$ the open set of points of X_o that are regular in codimension one.

We remark that being regular in codimension one is an open property. We also remark that $x \in X$ is regular in codimension one implies automatically that $x \in X_o$.

In this section, we prove the following theorem.

Theorem 10.2. *Let X be a compact prekähler tropical variety of dimension d equipped with a strongly semipositive form ω . Then, the orthogonality property holds on $\mathcal{R}$ in the following sense: for any lsc function $f: X \rightarrow \mathbb{R} \cup \{+\infty\}$ which is not constantly $+\infty$ on X_o , we have*

$$(10.1) \quad \int_{\mathcal{R}} (f - P_{\omega}(f)) \text{MA}_{\omega}(P_{\omega}(f)) = 0.$$

We have the following consequences.

Corollary 10.3. *Let μ be a nonnegative Radon measure which is compactly supported in X_o and has total volume $\deg(\omega^d)$. Let $\varphi \in \text{CPSH}_{\omega}(X)$ be a maximizer of F_{μ} . Then $\text{MA}_{\omega}(\varphi) \leq \mu$ on $\mathcal{R}$, the open subset consisting of points regular in codimension one.*

The proof is given in Section 10.2, along with the proof of the following.

Theorem 10.4. *Let X be a compact prekähler variety of dimension d equipped with a strongly semipositive form ω . Assume that $\mathcal{R} = X_o$, i.e. every point $x \in X_o$ is regular in codimension one. Then, the tropical Calabi–Yau theorem holds for (X, ω) .*

Remark 10.5. In particular, Theorem 10.4 applies to a tropical variety X which is a product of the form $\Gamma \times Z$ where Z is a compact integral affine manifold and Γ is a compact metric graph. Here, the tropical structure on Γ is locally matroidal [ACM25, § 5]. $\diamond$

10.1. A local result. Let $E_1 = \mathbb{R}^{n_1}$ and $E_2 = \mathbb{R}^{n_2}$, and set $E = E_1 \times E_2 = \mathbb{R}^n$, $n = n_1 + n_2$. Denote by π_1, π_2 the projection maps $E \rightarrow E_1$ and $E \rightarrow E_2$ respectively. Suppose either that $\mathcal{L} \subset E_1$ is a generalized tropical line or $\mathcal{L} = E_1 = \{0\}$. In the first case, without loss of generality, assume further that $\mathcal{L}$ generates E_1 linearly. Consider the fanfold $\mathcal{L} \times E_2$ in E .

We fix a basis $\{e_1, \dots, e_{n_1}\}$ of E_1 . Without loss of generality, we can assume that $\rho := \mathbb{R}_{\geq 0}e, \rho_1 := \mathbb{R}_{\geq 0}[e_1], \dots, \rho_{n_1} := \mathbb{R}_{\geq 0}[e_{n_1}]$ are all the $(n_1 + 1)$ rays of $\mathcal{L}$, where $e := -(e_1 + \dots + e_{n_1})$. The fanfold $\mathcal{L}$ is equipped with a canonical fan structure Σ_1 . We take $g_1 := \max\{0, -x_1, -x_2, \dots, -x_{n_1}\}$ and $h_1 := g_1|_{\Sigma_1}$.

Let Σ_2 be the standard complete fan on E_2 of the projective space. Take

$$h_2 := \max\{0, -x_1, -x_2, \dots, -x_{n_2}\},$$

which is a strictly convex conewise linear function on Σ_2 .

Therefore, $\Sigma := \Sigma_1 \times \Sigma_2$ is a fan supported on $\mathcal{L} \times E_2$. Define $h := h_1 \circ \pi_1 + h_2 \circ \pi_2 \in \text{LConv} \cap \text{PL}_{\text{hom}}(\Sigma)$. Then, $\omega := \partial\bar{\partial}h$ is a semipositive simplicial form on the canonical compactification $\bar{\Sigma}$. Proposition 5.7 implies that $[\omega]$ is a Kähler class on $\bar{\Sigma}$.

Theorem 10.6. *Notation as above, $(\bar{\Sigma}, \omega)$ verifies the equivalent properties (1)-(4) stated in Theorem 9.2.*

The remaining of this subsection is devoted to prove Theorem 10.6. We will show that $(\bar{\Sigma}, \omega)$ satisfies Property (1) in Theorem 9.2.

We start by defining a map

$$\text{DL}: \text{LConv}(\Sigma) \rightarrow \text{Conv}(E),$$

as follows. For each $u \in \text{LConv}(\Sigma)$, let $u': E \rightarrow \mathbb{R} \cup \{+\infty\}$ be the extension of u obtained by taking value $+\infty$ on $E \setminus |\Sigma|$. Let P denote the envelope operator given in Lemma 3.5, and set $\text{DL}(u) := P(u')$, that is,

$$\text{DL}(u)(x) := \inf \left\{ \sum_{i \in I} \nu_i u'(x_i) \mid (x, (x_i)_{i \in I}, (\nu_i)_{i \in I}) \text{ is a convex combination in } E \right\}, \forall x \in E.$$

Since the linear span of Σ equals to E , we have $\text{DL}(u)(x) < +\infty$ for any $x \in E$.

Now we prove two auxiliary lemmas.

Lemma 10.7. *For each $u \in \text{LConv}(\Sigma)$, we have $\text{DL}(u)|_{\Sigma} = u$. In particular, the map $\text{DL}: \text{LConv}(\Sigma) \rightarrow \text{Conv}(E)$ is an injection. Moreover, for each $v \in \text{Conv}(E)$, we have $\text{DL}(v|_{\Sigma}) \geq v$ and $\text{DL}(v|_{\Sigma})|_{\Sigma} = v|_{\Sigma}$.*

Proof. When $\mathcal{L} = \emptyset$, we have $\Sigma = \Xi$. Therefore, there is nothing to prove. In the remaining, we assume that $\mathcal{L}$ is a generalized tropical line.

Let $u: \Sigma \rightarrow \mathbb{R}$ be an element of $\text{LConv}(\Sigma)$, and set $\tilde{u} = \text{DL}(u)$. We need to show that $\tilde{u}|_{\Sigma} = u$, that is, to prove that for any convex combination $(x, (x_i)_{i \in I}, (\nu_i)_{i \in I})$ in E , we have

$$(10.2) \quad u(x) \leq \sum_{i \in I} \nu_i u(x_i).$$

Identify E_2 with $\{0\} \times E_2 \hookrightarrow E$. Two cases occur depending on whether $x \in E_2 \cap U$ or not.

(1) If $x \in E_2$, after applying an affine translation, we may assume that $x = 0 \in E$. Then, $(x, (x_i)_{i \in I}, (\nu_i)_{i \in I})$ is a polyhedral convex combination on Σ . Since $u \in \text{LConv}(U)$, by Proposition 3.6 the inequality (10.2) holds.

(2) If $x \notin E_2$, there exists a ray $\rho \in \mathcal{L}$ such that $x \in \text{relint}(\rho) \times E_2$. Let H_+ denote $\text{relint}(\rho) \times E_2$, and W denote the linear subspace of E generated by $\rho \times E_2$.

Partition the set $\{x_i\}_{i \in I}$ into $\{x_j\}_{j \in I_+} \subset H_+$ and $\{x_j\}_{j \in J} \subset \Sigma \setminus H_+$, $I = I_+ \sqcup J$ according to whether the points lie in H_+ or not. Since x lies in H_+ and it is a convex combination of x_i , the set I_+ is nonempty.

Note that $u|_{H_+} \in \text{Conv}(H_+)$. Therefore, if $J = \emptyset$, then the inequality in (10.2) follows automatically. Consequently, in the rest of the proof, we assume that J is nonempty. Set

$$\mu' := \sum_{j \in I_+} \nu_j, \quad \text{and} \quad \mu'' := \sum_{j \in J} \nu_j.$$

We do a normalization for the coefficients by

$$\nu'_j := \nu_j / \mu', \quad j \in I_+ \quad \text{and} \quad \nu''_j := \nu_j / \mu'', \quad j \in J.$$

Consider the points $x' \in H_+$ and $x'' \in E$ defined by

$$x' := \sum_{j \in I_+} \nu'_j x_j \quad \text{and} \quad x'' := \sum_{j \in J} \nu''_j x_j.$$

Then, we have the convex combination

$$x = \mu' x' + \mu'' x''.$$

Claim 10.8. *We have $x'' \in W \setminus H_+$.*

Proof of the claim. The fact that $x, x' \in W$ implies that $x'' \in W$. It remains to show that $x'' \notin H_+$.

Without loss of generality, we may assume that ρ is generated by e . For any $j \in J$, the projection of x_j onto E_1 has nonnegative coefficients in the basis $\{e_1, \dots, e_{n_1}\}$. Since x'' is a convex combination of the x_j , its E_1 component also has nonnegative coefficients. In contrast, the E_1 component of every element of H_+ has strictly negative coefficients under the same basis. Hence $x'' \notin H_+$. $\square$

Back to the proof of the lemma, note that E_2 is a hyperplane of W , and H_+ is one of the two half-spaces bounded by E_2 . Thus, Claim 10.8 implies that there exists $0 < t < \mu'$ such that $y := tx' + (1-t)x'' \in E_2$. After applying an affine translation, we may assume that $y = 0 \in E_2$. Therefore, $\left(y, (\{x_j\}_{j \in I_+}, \{x_j\}_{j \in J}), (\{t\nu'_j\}_{j \in I_+}, \{(1-t)\nu''_j\}_{j \in J})\right)$ is a polyhedral convex combination on Σ . Hence, Proposition 3.6 implies that

$$(10.3) \quad u(y) \leq \sum_{j \in I_+} t\nu'_j u(x_j) + \sum_{j' \in J} (1-t)\nu''_{j'} u(x_{j'}).$$

At the same time, we have the relation $x = \frac{\mu'(1-t)-t\mu''}{1-t}x' + \frac{\mu''}{1-t}y$. Note that $\frac{\mu'(1-t)-t\mu''}{1-t} + \frac{\mu''}{1-t} = 1$ since $\mu' + \mu'' = 1$. Moreover, $0 < \frac{\mu''}{1-t} < 1$ since $0 < \mu'' < 1$ and $t < \mu'$. Therefore, x is a convex combination of x' and y on $\overline{H_+}$. Since $u|_{\overline{H_+}}$ is convex, it follows that

$$(10.4) \quad u(x) \leq \frac{\mu'(1-t)-t\mu''}{1-t}u(x') + \frac{\mu''}{1-t}u(y).$$

Also, note that

$$(10.5) \quad u(x') \leq \sum_{j \in I_+} \nu'_j u(x_j).$$

Combining Equations 10.3, 10.4, and 10.5, we get Equation 10.2.

Therefore, the proof is complete. $\square$

Lemma 10.9. *Let Ξ be a complete simplicial fan on $E = \mathbb{R}^n$. Let h be a convex conewise linear function on Ξ . Suppose that $\mathcal{S} = \{(x_i, \lambda_i) \in E \times \mathbb{R}\}_{1 \leq i \leq k}$ is a finite collection of pairs in the sense of Example 4.19. We consider the function*

$$u_{\mathcal{S}} := \sup \{ u \in \text{Conv}(E) \mid |u - h| = O(1), u \leq f_{\mathcal{S}} \}.$$

Then $u_{\mathcal{S}} \in \text{PL}(E)$. Furthermore, there exists a complete simplicial polyhedral complex Π on E satisfying the following properties:

- (1) $\Pi_0 \subset \{x_i\}_{1 \leq i \leq k}$.
- (2) The recession fan of Π is Ξ , that is, for each $\sigma \in \Pi$, $\text{rec}(\sigma) \in \Xi$.
- (3) $u_{\mathcal{S}}$ is facewise linear on Π .

Proof. The first assertion that $u_S \in \text{PL}(E)$ is shown in [BFJ15, Prop.9.2]. For any affine function l on E , we denote $\sigma_l := \{x \in E \mid u(x) = l(x)\}$. Since u_S is PL, there exists a finite subset $\{l_i\}_{i \in I} \subset \text{Aff}(E)$ such that $E = \bigcup_{i \in I} \sigma_{l_i}$, and $\dim \sigma_{l_i} = n$ for any $i \in I$. Since u_S is convex, σ_{l_i} is a polyhedron for any $i \in I$. Therefore, $\{\sigma_{l_i}\}_{i \in I}$ generates a polyhedral structure Π' with support E . Then, u_S is facewise linear on Π' . Moreover, $|u_S - h| = O(1)$ implies Property (2) holds for Π' .

We now show that $\Pi'_0 \subset \{x_i\}_{1 \leq i \leq k}$. Let $x \in E \setminus \{x_i\}_{1 \leq i \leq k}$. Then, the domination principle (see Lemma 8.5 in [BFJ15]) implies that $\text{MA}_{\mathbb{R}}(u_S)(\{x\}) = 0$. Let $u \in \text{Conv} \cap \text{PL}_{\text{hom}}(E)$ such that there exist an affine transform $A: E \rightarrow E$ and an open neighborhood U of x satisfying $A(x) = 0$ and $u_S = u \circ A$ on U . Therefore, $\text{MA}_{\mathbb{R}}(u)(\{x\}) = \mu_{\text{leb}}(\text{dom}(u^*)) = 0$. Here, $u^* \in \text{Conv}(E^*)$ is the Legendre transform of u , and $\text{dom}(u^*) := \{u^* < +\infty\} \subset E^*$. This implies that $\text{dom}(u^*)$ is a polytope of dimension strictly less than n . Let $x' \in E$ such that $\langle x', \text{dom}(u^*) \rangle = 0$. Then we have $u = v \circ \pi_{x'}$ for $v \in \text{Conv} \cap \text{PL}_{\text{hom}}(\pi_{x'}(E))$, where $\pi_{x'}: E \rightarrow E/\mathbb{R}x'$ is the quotient map, and $\mathbb{R}x'$ is linear space of x' . Therefore, for any σ_{l_i} containing x , x is not an extremal point of σ_{l_i} . This implies that $x \notin \Pi'_0$, and hence $\Pi'_0 \subset \{x_i\}_{1 \leq i \leq k}$.

We take a refinement Π of Π' keeping $\Pi_0 = \Pi'_0$, breaking only the compact faces into simplices, to obtain a simplicial polyhedral structure Π . Then Π satisfies (1). Moreover, since we did not touch the recession cones, the recession fan of Π is Ξ and (2) is satisfied. Being a refinement of Π' , and u_S being facewise linear on Π' , Property (3) is also verified by Π . $\square$

We are now ready to prove Theorem 10.6.

Proof of Theorem 10.6. Let $\mathcal{S} = \{(x_i, \lambda_i) \in |\Sigma| \times \mathbb{R}\}_{1 \leq i \leq k}$, be a finite collection of pairs and set $\varphi = P_{\omega}(f_{\mathcal{S}}) = \sup \{\psi \in \text{CPSH}_{\omega} \mid \psi \leq f_{\mathcal{S}}\}$. Let S denote the finite subset $\{x_i\}_{1 \leq i \leq k} \subset |\Sigma|$. We will prove that $\text{Supp}(\text{MA}_{\omega}(\varphi)) \subset S$. By Theorem 4.8, the map

$$\text{CPSH}_{\omega}(\bar{\Sigma}) \longrightarrow \text{LConv}_h(\Sigma) \quad \psi \longmapsto \psi + h$$

is a bijection, where $\text{LConv}_h(\Sigma) := \{u \in \text{LConv}(\Sigma) \mid u - h = O(1)\}$. Hence, denoting $\mathcal{S}' = \{(x_i, \lambda_i + h(x_i)) \in |\Sigma| \times \mathbb{R}\}_{1 \leq i \leq k}$, and

$$u_0 := \sup \{u \in \text{LConv}_h(\Sigma) \mid u \leq f_{\mathcal{S}'}\},$$

we have $\varphi + h = u_0$.

Consider $\text{DL}(h) \in \text{Conv}(E)$. It is not hard to verify that $\text{DL}(h) = g$, where $g := g_1 \circ \pi_1 + h_2 \circ \pi_2 \in \text{Conv} \cap \text{PL}_{\text{hom}}(E)$. We define $\text{Conv}_g(E) := \{v \in \text{Conv}(E) \mid v - \text{DL}(h) = O(1)\}$, and

$$v_0 := \sup \{v \in \text{Conv}_g(E) \mid v \leq f_{\mathcal{S}'}\}.$$

We claim that $v_0|_{\Sigma} = u_0$ on Σ . It is clear that $v_0|_{\Sigma} \leq f_{\mathcal{S}'}$. Since $\text{DL}(h)|_{\Sigma} = g|_{\Sigma} = h$, we have $v_0|_{\Sigma} \in \text{LConv}_h(\Sigma)$. Therefore, by definition $v_0|_{\Sigma} \leq u_0$. On the other hand, by definition it is not hard to check that $\text{DL}(u_0) - \text{DL}(h) = O(1)$. Thus, $\text{DL}(u_0) \in \text{Conv}_g(E)$. By Lemma 10.7, $\text{DL}(u_0)|_{\Sigma} = u_0$. Hence, $\text{DL}(u_0) \leq f_{\mathcal{S}'}$, which implies that $\text{DL}(u_0) \leq v_0$ on E . We conclude that $u_0 = \text{DL}(u_0)|_{\Sigma} \leq v_0$ on Σ . Therefore, the claim is proved.

In the remaining part of the proof, we show that $\text{Supp}(\text{MA}(u_0)) \subset S$, where MA denotes the local tropical Monge–Ampère operator on Σ . Let Ξ_1 be the complete fan on E_1 defining the tropical projective space, and set $\Xi := \Xi_1 \times \Sigma_2$. Notice that g is a convex conewise linear function on Ξ . We apply Lemma 10.9 to g and $\mathcal{S}'$, and let Π be the complete simplicial polyhedral structure on E satisfying the properties stated in the lemma. Due to Property (1)

in the lemma, it is enough to prove that $\text{MA}(u_0)(E \setminus \Pi_0) = 0$. Let $\sigma \in \Pi$ such that $\dim \sigma \geq 1$, and let $x \in \text{int}(\sigma) \cap \Sigma$. Let E_σ be the linear subspace of E parallel to the affine span of σ , and let $\pi_\sigma : E \rightarrow E/E_\sigma$ denote the quotient map.

Locally around x , we have $u_0 = v \circ \pi_\sigma$ for some $v \in \text{PL}(\pi_\sigma(\Sigma))$. By assumption, $\dim \Sigma = n_2 + 1$. By [APW26, Prop. 5.6], we have $(\pi_\sigma)_*((\partial\bar{\partial}u_0)^{\wedge(n_2+1)}) = (\partial\bar{\partial}v)^{\wedge(n_2+1)}$ around $\pi_\sigma(x)$. Therefore, it suffices to show that $\dim \pi_\sigma(\Sigma) \leq n_2$ around x , as this will imply then $(\partial\bar{\partial}v)^{\wedge(n_2+1)} = 0$.

In the case $\dim \sigma \geq n_1$, it is clear that $\dim \pi_\sigma(\Sigma) \leq \dim E/E_\sigma \leq n_2$. On the other hand, we have the following claim.

Claim 10.10. *Suppose that $1 \leq \dim \sigma \leq n_1 - 1$. Then, either $\sigma \subset \Sigma$, or $\text{relint}(\sigma) \cap |\Sigma| = \emptyset$.*

Granted this claim, we explain how to finish the proof of the theorem. Indeed, when $\sigma \subset \Sigma$, we have $\dim \pi_\sigma(\Sigma) = n_2 + 1 - \dim \sigma \leq n_2$ around x , which concludes the proof. While in the other case, the condition $\text{relint}(\sigma) \cap |\Sigma| = \emptyset$ contradicts the assumption that $x \in \text{relint}(\sigma) \cap \Sigma$.

It remains to prove Claim 10.10. Observe that for any $\tau \in \Xi$, either $\tau \in \Sigma$ or $\text{relint}(\tau) \cap \Sigma = \emptyset$. Therefore, due to Property (2) in Lemma 10.9, it is enough to prove the claim when σ is bounded. Since Π is simplicial, σ is the convex hull of $\{z_j\}_{j \in J} \subset \Sigma$ with $|J| \leq n_1$. Therefore, without loss of generality, we may assume that $\pi_1(\{z_j\}_{j \in J}) \subset \bigcup_{1 \leq i \leq n_1} \rho_i$, where $\pi_1 : E \rightarrow E_1$ is the projection map, and ρ_i are the rays of $\mathcal{L}$ defined in the beginning of this subsection. Let τ be the cone generated by $\{\rho_i\}_{1 \leq i \leq n_1}$. We observe that $\pi_1(\sigma) \subset \tau$, $\tau \cap C = \bigcup_{1 \leq i \leq n_1} \rho_i$, and $\{\rho_i\}_{1 \leq i \leq n_1}$ are extremal rays of τ .

Two cases occur depending on whether all the z_j s belong to the same ray in $\mathcal{L}$ or not. In the former case, we have $\pi_1(\sigma) \subset \mathcal{L}$. While in the latter case, $\pi_1(\text{relint}(\sigma)) \subset \text{relint}(\pi_1(\sigma)) \subset E_1 \setminus \mathcal{L}$. Hence, the claim is proved. $\square$

Now, we prove the following corollary of Theorem 10.6.

Corollary 10.11. *Let U be a bounded open subset of $|\Sigma|$. Suppose that $u \in \text{LConv}(U)$, and it satisfies $\text{MA}(u)(U) > 0$. Then, there exists $v \in \text{LConv}(U)$ such that $v \geq u$ on U , and $\{u < v\}$ is a nonempty open subset compactly contained in U .*

Although this is a purely local statement, we will prove it by a global argument on $(\bar{\Sigma}, \omega)$.

Proof of Corollary 10.11. We observe first that there exists $x \in U$ such that for any open neighbourhood U' of x , we have $\text{MA}_{\text{Trop}, \bar{\Sigma}}(u)(U') > 0$. Indeed, otherwise, for each $x \in U$, take an open neighborhood U_x of x in U such that $\text{MA}_{\text{Trop}, \bar{\Sigma}}(u)(U_x) = 0$. Then, $U = \bigcup_{x \in U} U_x$. Since U is second-countable, there exists a countable subset A of U such that $U = \bigcup_{x \in A} U_x$. Therefore, $\text{MA}_{\text{Trop}, \bar{\Sigma}}(U) \leq \sum_{x \in A} \text{MA}_{\text{Trop}, \bar{\Sigma}}(U_x) = 0$, which contradicts with the assumption.

Take an affine function l on E such that $h' := h + l$ satisfies $h'(y) > 0$ for any $y \in |\Sigma| \setminus \{0\}$. Then, $\omega = \partial\bar{\partial}h = \partial\bar{\partial}h'$. Applying Proposition 5.19 to $u \in \text{LConv}(U)$, there exist an open neighborhood $W \Subset U$ of x and $\varphi \in \text{CPSH}_{C\omega}(\bar{\Sigma})$, for some $C > 0$, such that $\varphi + Ch' = u$ on W and $\varphi + Ch' \geq u$ on $U \setminus W$.

Take an open neighborhood $V \Subset W$ of x , and define the lsc function $f : \bar{\Sigma} \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$f(y) = \begin{cases} +\infty & \text{if } y \in V \\ \varphi(y) & \text{if } y \in \bar{\Sigma} \setminus V \end{cases}.$$

Since $\text{P}_{C\omega}(f) = \sup\{\text{P}_{C\omega}(f), \varphi\}$, we have $\text{P}_{C\omega}(f) = \varphi$ on $\bar{\Sigma} \setminus V$. Also, $\text{P}_{C\omega}(f) + Ch' \geq u$ on W . On the other hand, the orthogonality property of $(\bar{\Sigma}, C\omega)$ (Theorem 10.6) yields that

$\text{MA}_{C\omega}(\text{P}_{C\omega}(f)) = \text{MA}_{\text{Trop}, \bar{\Sigma}}(\text{P}_{C\omega}(f) + Ch')(V) = 0$. This implies that $\text{P}_{C\omega}(f) + Ch' \neq u$ on V . Hence, $\{u < \text{P}_{C\omega}(f) + Ch'\}$ is nonempty.

Now, we define

$$f(y) = \begin{cases} +\infty & \text{if } y \in V \\ \varphi(y) & \text{if } y \in \bar{\Sigma} \setminus V \end{cases}.$$

We take $v = (\text{P}_{C\omega}(f) + Ch')|_{\bar{U}} \in \text{LConv}(U)$, we have

$$v(y) = \begin{cases} (\text{P}_{C\omega}(f) + Ch')(y) & \text{if } y \in W \\ u(y) & \text{if } y \in U \setminus W \end{cases}.$$

Then, Lemma 4.13 implies that $v \in \text{LConv}(U)$. By construction, $v \geq u$ on U , $\{v < u\}$ is nonempty, and $\{v < u\} \subset W \Subset U$. $\square$

10.2. Proof of Theorem 10.2 and its consequences. We start with the main theorem.

Proof of Theorem 10.2. Arguing similar to the proof of (1) $\implies$ (2) in Theorem 9.2, to prove Theorem 10.2, it suffices to show that, for any finite collection of pairs $\mathcal{S} = \{(x_i, \lambda_i) \in X_o \times \mathbb{R}\}_{1 \leq i \leq k}$, the corresponding function $f_{\mathcal{S}}$ satisfies the orthogonality property on $\mathcal{R}$.

We prove this by contradiction. Suppose that there exists $f_{\mathcal{S}}$ such that the equality (10.1) is not satisfied. Then, there exists an open subset $U \Subset (\mathcal{R} \setminus \{x_i\}_{1 \leq i \leq k})$ such that $\text{MA}_{\omega}(\text{P}_{\omega}(f))(U) > 0$. Write $\omega = \partial\bar{\partial}h$ for some $h \in \text{LConv}(U)$. Set $\varphi := \text{P}_{\omega}(f_{\mathcal{S}})$, and let $u := \varphi + h \in \text{LConv}(U)$.

We may assume that there is a compatible chart $\phi: U \rightarrow W$ where W is a bounded open subset of $\mathcal{L} \times \mathbb{R}^{d-1}$ for a generalized tropical line $\mathcal{L} \subset \mathbb{R}^k$. Applying Corollary 10.11 to $u' := u \circ \phi^{-1} \in \text{LConv}(W)$, and get $v' \in \text{LConv}(W)$ satisfying the conditions in Corollary 10.11.

Since $\{\varphi + h < v' \circ \phi\} \Subset U$, we can define a function $\tilde{\varphi} := \sup\{\varphi, v' \circ \phi - h\}$ on X as in Construction 4.12. Then, Lemma 4.13 implies that $\tilde{\varphi} \in \text{CPSH}_{\omega}(X)$. By construction, $\tilde{\varphi} = \varphi$ on $\{x_i\}_{1 \leq i \leq k}$, which indicates that $\tilde{\varphi} \leq f_{\mathcal{S}}$. On the other hand, Corollary 10.11 implies that $\tilde{\varphi} \geq \varphi$ and $\tilde{\varphi}(x) > \varphi(x)$ for some $x \in U$. This contradicts with the assumption that $\varphi = \text{P}_{\omega}(f_{\mathcal{S}})$. $\square$

Proof of Corollary 10.3. We only need to prove that for any $g \in C^0(X)$ such that $g \geq 0$ and $\text{Supp}(g) \Subset \mathcal{R}$, we have

$$\int_X g \text{MA}_{\omega}(\varphi) \leq \int_X g d\mu.$$

Note that for any $t > 0$ $\varphi \leq \varphi + tg$, $\varphi + tg = \varphi$ on $X \setminus \text{Supp}(g)$. On the other hand, we have $\text{P}_{\omega}(\varphi + tg) = \sup\{\varphi + \text{P}_{\omega}(\varphi + tg)\}$. Therefore, $\text{P}_{\omega}(\varphi + tg) = \varphi$ on $X \setminus \mathcal{R} \subset X \setminus \text{Supp}(g)$. It follows that

$$\int_X (\varphi + tg - \text{P}_{\omega}(\varphi + tg)) \text{MA}_{\omega}(\text{P}_{\omega}(\varphi + tg)) = \int_{\mathcal{R}} (\varphi + tg - \text{P}_{\omega}(\varphi + tg)) \text{MA}_{\omega}(\text{P}_{\omega}(\varphi + tg)) = 0$$

where the second equality is induced by Theorem 10.2. Hence, $\varphi + tg$ verifies the orthogonality property for any $t > 0$.

The remaining part of the proof goes similar to the proof of (3) $\implies$ (4) in Theorem 9.2. Consider the function $G(t) := \text{F}_{\mu}(\varphi + tg)$. Then, $G(t) \leq G(0)$ for any $t > 0$ since φ is a

maximizer of F_μ . Applying Theorem 9.3, we get

$$\left. \frac{d}{dt} \right|_{t=0+} G(t) = \int_X g \, \text{MA}_\omega(\varphi) - \int_X g d\mu \leq 0.$$

The corollary is proved. $\square$

Proof of Theorem 10.4. We have proven the uniqueness part of tropical Calabi–Yau theorem in Theorem 8.1. It remains to show the existence part. Let μ be a nonnegative Radon measure of volume $\deg(\omega^d)$ compactly supported in X_o . Let φ be a maximizer of F_μ . Since $\mathcal{R} = X_o$, by Corollary 10.3, we have $\text{MA}(\varphi) \leq \mu$ on X_o . Note that $\text{MA}(\varphi)(\partial X) = 0$ (Theorem 6.6), and $\text{MA}(\varphi)(X) = \mu(X) = \deg(\omega^d)$ (Corollary 6.5 (2)). Thus, we obtain that $\text{MA}(\varphi) = \mu$ on X , and the existence of solutions follows. $\square$

11. FAILURE OF THE TROPICAL CALABI–YAU THEOREM

We define the following genericity condition.

Definition 11.1 (Condition (G) for a 2-dimensional tropical fanfold). Let Z be a 2-dimensional tropical fanfold embedded in a real vector space, and denote by E the linear subspace generated by Z . Let n denote the dimension of E . We say Z verifies Condition (G) if the following holds: we have $n \geq 3$ and for any $(n-2)$ -dimensional subspace $F \subset E$, we have $\dim \pi(Z) = 2$, where π denotes the projection map $E \rightarrow E/F$. We say that a tropical fan Σ verifies (G) if the corresponding fanfold $|\Sigma|$ does.

Remark 11.2. Condition G is an intrinsic property of the tropical fanfold, it does not depend on the real vector space into which the fanfold embeds. $\diamond$

Remark 11.3. A 2-dimensional tropical fanfold Z in $E = \mathbb{R}^3$ fails to satisfy Condition (G) if and only if Z is the product of $\mathbb{R}$ and a tropical fanfold of dimension one, that is, if and only if E admit a decomposition of the form $E = E_1 \times E_2$ for $E_1 \simeq \mathbb{R}$ and $E_2 \simeq \mathbb{R}^2$, and there exists a 1-dimensional tropical fanfold $\Gamma \subset E_2$ such that $Z = E_1 \times \Gamma$. $\diamond$

The main purpose of this section is to prove the following theorem.

Theorem 11.4. *Let X be a compact prekähler tropical variety of dimension d . Suppose that there exist a polyhedral structure $\bar{\Pi}$ on X with induced polyhedral structure Π on X_o , and $\sigma \in \Pi_{d-2}$ such that the star fan Π^σ satisfies the genericity condition (G). Then, for any strongly semipositive form ω on X , the pair (X, ω) does not verify the tropical Calabi–Yau theorem.*

An explicit example is treated in Section 11.3.

11.1. A combinatorial result. We start by the following definition.

Definition 11.5. A finite collection $\{e_1, \dots, e_k\}$ of vectors in a real vector space E is *nonnegatively* (resp. *positively*) dependent if there exist reals $a_i \geq 0$ (resp. $a_i > 0$), not all equal to 0, such that $\sum_{i=1}^k a_i e_i = 0$. Otherwise, we say that the collection is nonnegatively (resp. positively) independent.

A finite collection of rays $\{\rho_1, \dots, \rho_k\}$ in E is nonnegatively (resp. positively) independent if a (equivalently, any) collection of vectors $\{e_1, \dots, e_k\}$ with e_j generator of ρ_j is nonnegatively (resp. positively) independent. $\diamond$

Proposition 11.6. *Suppose that Z is a 2-dimensional tropical fan in $E = \mathbb{R}^n$ satisfying*

- (1) *Z satisfies the genericity condition (G).*
- (2) *$\text{span}(Z) = E$.*
- (3) *Z is connected through codimension 1.*

Then, there exist an $(n - 2)$ -dimensional subspace $F \subset E$ and a collection $\{\rho_1, \dots, \rho_{n-1}\}$ of $(n - 1)$ nonnegatively dependent rays inside $F \cap Z$.

We postpone the proof of this proposition to Section 11.4, and first explain how deduce the main result of the section.

11.2. Proof of Theorem 11.4. We start by proving a local result.

Proposition 11.7. *Let E_1 and E_2 be real vector spaces of dimension d_1 and d_2 respectively, and $E = E_1 \times E_2$. Let $Z \subset E_1$ be a 2-dimensional tropical fanfold satisfying the genericity condition (G), and let $Z' := Z \times E_2 \subset E$. Then, there exists $u \in \text{LConv} \cap \text{PL}(Z')$ such that*

- (1) *$\text{MA}_Z(u)(\{0\}) > 0$. Here, MA_Z denotes the local tropical Monge–Ampère operator on Z .*
- (2) *u is not strictly convex at 0. More precisely, for any open neighborhood U of 0 in Z' , there exists a finite subset $\{x_i\}_{i \in I} \subset U \setminus \{0\}$ such that for every $v \in \text{LConv}(U)$ with $v(x_i) \leq u(x_i)$ for all $i \in I$, we have $v(0) \leq u(0)$.*

Proof. We first show that it is enough to prove the statement when $Z' = Z$, that is, when $E_2 = (0)$. Suppose that $u_1 \in \text{LConv} \cap \text{PL}_{\text{hom}}(Z)$ is a function satisfying the properties (1) and (2) in the proposition for $Z' = Z$, and $E_2 = (0)$. In particular, $\text{MA}_Z(u_1)(\{0\}) > 0$. Take any $u_2 \in \text{Conv} \cap \text{PL}_{\text{hom}}(E_2)$ such that $\text{MA}_{E_2}(u_2)(\{0\}) > 0$. Then, we define $u := u_1 \circ \pi_1 + u_2 \circ \pi_2$, where π_1 and π_2 are the projection maps from Z' to Z and E_2 , respectively. We claim that u satisfies the two properties in the proposition. Fix simplicial fan structures Σ and Ξ on Z and E_2 , respectively, such that u_1 and u_2 are conewise linear on Σ and Ξ , respectively. The product $\Sigma' = \Sigma \times \Xi$ is a fan structure on Z' , and u is conewise linear on Σ' . The Künneth formula for the Chow rings of fans says that $A^\bullet(\Sigma') = A^\bullet(\Sigma) \otimes A^\bullet(\Xi)$, and the degree map on the left is the product of the two degree maps on the right under this decomposition (see [AP26, Prop. 3.5]). Therefore, $\text{MA}_{Z'}(u)(\{0\}) = \text{MA}_Z(u_1)(\{0\}) \cdot \text{MA}_{E_2}(u_2)(\{0\}) > 0$. At the same time, for any open neighborhood $U = U_1 \times U_2$ of 0 in Z' , where U_1, U_2 are open neighborhood of 0 in Z and E_2 , respectively, take a finite subset $\{x_i\}_{i \in I} \subset U_1 \setminus \{0\}$ which satisfies property (2) for $u_1 \in \text{LConv} \cap \text{PL}(\Sigma)$. Notice that for any $v \in \text{LConv}(U)$, we have $v|_{U_1 \times \{0\}} \in \text{LConv}(U_1 \times \{0\})$. Hence, by assumption, $v((x_i, 0)) \leq u((x_i, 0))$ for each i would imply $v(0) \leq u(0)$. Therefore, the finite subset $\{(x_i, 0)\}_{i \in I} \subset U \setminus \{0\}$ satisfies the property (2) for u , and hence the reduction step is completed.

In the remainder of the proof, we assume that $Z' = Z$, and $E = E_1$. We prove the proposition by dividing into two cases.

Case 1: Z is not connected through codimension 1. In this case, since $\dim(Z) = 2$, there exist tropical fanfolds Z_1 and Z_2 in E such that $Z = Z_1 \cup Z_2$, and $Z_1 \cap Z_2 = \{0\}$. By Lemma 8.5, there exists $u_1 \in \text{LConv} \cap \text{PL}_{\text{hom}}(Z_1)$ such that $\text{MA}_{Z_1}(u_1) > 0$. Take an affine support function l of u_1 at 0, and define $u \in \text{LConv} \cap \text{PL}_{\text{hom}}(Z)$ by $u := u_1$ on Z_1 and $u := l$ on Z_2 . It remains to check that u satisfies the two properties stated in the proposition. Observe that $\text{MA}_Z(u)(\{0\}) = \text{MA}_{Z_1}(u_1)(\{0\}) + \text{MA}_{Z_2}(l)(\{0\}) = \text{MA}_{Z_1}(u_1)(\{0\}) > 0$. On the other hand, for any open neighborhood U of 0 in Z , take a finite subset $\{x_i\}_{i \in I} \subset Z_2 \cap (U \setminus \{0\})$ such that

$0 \in \text{conv}(\{x_i\}_{i \in I}) \subset U \cap Z$, where $\text{conv}(\{x_i\}_{i \in I})$ denotes the convex hull of $\{x_i\}_{i \in I}$ in E . Then, for any $v \in \text{LConv}(U)$ satisfying $v(x_i) \leq u(x_i) = l(x_i)$ for each $i \in I$, Proposition 3.6 implies that $v(0) \leq u(0) = 0$. Therefore, the proposition is concluded in this case.

Case 2: Z is connected through codimension 1. Without loss of generality, may assume that $E = \text{span}(Z)$ and $\dim E = n$. Then, Z verifies all the three conditions in Proposition 11.6. Let $F \subset E$ be the $(n-2)$ -dimensional subspace and $\{\rho_1, \dots, \rho_{n-1}\}$ the $(n-1)$ rays in $F \cap Z$ given by that proposition. Since Z satisfies the genericity condition (G), and is a tropical fanfold, we have $\pi(Z) = E/F$, where π denote the quotient map $E \rightarrow E/F$. We equip E/F with a tropical fanfold structure as the pushforward of Z , as in Section 2.1.7 (the corresponding tropical weight function is a positive constant). Let $u' \in \text{Conv} \cap \text{PL}_{\text{hom}}(E/F)$ such that $\text{MA}_{E/F}(u')(\{0\}) > 0$. Then, $u := u' \circ \pi \in \text{LConv} \cap \text{PL}_{\text{hom}}(Z)$. Since $\text{MA}_Z(u)$ is a Dirac mass at 0, [APW26, Prop. 5.20] implies that $\text{MA}_Z(u)(\{0\}) = \text{MA}_{E/F}(u')(\{0\}) > 0$. On the other hand, let U be an open neighborhood of 0 in Z . Take $x_i \in \text{relint}(\rho_i)$ for each $1 \leq i \leq n-1$ such that $\text{conv}(\{x_i\}_{1 \leq i \leq n-1}) \subset U \cap Z$, where $\text{conv}(\{x_i\}_{1 \leq i \leq n-1})$ denotes the convex hull of $\{x_i\}_{1 \leq i \leq n-1}$ in E . Since $\{\rho_i\}_{1 \leq i \leq n-1}$ is nonnegatively dependent in F , $(0, (x_i)_{1 \leq i \leq n-1}, (\nu_i)_{1 \leq i \leq n-1})$ is a polyhedral convex combination for some real numbers $0 \leq \nu_i \leq 1$. Notice that $u|_{F \cap \Sigma} = 0$. Thus, by Proposition 3.6, for any $v \in \text{LConv}(U)$ satisfying $v(x_i) \leq u(x_i) = 0$ for each $1 \leq i \leq n-1$, we have $v(0) \leq 0 = u(0)$. Therefore, u satisfies the two properties in the proposition, and the proposition is proven in this case.

This finishes the proof. $\square$

We are now ready to prove the main theorem of this section.

Proof of Theorem 11.4. Notation as in the statement of the theorem, let $\sigma \in \Pi_{d-2}$ such that Π^σ satisfies Condition (G). We take a point $x \in \text{relint}(\sigma)$. Writing $\Pi^x = \Pi^\sigma \times \mathbb{R}^{d-2}$, and applying Proposition 11.7 to $Z = |\Pi^x|$, we obtain $u \in \text{LConv} \cap \text{PL}(|\Pi^x|)$ that satisfies (1) and (2) in that proposition.

Take an open neighborhood W of x such that W is isomorphic to an open neighborhood of 0 in $|\Pi^x|$. Since ω is a strongly semipositive form on X , by definition (see Definition 5.16) there exists $\psi \in \text{PSH}_\omega \cap \text{PL}_\omega(X)$ such that $\psi + h = Cu + l$ on an open neighborhood U of x in W , where h is the local potential of ω on U , C is a positive constant, and l is an affine function on U . Therefore, $\text{MA}_\omega(\psi)(\{x\}) = \text{MA}_Z(Cu)(\{x\}) > 0$. Furthermore, there exists a finite subset $\{x_i\}_{i \in I} \subset U$ such that for every $v \in \text{LConv}(U)$ with $v(x_i) \leq (\psi + h)(x_i)$ for all $i \in I$, we have $v(x) \leq (\psi + h)(x)$.

We choose a refinement $\bar{\Pi}'$ of $\bar{\Pi}$ on X , inducing a refinement Π' of Π on X_o , such that $\{x\} \cup \{x_i\}_{i \in I} \subset \Pi'_0$, and ψ is ω -facewise affine on $\bar{\Pi}'$. Set the finite collection of pairs $\mathcal{S} := \{(x, \psi(x)) \mid x \in \Pi'_0\}$. Using Lemma 4.21, we have $\psi = P_\omega(f_{\mathcal{S}})$. Then, by assumption, $\psi = P_\omega(f_{\mathcal{S}'})$ for $\mathcal{S}' := \{(x, \psi(x)) \mid x \in \Pi'_0 \setminus \{x\}\}$. Therefore, $\text{Supp}(\text{MA}_\omega(f_{\mathcal{S}'})) \not\subset \Pi'_0 \setminus \{x\}$. Hence, by Theorem 9.2, (X, ω) does not verify the tropical Calabi–Yau theorem. $\square$

11.3. An explicit example for the failure of the tropical Calabi–Yau theorem. We present an explicit example that elucidates the proof of Theorem 11.4.

Example 11.8. Let Z be the fanfold of dimension 2 in $E = \mathbb{R}^3$ associated to the uniform rank 3 matroid $U_{3,4}$ on 4 elements. It is given as follows. Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be the standard basis of $\mathbb{R}^3$ and $\mathbf{e}_0 = -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$. Then, denoting $\rho_i := \mathbb{R}_{\geq 0}\mathbf{e}_i$, we have $Z = \bigcup_{0 \leq i < j \leq 3} \rho_i + \rho_j$.

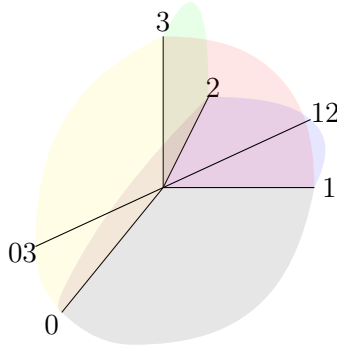


FIGURE 1. The fan Σ with support the Bergman fanfold Z of the uniform matroid $U_{3,4}$.

We define a unimodular fan Σ with support Z whose rays are the ρ_J for J labeled as in Figure 1. For instance, for $J = 03$, the corresponding ray is $\rho_{03} = \mathbb{R}_{\geq 0}\mathfrak{e}_{03}$ where $\mathfrak{e}_{03} := \mathfrak{e}_0 + \mathfrak{e}_3$. Let $X = \overline{\Sigma}$ be the canonical compactification of Σ , which is a compact tropical variety with one single chart given by $\overline{\Sigma}$. Note that we have $X_o = Z$.

Let $h \in \text{PL}_{\text{hom}}(Z)$ defined by the following table on the generators of the rays of Σ

J	0	1	2	3	03	12
$h(\mathfrak{e}_J)$	1	1	1	1	2	2

extended linearly on each cone of Σ . Notice that h is a convex conewise linear function on Σ . Moreover, Proposition 5.19 shows that $\omega = \partial\bar{\partial}h$ is a strongly semipositive simplicial form on X . A direct calculation shows that $\omega^2 = 16\delta_0$.

Observe that $\mathfrak{e}_{03} = -\mathfrak{e}_{12}$. Let F be the line containing $\mathfrak{e}_{03}$ and $\mathfrak{e}_{12}$. Then $F \subset E$ is a line satisfying the conditions in Proposition 11.6. Let $\pi: E \rightarrow E/F$ be the quotient map. It is clear that $\dim(\pi(Z)) = 2$.

We define $g \in \text{PL}_{\text{hom}}(Z)$ by the following table on the generators of the rays of Σ ,

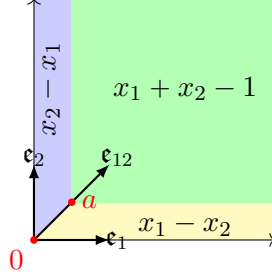
J	0	1	2	3	03	12
$g(\mathfrak{e}_J)$	1	1	1	1	0	0

extended linearly to each cone of Σ . Then it is not hard to see that $g = g' \circ \pi$ for some strictly convex homogeneous piecewise linear function g' on E/F . Therefore, by [APW26, Prop. 5.2] we have $\text{MA}(g)\{0\} > 0$. In fact one can check that $\text{MA}(g)\{0\} = 8$ via a direct calculation in the Chow ring of Σ . Furthermore, since $g|_F = 0$, $g \in \text{LConv} \cap \text{PL}(\Sigma)$ satisfies the conditions in Proposition 11.7.

We now make use of this function g to construct an explicit counterexample to tropical Calabi–Yau theorem. Observe that

$$\max\{g, h - 1\} \in \text{LConv}(X_o), \quad |\max\{g, h - 1\} - h| = O(1).$$

Therefore, by Theorem 4.8, $\max\{g, h - 1\} - h$ can be extended to $\varphi \in \text{PSH}_\omega \cap \text{PL}(X)$. The restriction of $\varphi + h$ on the cones $\rho_0 + \rho_1$, $\rho_0 + \rho_2$, $\rho_1 + \rho_3$, $\rho_2 + \rho_3$ equals to g . The restriction of $\varphi + h$ on the cone $\rho_1 + \rho_2$ is shown in Figure 2, so one can observe that $\varphi + h = g$ in the blue and yellow region, and $\varphi + h = h$ in the green region. The figure for $\rho_0 + \rho_3$ is symmetric. Therefore, $\varphi + h = g$ in a neighborhood of $\bigcup_{0 \leq i \leq 3} \rho_i$, and $\text{MA}_\omega(\varphi)$ is supported

FIGURE 2. The values of the function $\phi + h$ on the cone $\rho_1 + \rho_2$.

on $\{0, a, b\}$, where a and b denote $\frac{1}{2}\mathbf{e}_{12}$ and $\frac{1}{2}\mathbf{e}_{03}$ respectively. A direct calculation shows that $\text{MA}_\omega(\varphi) = 8\delta_0 + 4\delta_a + 4\delta_b$.

We consider the finite collection of pairs $\mathcal{S} = \{(a, 0), (b, 0)\} \subset \Sigma_o \times \mathbb{R}$. Then $\text{P}_\omega(f_{\mathcal{S}}) = \varphi$, and therefore $f_{\mathcal{S}}$ does not verify the orthogonality property. Following the proof of Theorem 9.2, one can show that the measure

$$\mu = 8\delta_a + 8\delta_b$$

does not admit any solution in $\text{PSH}_\omega(X)$. $\diamond$

Remark 11.9. Take $\omega' := \partial\bar{\partial}g$ in the above example. Then, ω is a semipositive form on Z with $\deg(\omega^2) = 8 > 0$. However, ω' is not strongly semipositive. To see this, observe that $\omega'|_F = 0$, where F is the line in E containing $\mathbf{e}_{03}$ and $\mathbf{e}_{12}$. This implies that $\varphi|_F$ is constant for any $\varphi \in \text{CPSH}_\omega(Z)$. On the other hand, one can easily find an open neighborhood W of 0 in Z and $f \in \text{LConv}(W)$ such that $f|_{U \cap F}$ is non-constant. Therefore, the condition in Definition 5.16 cannot be satisfied for ω' and $(W, f, 0, g)$. $\diamond$

Remark 11.10. Notice that if a 2-dimensional fanfold Z can be written as a product $Z = \Gamma_1 \times \Gamma_2$ for two 1-dimensional tropical fanfolds Γ_1, Γ_2 , then Z does not verify the genericity condition (G). However, if neither Γ_1 nor Γ_2 is equal to $\mathbb{R}^1$, we can still choose an $(n-2)$ -dimensional subspace $F \subset E$ satisfying the conditions in Proposition 11.6, and moreover verifying that $\dim(\pi(Z)) = 2$, for the quotient map $\pi: E \rightarrow E/F$. We illustrate this in Example 11.11 below. In this situation, the arguments in Proposition 11.7 and Theorem 11.4 apply verbatim, so Theorem 11.4 still holds for any projective compactification of Z . $\diamond$

Example 11.11. Suppose that Σ_1 is a tropical fan in $\mathbb{R}^2$ with 3 rays $\mathbb{R}_{\geq 0}\mathbf{e}_1, \mathbb{R}_{\geq 0}\mathbf{e}_2, \mathbb{R}_{\geq 0}\mathbf{e}$, where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a base of $\mathbb{R}_2$, and $\mathbf{e} := -\mathbf{e}_1 - \mathbf{e}_2$. Suppose further that $\Sigma_2 \cong \Sigma_1$, where we denote the rays of Σ_2 by $\mathbb{R}_{\geq 0}\mathbf{e}'_1, \mathbb{R}_{\geq 0}\mathbf{e}'_2, \mathbb{R}_{\geq 0}\mathbf{e}$ with $\mathbf{e}' := -\mathbf{e}'_1 - \mathbf{e}'_2$. Let $\Sigma = \Sigma_1 \times \Sigma_2 \subset E = \mathbb{R}^4$. Then, one can take nonnegatively dependent rays $\rho_1 := \mathbb{R}_{\geq 0}(\mathbf{e}_1 + \mathbf{e}'_1)$, $\rho_2 := \mathbb{R}_{\geq 0}(\mathbf{e}_2 + \mathbf{e}'_2)$, $\rho_3 := \mathbb{R}_{\geq 0}(\mathbf{e} + \mathbf{e}')$ inside Σ , and let F be the subspace generated by them. Then, it is not difficult to check that $\dim F = 2$ and $\dim \pi(\Sigma) = 2$, for the quotient map $\pi: E \rightarrow E/F$. $\diamond$

Actually, we have the following conjecture.

Conjecture 11.12. *Let (X, ω) be a compact prekähler tropical variety equipped with a strongly semipositive form ω . Then, (X, ω) verifies the tropical Calabi–Yau theorem if and only if $\mathcal{R} = X_o$, i.e. every point $x \in X_o$ is regular in codimension one (see Definition 10.1).*

11.4. Proof of Proposition 11.6. It remains to prove the combinatorial statement in Proposition 11.6. We will use several polyhedral geometric results in the proof. In order to increase the readability, we gather them all in Appendix B, to which we refer in the course of the proof. We also recall the following result in topology.

Lemma 11.13. *Let D^{n+1} be the $(n+1)$ -dimensional disk, and $S^n := \partial D^{n+1}$ the n -dimensional sphere. Define the open northern and southern hemisphere by*

$$S_+^n := \{x \in S^n \mid x_{n+1} > 0\}, \quad S_-^n := \{x \in S^n \mid x_{n+1} < 0\}.$$

Let $f: D^n \rightarrow D^{n+1}$ be a continuous map whose restriction $f|_{\partial D^n}$ induces a homeomorphism from ∂D^n to the equator $S^{n-1} \subset S^n$. Let $\gamma: [0, 1] \rightarrow D^{n+1}$ be a curve such that $\gamma(0) = x_+ \in S_+^n$ and $\gamma(1) = x_- \in S_-^n$. Then, there exist $t_0 \in [0, 1]$ and $x_0 \in D^n$ such that $\gamma(t_0) = f(x_0)$.

Proof. This is a standard application of the Borsuk-Ulam Theorem [Mat03, Thm. 2.1.1]. Since we were not able to find a reference for this precise statement in the literature, we sketch a proof here. Proceeding by contradiction, suppose the statement is not correct. Define the continuous map $g: D^n \times [0, 1] \rightarrow S^n$ by $(x, t) \mapsto (f(x) - \gamma(t)) / \|f(x) - \gamma(t)\|$. Notice that $D^n \times [0, 1]$ is homeomorphic to D^{n+1} . It is not hard to show that the map $g|_{\partial(D^n \times [0, 1])}: \partial(D^n \times [0, 1]) \rightarrow S^n$ has degree 1. But at the same time, g is nullhomotopic since $D^n \times [0, 1]$ is retractable. This yields a contradiction. $\square$

We are prepared to prove Proposition 11.6. We proceed in several steps.

• **Step 1. Reduction to tropical curves.**

We fix a fan structure Σ on Z . Also, we fix an Euclidean metric on E .

Consider the dual space $E^\vee$, and take a generic rational element $l \in E^\vee$ such that the $(n-1)$ -dimensional vector subspace $A := \{l = 0\} \subset E$ does not contain any nonzero cone σ in Σ . Then, $\{l = 1\} \cap Z$ and $\{l = -1\} \cap Z$ are both tropical curves. Let $\pi: E \rightarrow A$ be the orthogonal projection (with respect to the fixed Euclidean metric). Denote the image of these two tropical curves in A by

$$\Gamma := \pi(\{l = 1\} \cap Z) \text{ and } \Gamma' := \pi(\{l = -1\} \cap Z), \text{ and set } -\Gamma' := -\pi(\{l = -1\} \cap Z).$$

Let $\mathfrak{C}$ be the subset of A defined by

$$\mathfrak{C} := \left\{ x = \sum_{i=1}^{n-2} \lambda_i x_i \in A \mid x_i \in \Gamma, \lambda_i \geq 0 \text{ for each } 1 \leq i \leq n-2, \text{ and } \sum_{i=1}^{n-2} \lambda_i = 1 \right\}.$$

In the rest of the proof, we show that $\mathfrak{C} \cap (-\Gamma') \neq \emptyset$. This implies the proposition, by taking the $(n-2)$ rays generated by $x_i \in \Gamma$, and the one generated by x with $x = \sum_i \lambda_i x_i \in \mathfrak{C} \cap (-\Gamma')$.

• **Step 2. Selection of rays.**

Set $\Upsilon = \text{rec}(\Gamma)$, the recession cone of Γ , and note that $\Upsilon = \text{rec}(\Gamma')$. Notice that Υ is balanceable (that is, the rays of Υ are positively dependent), and $\text{span}(\Upsilon) = A$. Applying Lemma B.2 to the balanceable 1-dimension fan Υ and the simplicial fan $\{0\}$ supported on A , we obtain a complete simplicial fan Ξ'' supported on A such that the set of rays Ξ'' and that of Υ coincide, that is, $\Xi''_1 = \Upsilon_1$.

For $\rho \in \Upsilon_1$, we use $\mathbf{e}_\rho$ for the primitive vector on ρ . Fix $\rho_0 \in \Upsilon_1$. Let $\sigma \in \Xi''$ such that $-\mathbf{e}_{\rho_0} \in \text{relint}(\sigma)$. Let $\rho_1, \dots, \rho_m \in \Upsilon_1$ be the rays generating σ , where $m = \dim \sigma \leq n-1$. Then, we have $-\mathbf{e}_{\rho_0} = \sum_{1 \leq j \leq m} a_j \mathbf{e}_{\rho_j}$ for positive $a_j > 0$, $j = 1, \dots, m$. Since σ is simplicial, $\mathbf{e}_{\rho_1}, \dots, \mathbf{e}_{\rho_m}$ are linearly independent, and generate an m -dimensional vector subspace $A_0 \subset A$.

Let $l := \mathbb{R}\mathbf{e}_{\rho_0} \subset A$ be the linear span of ρ_0 . Consider the projection map $p: A \rightarrow B := A/l$. Denote $\Upsilon' = p(\Upsilon)$. Let B_0 be the image of A_0 under the quotient map p . We denote $\rho'_j := p(\rho_j) \in \Upsilon'_1$, and $\mathbf{e}_{\rho'_j} := p(\mathbf{e}_{\rho_j})$. Then, B_0 is an $(m-1)$ -dimensional vector subspace in B , and we have

$$\text{span}_{1 \leq j \leq m}(\{\mathbf{e}_{\rho'_j}\}) = B_0, \quad \text{and} \quad 0 = \sum_{1 \leq j \leq m} a_j \mathbf{e}_{\rho'_j} \quad \text{in } B_0.$$

Therefore, the rays $\rho'_1, \dots, \rho'_m$ generate a complete simplicial fan supported on B_0 that we denote by Ξ' . Each cone of Ξ' is generated by a proper subsets of $\{\rho'_1, \dots, \rho'_m\}$.

Notice that Υ' is a balanceable 1-dimensional fan supported on B . Then, the subfan of Υ' defined by $\mathcal{L} := \{0\} \cup \{\rho'_j \mid 1 \leq j \leq m\} \cup \{\xi \mid \xi \in \Upsilon'_1, \text{ and } \xi \not\subset B_0\}$ is also balanceable. Applying Lemma B.2 to $\mathcal{F}' = \Xi'$ and $\mathcal{L}$ defined above, we obtain a complete simplicial fan Ξ supported on B such that Ξ' is a subfan of Ξ , and $\Xi_1 \subset \Upsilon'_1$.

We choose a map $s_{(1)}: \Xi_1 \rightarrow \Upsilon_1$ such that $p_{(1)}(s_{(1)}(\xi)) = \xi$ for each $\xi \in \Xi_1$, and $s_{(1)}(\rho'_j) = \rho_j$ for each $1 \leq j \leq m$. Here $p_{(1)}$ denote the map induced by p , which sends the set of rays in A to the set of rays in B . This induces a continuous map $s: B \rightarrow A$ by extending $s_{(1)}$ linearly on each cone of Ξ . Therefore, s is a conewise linear map on Ξ , and $p \circ s = \text{id}_B$.

Define the open subsets

$$U_+ := \{x \in A \mid x = s(p(x)) + a \cdot \mathbf{e}_{\rho_0} \text{ for some } a > 0\},$$

$$U_- := \{x \in A \mid x = s(p(x)) + a \cdot \mathbf{e}_{\rho_0} \text{ for some } a < 0\}$$

Since $s(p(-\mathbf{e}_{\rho_0})) = 0$, we have $-\mathbf{e}_{\rho_0} \in U_-$. On the other hand, $-\mathbf{e}_{\rho'_1} = \sum_{2 \leq j \leq m} \frac{a_j}{a_1} \mathbf{e}_{\rho'_j}$ with $\frac{a_j}{a_1} > 0$. Hence, $-\mathbf{e}_{\rho'_1} \in \text{Cone}(\rho'_2, \dots, \rho'_m) \in \Xi'$. Since Ξ' is a subfan of Ξ and s is conewise linear on Ξ , we have

$$s(p(-\mathbf{e}_{\rho_1})) = s(-\mathbf{e}_{\rho'_1}) = \sum_{2 \leq j \leq m} \frac{a_j}{a_1} s(\mathbf{e}_{\rho'_j}) = \sum_{2 \leq j \leq m} \frac{a_j}{a_1} \mathbf{e}_{\rho_j} = -\mathbf{e}_{\rho_1} - \frac{1}{a_1} \mathbf{e}_{\rho_0}.$$

This implies that $-\mathbf{e}_{\rho_1} \in U_+$.

• **Step 3. Constructions of $P \subset B$ and $\tilde{P} \subset B \times \mathbb{R}$.**

Recall from the previous step that $\dim A = n-1$, $\dim B = n-2$, and Ξ'' and Ξ are complete simplicial fans supported on A and B , respectively.

We first construct a subset $P \subset B$. For each $\xi \in \Xi_1$, let $\mathbf{u}_\xi$ be the vector of norm one on ξ , that is, the unique point of $B(0, 1) \cap \xi$ in B . Consider the simplicial polyhedral complex

$$\Delta_P := \{ \text{conv}(0, \mathbf{u}_{\xi_1}, \mathbf{u}_{\xi_2}, \dots, \mathbf{u}_{\xi_k}) \mid \text{Cone}(\xi_1, \xi_2, \dots, \xi_k) \in \Xi \},$$

and denote by $P \subset B$ its support. Since Ξ is complete, P is homeomorphic to an $(n-2)$ -dimensional disk D^{n-2} . The set of vertices of Δ_P is given by $(\Delta_P)_0 = \{0, \mathbf{u}_\xi \mid \xi \in \Xi_1\}$.

Fix $N \in \mathbb{N}_+$ such that $N > |\Gamma_0|$, with Γ_0 the set of vertices of the tropical curve Γ in A . For each $\xi \in \Xi_1$ and $0 \leq i \leq N$, denote $\mathbf{u}_{\xi,i} := B(0, i/N) \cap \xi$. The segment $[0, \mathbf{u}_\xi]$ is thus subdivided into N segments by the points $\mathbf{u}_{\xi,i}$. Applying Lemma B.1, we can subdivide Δ_P to a simplicial polyhedral complex Δ'_P such that $(\Delta'_P)_0 = \{0, \mathbf{u}_{\xi,i} \mid \xi \in \Xi_1, 1 \leq i \leq N\}$.

Claim 11.14. *Let $\sigma \in \Delta'_P$ be an $(n-2)$ -dimensional simplex. Then, there exists rays $\xi_1, \xi_2, \dots, \xi_{n-2} \in \Xi_1$ such that $\text{Cone}(\xi_1, \xi_2, \dots, \xi_{n-2}) \in \Xi_{n-2}$, and*

$$\sigma = \text{conv}(\mathbf{u}_{\xi_1, i_1}, \mathbf{u}_{\xi_1, i_1+1}, \mathbf{u}_{\xi_2, i_2}, \mathbf{u}_{\xi_3, i_3}, \dots, \mathbf{u}_{\xi_{n-2}, i_{n-2}})$$

Proof of Claim 11.14. Since $\Delta_{P'}$ is a subdivision of Δ_P , we may assume that

$$\sigma \subset \text{conv}(0, u_{\xi_1}, u_{\xi_2}, \dots, u_{\xi_{n-2}}), \quad \text{with } \text{Cone}(\xi_1, \xi_2, \dots, \xi_{n-2}) \in \Xi_{n-2}$$

Since σ is an $(n-2)$ -dimensional simplex, one can write $\sigma = \text{conv}(\mathfrak{S})$, where $\mathfrak{S}$ is a subset of $\{0\} \cup \{u_{\xi_j, i} \mid 1 \leq j \leq n-2, 1 \leq i \leq N\}$ of cardinality $n-1$.

Suppose that there exists $1 \leq j_0 \leq n-2$ such that $\mathfrak{S} \cap \{u_{\xi_{j_0}, i} \mid 1 \leq i \leq N\}$ is empty. Then, $\sigma \subset \text{Cone}(\xi_1, \dots, \widehat{\xi_{j_0}}, \dots, \xi_{n-2})$, which implies that $\dim \sigma \leq n-3$, and hence contradicts with our assumption. Therefore, after possibly permutating $\{u_j\}_{1 \leq j \leq n-2}$, one can write $\mathfrak{S} = \{u_{\xi_1, i'_1}, u_{\xi_1, i_1}, u_{\xi_2, i_2}, u_{\xi_3, i_3}, \dots, u_{\xi_{n-2}, i_{n-2}}\}$, where $1 \leq i_j \leq N$ for each $1 \leq j \leq n-2$, and $0 \leq i'_1 < i_1 \leq N$.

By assumption, for any $i''_1 \in \{0, \dots, n-2\} \setminus \{i_1, i'_1\}$, the vertex u_{ξ_1, i''_1} is not contained in σ . This implies that $u_{\xi_1, i''_1} \notin \text{conv}(u_{\xi_1, i_1}, u_{\xi_1, i'_1})$, and hence we have $i_1 = i'_1 + 1$. Therefore, the claim is concluded. $\square$

We now construct a subset $\tilde{P} \subset B \times \mathbb{R}$. Denote $u_+ := (0, 1) \in B \times \mathbb{R}$, and $u_- := (0, -1) \in B \times \mathbb{R}$. Consider the simplicial polyhedral complex

$$\Delta_{\tilde{P}} := \{ \sigma, \text{conv}(\sigma, u_+), \text{conv}(\sigma, u_-) \mid \sigma \in \Delta_P \}$$

and denote by $\tilde{P}$ its support in $B \times \mathbb{R}$. We set $(\partial \tilde{P})_{\pm} := \partial \tilde{P} \cap (B \times \mathbb{R}_{\pm})$. We denote $i: P \rightarrow \tilde{P}$ the inclusion map given by $x \mapsto (x, 0)$. Then, it is not hard to see that there exists a homeomorphism from $\tilde{P}$ to an $(n-1)$ -dimensional disk D^{n-1} with ∂P mapped to the equator S^{n-3} of D^{n-1} , and $(\partial \tilde{P})_{\pm}$ mapped to the open northern and southern hemisphere respectively.

We define a subdivision $\Delta'_{\tilde{P}}$ of $\Delta_{\tilde{P}}$ by:

$$\Delta'_{\tilde{P}} := \{ \text{conv}(\sigma', u_+), \text{conv}(\sigma', u_-) \mid \sigma' \in \Delta'_P \}.$$

• **Step 4. Constructions of $Q \subset A$ and a homeomorphism $h: Q \rightarrow \tilde{P}$.**

Denote by $\Gamma_{1,\infty}$ the set of infinite edges of Γ . Recall that Υ is the recession cone of Γ , and we have chosen a map $s_{(1)}: \Xi_1 \rightarrow \Upsilon_1$ in Step 2. For each ray $\xi \in \Xi_1$, we choose an infinite edge $\text{Leg}(s_{(1)}(\xi)) \in \Gamma_{1,\infty}$ such that $\text{Leg}(s_{(1)}(\xi)) = x + s_{(1)}(\xi)$ for some $x \in \Gamma$, with $s_{(1)}(\xi) \in \Upsilon_1$. Fix a vertex v of Γ , and fix $r \gg 0$. Denote $v_{\xi} := B(v, r) \cap \text{Leg}(s_{(1)}(\xi))$.

Recall that ρ_0 is the ray of Υ_1 which we fixed in Step 2. We choose an infinite edge $\text{Leg}(-\rho_0) \in (-\Gamma')_{1,\infty}$ such that $\text{Leg}(-\rho_0) = x_0 - \rho_0$ for some $x_0 \in -\Gamma'$. Let $\rho_{0,v} = v + \rho_0$ be a ray originated at v with direction ρ_0 . Denote $v_+ := B(v, r) \cap \rho_{0,v} \in A$, and $v_- := B(v, r) \cap \text{Leg}(-\rho_0) \in A$.

Denote the finite set $S := \{0, +, -\} \cup \Xi_1$. We define an abstract simplicial complex on S by

$$K := \{(0, \xi_1, \dots, \xi_k), (0, +, \xi_1, \dots, \xi_k), (0, -, \xi_1, \dots, \xi_k) \mid \text{Cone}(\xi_1, \dots, \xi_k) \in \Xi\}$$

We view $u: S \rightarrow B \times \mathbb{R}$ by setting $u(0) = 0$, $u(\pm) = u_{\pm}$, and $u(\xi) = u_{\xi}$; and we define $v: S \rightarrow A$ by setting $v(0) = v$, $v(\pm) = v_{\pm}$, and $v(\xi) = v_{\xi}$.

For any $T \in K$, we use $u_T \subset B \times \mathbb{R}$ and $v_T \subset A$ to denote the image of T under u and v respectively. We define simplices $|T|_{\tilde{P}} := \text{conv}(u_T) \subset B \times \mathbb{R}$, and $|T|_Q := \text{conv}(v_T) \subset A$. Notice that the simplicial polyhedral complex $\Delta_{\tilde{P}}$ equals to $\{|T|_{\tilde{P}} \mid T \in K\}$.

Now we define a set of simplices in A by

$$\Delta_Q := \{|T|_Q \mid T \in K\}^8$$

with

$$Q := \left(\bigcup_{T \in K} |T|_Q \right) \subset A.$$

We define a continuous map $h: \tilde{P} \rightarrow Q$ which is facewise linear on $\Delta_{\tilde{P}}$ by $h(u(\mathfrak{s})) = v(\mathfrak{s})$, for $\mathfrak{s} \in S$, extended linearly on each face of $\Delta_{\tilde{P}}$.

Claim 11.15. *When $r \gg 0$, Δ_Q is a simplicial polyhedral complex supported on $Q \subset A$, and $h: \tilde{P} \rightarrow Q$ is a homeomorphism.*

Proof of Claim 11.15. For each ray $\xi \in \Xi_1$, denote $v'_\xi := B(v, 1) \cap s_{(1)}(\xi) \in A$ with $s_{(1)}(\xi) \in \Upsilon_1$. Furthermore, we denote $v' := 0$, $v'_+ := e_{\rho_0}$, and $v'_- := -e_{\rho_0}$. We view v' as a map from S to A by setting $v'(0) = v'$, $v'(\pm) = v'_\pm$, $v'(\xi) = v'_\xi$. Set $|T|_{Q'} := \text{conv}(v'_T) \subset A$ for $T \in K$, where v'_T denotes the image of T under v' . We define an auxiliary set of simplices in A by $\Delta_{Q'} := \{|T|_{Q'} \mid T \in K\}$, with support $Q' := (\bigcup_{T \in K} |T|_{Q'}) \subset A$.

For each $\xi \in \Xi_1$, denote $u'_\xi := p(v'_\xi)$, and denote $P' := p(Q') \subset B$. Notice that $u'_\xi \in \mathbb{R}_+ \cdot u_\xi$ for each $\xi \in \Xi_1$. Therefore, the simplicial polyhedral complex $\Delta_{P'} := \{|T|_{P'} \mid T \in K, T \subset \Xi_1\}$ is isomorphic to P , and it has support P' . Here $|T|_{P'} := \text{conv}(u'_T) \subset B$ for any $T \in K$ such that $T \subset \Xi_1$. Denote s' the restriction of the continuous injection $s: B \rightarrow A$ to P' . Notice that $s'(P') \subset Q'$, $s'(\partial P') \subset \partial Q'$, and $|T|_{Q'} = s'(|T|_{P'})$ for any $T \subset \Xi_1 \subset S$.

We show that for any $T_1, T_2 \in K$, we have $|T_1 \cap T_2|_Q = |T_1|_Q \cap |T_2|_Q$. When $T_1, T_2 \subset \Xi_1 \subset S$, the equality follows from the facts that $|T_1 \cap T_2|_{P'} = |T_1|_{P'} \cap |T_2|_{P'}$ and s' is injective. Otherwise, we may assume that $+$ $\in T_1$ without loss of generality. Remark that each of T_1 and T_2 contains at most one of $\pm$. In particular, we have $- \notin T_1$, and hence $|T_1|_Q \subset \overline{U_+}$. Therefore, there are two possible cases to consider.

- (1) If $+$ $\notin T_2$, then $|T_2|_Q \subset \overline{U_-}$. Thus, $|T_1|_Q \cap |T_2|_Q \subset s'(P') \subset \overline{U_+} \cap \overline{U_-}$. Notice that $|T_1|_Q \cap s'(P') = |T_1 \setminus \{+\}|_{Q'}$, and $|T_1|_Q \cap s'(P') = |T_2 \setminus \{-\}|_{Q'}$. Hence, the equality follows from the previous case.
- (2) Suppose $+$ $\in T_2$, and $x \in |T_1|_Q \cap |T_2|_Q$. Denote $T'_i := T_i \setminus +$ for each $i \in \{1, 2\}$. Notice that $|T_i|_{Q'} = \text{conv}(|T'_i|_{P'}, v'_+)$. Thus, there exist $y_i \in |T'_i|_{P'}$ and $\lambda_i \geq 0$, for each $i \in \{1, 2\}$, such that $x = s'(y_1) + \lambda_1 \cdot v'_+ = s'(y_2) + \lambda_2 \cdot v'_+$. Equivalently, $s'(y_1) - s'(y_2) = (\lambda_2 - \lambda_1) \cdot v'_+$. Recall that $p \circ s' = \text{id}_{P'}$. So we have $y_1 - y_2 = (\lambda_1 - \lambda_2)p(v'_+) = 0$. This implies that $y_1 = y_2 \in |T'_1|_{P'} \cap |T'_2|_{P'} = |T'_1 \cap T'_2|_{P'}$, and hence $x \in \text{conv}(|T'_1 \cap T'_2|_{P'}, v'_+) = |T_1 \cap T_2|_{Q'}$. Therefore, the equality is proved.

Now fix $\epsilon \ll 1$. We can choose $r \gg 0$ such that for any $T_1, T_2 \in K$, one has $d(\frac{1}{r}|T_i|_Q - v, \frac{1}{r}|T_i|_Q) < \epsilon$ for each $i \in \{1, 2\}$. Therefore, by Lemma B.4, $|T_1 \cap T_2|_Q = |T_1|_Q \cap |T_2|_Q$. Thus, we proved that Δ_Q is a simplicial polyhedral complex when $r \gg 0$. Hence, $h: \tilde{P} \rightarrow Q$ is a homeomorphism whose inverse is given by the facewise linear extension of the map $v(\mathfrak{s}) \mapsto u(\mathfrak{s})$ to the polyhedral complex Δ_Q . The claim follows. $\square$

• **Step 5. Construction of a continuous map $g: P \rightarrow Q$ such that $g(P) \subset \mathfrak{C} \cap Q$, and $(h^{-1} \circ g)|_{\partial P} = i|_{\partial P}$.**

⁸We will prove Δ_Q is a simplicial polyhedral complex in Claim 11.15.

Recall that Γ is a connected tropical curve. Therefore, when $r \gg 0$, we can take $\mathbf{v}_{\xi,0}, \dots, \mathbf{v}_{\xi,N}$ in the vertex set Γ_0 , for each $\xi \in \Xi_1$, such that $\mathbf{v}_{\xi,0} = \mathbf{v}$, $\mathbf{v}_{\xi,N} = \mathbf{v}_\xi$, and $\mathbf{v}_{\xi,i} \sim \mathbf{v}_{\xi,i+1}$ is an edge in Γ for any $0 \leq i \leq N-1$.

We define a continuous map $g: P \rightarrow Q$ which is facewise linear on Δ'_P by setting $g(\mathbf{u}_{\xi,i}) := \mathbf{v}_{\xi,i}$, and then extend it linearly on each face of Δ'_P .

Let $\sigma \in \Delta'_P$. By Claim 11.14, $\sigma = \text{conv}(\mathbf{u}_{\xi_1,i_1}, \mathbf{u}_{\xi_1,i_1+1}, \mathbf{u}_{\xi_2,i_2}, \mathbf{u}_{\xi_3,i_3}, \dots, \mathbf{u}_{\xi_{n-2},i_{n-2}})$, where $(\xi_1, \xi_2, \dots, \xi_{n-2})$ generate an $(n-2)$ -dimensional cone in Ξ .

For $x \in \sigma$, we have

$$g(x) \in \text{conv}(\mathbf{v}_{\xi_1,i_1}, \mathbf{v}_{\xi_1,i_1+1}, \mathbf{v}_{\xi_2,i_2}, \mathbf{v}_{\xi_3,i_3}, \dots, \mathbf{v}_{\xi_{n-2},i_{n-2}}).$$

Since $\mathbf{v}_{\xi_1,i_1} \sim \mathbf{v}_{\xi_1,i_1+1}$ is an edge in Γ , there exists a point y inside this edge such that

$$g(x) \in \text{conv}(y, \mathbf{v}_{\xi_2,i_2}, \mathbf{v}_{\xi_3,i_3}, \dots, \mathbf{v}_{\xi_{n-2},i_{n-2}}) \subset \mathfrak{C}.$$

This shows that $g(P) \subset \mathfrak{C}$.

It remains to characterize $(h^{-1} \circ g)|_{\partial P}$. Notice that for any $x \in \partial P$, $x \in \tau = \text{conv}(\mathbf{u}_{\xi_1}, \dots, \mathbf{u}_{\xi_k})$ where $\text{Cone}(\xi_1, \dots, \xi_k) \in \Xi$. Then, by definition, $h^{-1} \circ g(\mathbf{u}_{\xi_i}) = \mathbf{u}_{\xi_i}$, and both $g|_\tau$ and $h^{-1}|_{g(\tau)}$ are linear maps. Therefore, $(h^{-1} \circ g)|_\tau$ is an identity map and coincides with $i|_\tau$.

• **Step 6. Concluding Proposition 11.6 by applying Lemma 11.13.**

We choose an infinite edge $\text{Leg}(-\rho_1) \in (-\Gamma')_{1,\infty}$ such that $\text{Leg}(-\rho_1) = x_1 - \rho_1$ for some $x_1 \in -\Gamma'$. Recall that $(\partial \tilde{P})_\pm := \partial \tilde{P} \cap (B \times \mathbb{R}_\pm)$, and $(\partial Q)_\pm := h^{-1}((\partial \tilde{P})_\pm)$.

Claim 11.16. *When $r \gg 0$, $\text{Leg}(-\rho_1) \cap (\partial Q)_+$ is nonempty.*

Proof of Claim 11.16. Recall that in the proof of Claim 11.15, we constructed $Q' \subset A$, $P' \subset B$, and a continuous map $s': P' \rightarrow Q'$ satisfying $p \circ s' = \text{id}_{P'}$. Let $h': \tilde{P} \rightarrow Q'$ be the facewise linear extension of the map $\mathbf{u}(t) \mapsto \mathbf{v}'(t)$ to $\Delta_{\tilde{P}}$. Denote $(\partial Q')_+ := h'((\partial \tilde{P})_+)$. We first show that $-\rho_1 \cap (\partial Q')_+$ is nonempty. One can define a continuous PL function $u'_+: P' \rightarrow \mathbb{R}$ such that $s'(y) + u'_+(y) \cdot e_{\rho_0} \in (\partial Q')_+$ for any $x \in \text{int}(P')$. Remark that $u'_+(0) = 1$, and $u'_+(y) = 0$ for any $y \in \partial P'$.

Let $y'_0 := -\xi_1 \cap \partial P' \subset P'$ where $\xi_1 = p(\rho_1)$, and let I' be the interval connecting 0 and y'_0 . Define linear function $l'_+: I' \rightarrow \mathbb{R}$ such that $s'(y) + l'_+(y) \in -\rho_1$. Recall that in Step 2, we proved $-e_{\rho_1} \in U_+$. This implies that $l'_+(y'_0) > 0$. On the other hand, $l'_+(0) = 0$. Therefore, there exists $y'_1 \in I'$ such that $l'_+(y'_1) = u'_+(y'_1)$, and hence $s'(y'_1) + u'_+(y'_1) \in -\rho_1 \cap (\partial Q')_+$.

Now we prove the claim. Fix $\epsilon \ll 1$, and we assume $r \gg 0$. Recall that in the proof of Claim 11.15, we showed that $d(\frac{1}{r}Q - \mathbf{v}, Q') < \epsilon$, where d denotes the Gromov-Hausdorff distance with respect to the Euclidean metric. Meanwhile, we have $d(\frac{1}{r}\text{Leg}(-\rho_1) - \mathbf{v}, -\rho_1) < \epsilon$. This implies that $d(p(\frac{1}{r}Q - \mathbf{v}), P') < \epsilon$, $d(I, I') < \epsilon$, $\|u_+ - u'_+\|_{C^0} < \epsilon$ and $\|l_+ - l'_+\|_{C^0} < \epsilon$, where I, u_+, l_+ is defined with respect to Q in a similar way that I', u'_+, l'_+ is defined with respect to Q' . Thus, we can conclude that when $r \gg 0$, there exists $y_1 \in I$ such that $u_+(y_1) = l_+(y_1)$, and $\text{Leg}(-\rho_1) \cap (\partial Q)_+$ is nonempty. $\square$

Now we finish the proof of Proposition 11.6. By Claim 11.16, there exists $\mathbf{w}_+ \in \text{Leg}(-\rho_1) \cap (\partial Q)_+$. Since Γ' is a connected tropical curve, when $r \gg 0$, there exists a continuous path $\gamma': [0, 1] \rightarrow Q \cap (-\Gamma')$ such that $\gamma'(0) = \mathbf{w}_+$ and $\gamma'(1) = \mathbf{v}_-$.

We define continuous maps $f: P \rightarrow \tilde{P}$ by $f := h^{-1} \circ g$, and $\gamma: [0, 1] \rightarrow \tilde{P}$ by $\gamma := h^{-1} \circ \gamma'$. Recall that we proved $f|_{\partial P} = i|_{\partial P}$ in Step 5. Furthermore, $\gamma(0) = h^{-1}(\mathbf{w}_+) \in (\partial \tilde{P})_+$; and by definition, $\gamma(1) = \mathbf{u}_- \in (\partial \tilde{P})_-$. Therefore, by applying Lemma 11.13 to f and γ , there exists

$x' \in \tilde{P}$ such that $x' \in f(P) \cap \gamma([0, 1])$. Hence, $h(x') \in g(P) \cap \gamma'([0, 1]) \subset \mathfrak{C} \cap (-\Gamma')$. Thus, we conclude that $\mathfrak{C} \cap (-\Gamma')$ is nonempty, and hence the proposition is proven.

12. RELATION TO NON-ARCHIMEDEAN SOLUTIONS

In this section, $\mathbb{K} = \mathbb{C}((t))$ denotes the valued field of Laurent power series equipped with the t -adic valuation, and $\mathbb{K}^\circ$ denotes its valuation ring. We have $\kappa := \mathbb{K}/\mathbb{K}^\circ \cong \mathbb{C}$. We fix a smooth projective variety $\mathbf{X}$ over $\mathbb{K}$, and we use $\mathbf{X}^{\text{an}}$ to denote its Berkovich analytification over $\mathbb{K}$.

12.1. SNC models and dual complexes. A *model* $\mathfrak{X}$ of $\mathbf{X}$ is a flat and proper scheme $\mathfrak{X}$ over $\text{Spec}(\mathbb{K}^\circ)$ together with an identification of its generic fibre with $\mathbf{X}$. We say that $\mathfrak{X}'$ *dominates* $\mathfrak{X}$ if there exists a proper birational morphism $\pi: \mathfrak{X}' \rightarrow \mathfrak{X}$ over $\text{Spec} \mathbb{K}^\circ$ satisfying $\pi|_{\mathfrak{X}} = \text{id}_{\mathfrak{X}}$.

Let $\mathfrak{X}$ be a model of $\mathbf{X}$. We view the special fibre $\mathfrak{X}_s = \text{div}(t)$ as a Cartier divisor of $\mathfrak{X}_s$. Let $\{E_i\}_{i \in I}$ be the finite set of irreducible component of $\mathfrak{X}_s$. Denote $E_J := \bigcap_{j \in J} E_j$ for any subset $J \subset I$. Following the convention in [BFJ16b], we say that $\mathfrak{X}$ is an SNC model if the special fiber $\mathfrak{X}_s$ is a simple normal crossing divisor in $\mathfrak{X}$, and E_J is either irreducible or empty for each subset $J \subset I$. It follows from [Tem08] that for any model $\mathfrak{X}$ of $\mathbf{X}$, there exists an SNC model $\mathfrak{X}'$ dominating $\mathfrak{X}$.

Let $\mathfrak{X}$ be an SNC model. Suppose $\mathfrak{X}_s = \text{div}(t) = \sum_{i \in I} b_i E_i$ with $b_i \in \mathbb{Z}_{>0}$. The *dual complex* $\Delta_{\mathfrak{X}}$ of $\mathfrak{X}$ is the simplicial complex whose set of vertices equals to I , and whose simplices are subsets $J \subset I$ such that $E_J \neq \emptyset$. We associate to each such J , the simplex σ_J consisting of the tuples $w = (w_j)_{j \in J}$ with $w_j \geq 0$ and $\sum_{j \in J} b_j w_j = 1$. Geometrically, $\Delta_{\mathfrak{X}}$ is realized as the union of σ_J with $E_J \neq \emptyset$ in $\mathbb{R}^I$. By an abuse of notation, we use the same symbol $\Delta_{\mathfrak{X}}$ for this realization.

Let $J \subset I$ be a simplex in $\Delta_{\mathfrak{X}}$ and take $w = (w_j)_{j \in J} \in \sigma_J$. Let ξ_J be the generic point of E_J . Take a system of local coordinates $\{z_j\}_{j \in J} \subset \mathcal{O}_{\mathfrak{X}, \xi_J}$ with z_j defining E_j around ξ_J . By Cohen's structure theorem, $\hat{\mathcal{O}}_{\mathfrak{X}, \xi_J} \simeq \kappa(\xi_J)[[z_j, j \in J]]$, where $\kappa(\xi_J)$ denote the residue field of $\mathcal{O}_{\mathfrak{X}, \xi_J}$. Under this identification, denote by $v_{J,w}$ the restriction to $\mathcal{O}_{\mathfrak{X}, \xi_J} \subset \hat{\mathcal{O}}_{\mathfrak{X}, \xi_J}$ of the monomial valuation with weight w_j on the power series ring, i.e.

$$v_{J,w} \left(\sum_{\alpha \in \mathbb{N}^J} c_\alpha z^\alpha \right) = \min \left\{ \sum_{j \in J} w_j \alpha_j \mid c_\alpha \neq 0 \right\}, \quad \text{where } \alpha = (\alpha_j)_{j \in J}.$$

On the other hand, we have $t = u \cdot \prod_{j \in J} z_j^{b_j}$ in $\mathcal{O}_{\mathfrak{X}, \xi_J}$, where u is a unit in $\mathcal{O}_{\mathfrak{X}, \xi_J}$. Thus, we get $v_{J,w}(t) = 1$, and therefore $v_{J,w}|_{\mathbb{K}} = v_{\mathbb{K}}$. This implies that $v_{J,w} \in \mathbf{X}^{\text{an}}$, and hence yields an embedding of $\text{emb}_{\mathfrak{X}}: \Delta_{\mathfrak{X}} \rightarrow \mathbf{X}^{\text{an}}$, for the geometric realization of $\Delta_{\mathfrak{X}}$. Moreover, the parameters w equip $\Delta_{\mathfrak{X}}$ with an integral affine structure.

Now, we recall the retraction map $p_{\mathfrak{X}}: \mathbf{X}^{\text{an}} \rightarrow \Delta_{\mathfrak{X}}$. Let $x \in \mathbf{X}^{\text{an}}$, and let v_x denote the semivaluation defined by x . Then, the *center* of v_x , denoted by $c_{\mathfrak{X}}(x)$, is defined to be the unique (scheme) point $\xi \in \mathfrak{X}_s$ satisfying $v_x(f) \geq 0$ for any $f \in \mathcal{O}_{\mathfrak{X}, \xi}$, and $v_x(f) > 0$ for any $f \in \mathfrak{m}_{\mathfrak{X}, \xi}$. The valuative criterion of properness guarantees the existence and the uniqueness of $c_{\mathfrak{X}}(x)$. Let $J \subset I$ be the maximal subset such that $c_{\mathfrak{X}}(x) \in E_J$. Then $p_{\mathfrak{X}}(x) \in \sigma_J \subset \Delta_{\mathfrak{X}}$ corresponds to the monomial valuation with weight $v_x(z_j)$, $j \in J$.

By definition, $p_{\mathfrak{X}}|_{\Delta_{\mathfrak{X}}} = \text{id}$. If $\mathfrak{X}'$ is an SNC model dominating $\mathfrak{X}$, then we have $\Delta_{\mathfrak{X}} \subset \Delta_{\mathfrak{X}'}$, and $p_{\mathfrak{X}} \circ p_{\mathfrak{X}'} = p_{\mathfrak{X}}$. We recall the following result.

Proposition 12.1 ([BFJ16b, Cor. 2.5]). $\mathbf{X}^{\text{an}}$ is homeomorphic to $\varprojlim_{\mathfrak{X}} \Delta_{\mathfrak{X}}$. Here, $\mathfrak{X}$ ranges over all SNC models of $\mathbf{X}$.

12.2. Non-Archimedean Calabi–Yau theorem. In [BFJ15, BJ22], a non-Archimedean pluripotential theory has been developed. We begin by introducing very briefly the conventions from this theory that will be used throughout this section. For further details on these conventions, we refer to [APW26, § 6], and for a more comprehensive introduction to the theory, we refer to [BFJ16a].

Let $\mathbf{L}$ be a $\mathbb{Q}$ -line bundle on $\mathbf{L}$. A *model* of $(\mathbf{X}, \mathbf{L})$ is a pair $(\mathfrak{X}, \mathfrak{L})$ consisting of a model $\mathfrak{X}$ of $\mathbf{X}$ and a $\mathbb{Q}$ -line bundle $\mathfrak{L}$ on $\mathfrak{X}$ whose restriction on $\mathbf{X}$ (as the generic fibre of $\mathfrak{X}$) is isomorphic to $\mathbf{L}$. We say that $(\mathfrak{X}', \mathfrak{L}')$ *dominates* $(\mathfrak{X}, \mathfrak{L})$ if $\mathfrak{X}'$ dominates $\mathfrak{X}$ by a birational proper morphism $\pi: \mathfrak{X}' \rightarrow \mathfrak{X}$, and $\mathfrak{L}' = \pi^* \mathfrak{L}$. We say that $(\mathfrak{X}', \mathfrak{L}')$ and $(\mathfrak{X}'', \mathfrak{L}'')$ are *equivalent* if there exists a model $(\mathfrak{X}, \mathfrak{L})$ dominating both $(\mathfrak{X}', \mathfrak{L}')$ and $(\mathfrak{X}'', \mathfrak{L}'')$. A model $(\mathfrak{X}, \mathfrak{L})$ is called *nef* if $\mathfrak{L}$ is relatively nef on $\mathfrak{X}/\text{Spec } \mathbb{K}^\circ$.

A *closed* $(1, 1)$ -form is an equivalence class of models of $(\mathbf{X}, \mathbf{L})$ for some $\mathbf{L} \in \text{Pic}_{\mathbb{Q}}(\mathbf{X})$. A closed $(1, 1)$ -form ω_{NA} is called *semipositive* if some (equivalently, any) model representing ω_{NA} is nef. The first Chern class $c_1(\omega_{\text{NA}}) \in N^1(\mathbf{X})$ is defined to be $c_1(\mathfrak{L}|_{\mathbf{X}}) \in N^1(\mathbf{X})$ for some model $(\mathfrak{X}, \mathfrak{L})$ representing ω_{NA} .

Let ω_{NA} be a closed semipositive $(1, 1)$ -form on $\mathbf{X}^{\text{an}}$. For any PL function φ on $\mathbf{X}^{\text{an}}$ (denoted by $\varphi \in \text{PL}(\mathbf{X}^{\text{an}})$), one can associate a closed $(1, 1)$ -form $\omega_{\text{NA}, \varphi}$ on $\mathbf{X}^{\text{an}}$, which serves as the non-Archimedean analogue of $\omega_{\text{NA}} + dd^c \varphi$ (see [APW26, § 6.2]). A PL function φ on $\mathbf{X}^{\text{an}}$ is called ω_{NA} -psh (written $\varphi \in \text{PSH}_{\omega, \text{NA}} \cap \text{PL}(\mathbf{X}^{\text{an}})$) if $\omega_{\text{NA}, \varphi}$ is semipositive.

Following [BJ22, APW26], when $c_1(\omega_{\text{NA}}) \in N^1(\mathbf{X})$ is an ample class, a function $\varphi: \mathbf{X}^{\text{an}} \rightarrow \mathbb{R} \cup \{-\infty\}$ is called ω -psh (written $\varphi \in \text{PSH}_{\omega, \text{NA}}(\mathbf{X}^{\text{an}})$) if φ is generically finite (*i.e.* φ is not constantly equal to $-\infty$ on each connected component of $\mathbf{X}^{\text{an}}$) and φ can be written as the pointwise limit of a decreasing sequence of ω_{NA} -psh PL functions. In general, when $c_1(\omega_{\text{NA}}) \in N^1(\mathbf{X})$ is a nef class, we say that $\varphi \in \text{PSH}_{\omega, \text{NA}}(\mathbf{X}^{\text{an}})$ if $\varphi \in \text{PSH}_{\omega + \epsilon \theta, \text{NA}}(\mathbf{X}^{\text{an}})$ for each $\epsilon \in \mathbb{Q}_{>0}$, where θ_{NA} is a closed semipositive $(1, 1)$ -form on $\mathbf{X}^{\text{an}}$ such that $c_1(\theta_{\text{NA}})$ is ample. We use $\text{CPSH}_{\omega, \text{NA}}(\mathbf{X}^{\text{an}})$ to denote the space of continuous ω_{NA} -psh functions on $\mathbf{X}^{\text{an}}$.

Following [CL06], for $\varphi \in \text{PSH}_{\omega, \text{NA}} \cap \text{PL}(\mathbf{X}^{\text{an}})$, the non-Archimedean Monge–Ampère measure $\text{MA}_{\omega, \text{NA}}(\varphi)$ is defined in terms of intersection numbers on a model representing ω_{NA} . Using the Chern–Levine–Nirenberg inequality, one can extend the definition of the non-Archimedean Monge–Ampère measure $\text{MA}_{\omega, \text{NA}}(\varphi)$ to $\varphi \in \text{CPSH}_{\omega, \text{NA}}(\mathbf{X}^{\text{an}})$. This measure is a nonnegative Radon measure on $\mathbf{X}^{\text{an}}$ with total mass $\deg(\mathbf{L})^d$.

We now recall the non-Archimedean Calabi–Yau theorem.

Theorem 12.2 (NA Calabi–Yau Theorem, [BFJ15, BJ22]). *Let $\mathbf{X}$ be a smooth polarized variety over $\mathbb{K}$, and let ω_{NA} be a closed semipositive $(1, 1)$ -form on $\mathbf{X}^{\text{an}}$ such that $c_1(\omega_{\text{NA}})$ is an ample class on $\mathbf{X}$. Let $\mathfrak{X}$ be an SNC model of $\mathbf{X}$. Let μ be a nonnegative Radon measure supported on $\Delta_{\mathfrak{X}}$ of total volume $\deg(\mathbf{L})^d$. Then, there exists $\varphi \in \text{CPSH}_{\omega, \text{NA}}(\mathbf{X}^{\text{an}})$, unique up to a constant, such that*

$$\text{MA}_{\omega, \text{NA}}(\varphi) = \mu.$$

We also recall the following result.

Proposition 12.3 ([BJ22, Lem. 5.13 + Thm. 5.20]). *Let $\mathbf{X}$ be a smooth polarized variety over $\mathbb{K}$, and let ω_{NA} be a closed semipositive $(1, 1)$ -form on $\mathbf{X}^{\text{an}}$ such that $c_1(\omega_{\text{NA}})$ is an ample class*

on $\mathbf{X}$. Suppose that $\pi: \tilde{\mathbf{X}} \rightarrow \mathbf{X}$ be a proper birational morphism. Then, $\pi^*: \text{PSH}_{\omega, \text{NA}}(\mathbf{X}^{\text{an}}) \rightarrow \text{PSH}_{\pi^*\omega, \text{NA}}(\tilde{\mathbf{X}}^{\text{an}})$ is bijective.

12.3. Comparison Theorem. Let $\iota: \mathbf{X} \hookrightarrow \mathbb{P}_{\Xi}$ be an embedding of $\mathbf{X}$ into a projective toric variety $\mathbb{P}_{\Xi}$. Denote $\mathbf{Y} := \mathbf{X} \cap \mathbf{T}$, where $\mathbf{T}$ is the open torus orbit in $\mathbb{P}_{\Xi}$. Let $X = \text{Trop}(\mathbf{X}, \iota) \subset \text{TP}_{\Xi}$ and $Y = \text{Trop}(\mathbf{Y}, \iota) \subset \mathbf{T}$ be the tropicalization of $\mathbf{X}$ and $\mathbf{Y}$, respectively, where TP_{Ξ} is the tropical toric variety associated to Ξ , and $\mathbf{T}$ is its open tropical torus orbit.

Let $\mathbf{L}$ be an ample toric line bundle over $\mathbb{P}_{\Xi}$, and let h be the convex conewise linear function on Ξ associated to $\mathbf{L}$. The trivial model $((\mathbb{P}_{\Xi})_{\mathbb{K}^{\circ}}, \mathbf{L}_{\mathbb{K}^{\circ}})$ of $(\mathbb{P}_{\Xi}, \mathbf{L})$ then induces a non-Archimedean closed semipositive $(1, 1)$ -form on $\mathbb{P}_{\Xi}^{\text{an}}$, whose restriction to $\mathbf{X}^{\text{an}}$ is denoted by ω_{NA} .

Theorem 12.4. Suppose that there exists a bounded function $\varphi_o: Y \rightarrow \mathbb{R}$ such that $\varphi_o + h \in \text{LConv}(Y)$. Then, $\varphi_o \circ \text{Trop}$ can be extended to $\varphi_{\text{NA}} \in \text{CPSH}_{\omega, \text{NA}}(\mathbf{X}^{\text{an}})$. Furthermore, $\text{MA}_{\omega, \text{NA}}(\varphi_{\text{NA}})$ has zero mass on $\mathbf{X}^{\text{an}} \setminus \mathbf{Y}^{\text{an}}$, and $\text{Trop}_*(\text{MA}_{\omega, \text{NA}}(\varphi_{\text{NA}})) = \text{MA}_{\text{Trop}}(\varphi_o + h)$ on Y .

Proof. By Corollary 5.9, there exist a refinement $\tilde{\Xi}$ of Ξ and a polyhedral structure Π on Y such that $\text{TP}_{\tilde{\Xi}}$ is a projective tropical toric variety, $\text{rec}(\Pi)$ is a subfan of $\tilde{\Xi}$, and the closure of Y in $\text{TP}_{\tilde{\Xi}}$ denoted $\tilde{X}$ is a tropical subvariety of $\text{TP}_{\tilde{\Xi}}$. Then, $\tilde{X}$ is a tropical variety in the sense of this paper. Let $\tilde{\mathbf{X}}$ denote the closure of $\mathbf{Y}$ in $\mathbb{P}_{\tilde{\Xi}}$, and $\tilde{\iota}: \tilde{\mathbf{X}} \hookrightarrow \mathbb{P}_{\tilde{\Xi}}$ the corresponding embedding. Let $\pi: \mathbb{P}_{\tilde{\Xi}} \rightarrow \mathbb{P}_{\Xi}$ be the toric birational morphism. Then, we have $\iota \circ \pi = \pi \circ \tilde{\iota}$. Note that $\tilde{X} = \text{Trop}(\tilde{\mathbf{X}}, \tilde{\iota})$, with the open part $\tilde{X}_o = Y$.

Remark that h is conewise linear on $\tilde{\Xi}$. By Example 4.9, $\tilde{\omega} := \partial\bar{\partial}h$ is a semipositive simplicial form on $\tilde{X}$. Let g be a strictly convex conewise linear function on $\tilde{\Xi}$. By Proposition 5.7, $[\tilde{\theta}]$ is a tropical Kähler class on $\tilde{X}$ for $\tilde{\theta} := \partial\bar{\partial}g$. Since φ_o is bounded on Y and $\varphi_o + h \in \text{LConv}(Y)$, by Theorem 4.8, φ_o extends uniquely to $\tilde{\varphi} \in \text{CPSH}_{\tilde{\omega}}(\tilde{X})$. Let $\{\alpha_i\}_{i \geq 1}$ be a decreasing sequence of positive numbers $\{\alpha_i\}_{i \geq 1}$ such that $\lim_{i \rightarrow +\infty} \alpha_i = 0$. Denote $\tilde{\theta}_i := \tilde{\omega} + \alpha_i \tilde{\theta}$ for each $i \geq 1$. Now, applying Theorem 5.13, we obtain a sequence of functions $\{\tilde{\varphi}_i\}_{i \geq 1} \subset \text{CPSH}_{\tilde{\theta}_i} \cap \text{PL}(\tilde{X})$ such that $\tilde{\varphi}_i$ converges uniformly to $\tilde{\varphi}$ on $\tilde{X}$.

Let $\tilde{\mathbf{L}}_h$ and $\tilde{\mathbf{L}}_g$ denote the nef toric line bundles on $\mathbb{P}_{\tilde{\Xi}}$ associated to h and g respectively. Let $\tilde{\omega}_{\text{NA}}, \tilde{\theta}_{\text{NA}}$ denote their corresponding closed semipositive $(1, 1)$ -forms on $\tilde{\mathbf{X}}^{\text{an}}$ respectively. Then, by [APW26, Thm. 6.13], we have for each $i \geq 1$, $\tilde{\varphi}_i \circ \text{Trop} \in \text{PSH}_{\tilde{\theta}_i, \text{NA}} \cap \text{PL}(\tilde{\mathbf{X}}^{\text{an}})$, and $\text{Trop}_*(\text{MA}_{\tilde{\theta}_i, \text{NA}}(\tilde{\varphi}_i \circ \text{Trop})) = \text{MA}_{\tilde{\theta}_i, \text{Trop}}(\tilde{\varphi}_i)$. Then, we define $\tilde{\varphi}_{\text{NA}} := \tilde{\varphi} := \lim_{i \rightarrow +\infty} \tilde{\varphi}_i \circ \text{Trop} \in \text{CPSH}_{\tilde{\omega}, \text{NA}}(\tilde{\mathbf{X}}^{\text{an}})$. By construction, we have $\tilde{\varphi}_{\text{NA}}|_{\mathbf{Y}^{\text{an}}} = \varphi_o \circ \text{Trop}$. Moreover, the non-Archimedean Bedford–Taylor theory [BJ22, Thm. 7.18] combined with the tropical analogue (Theorem 6.2) imply that $\text{Trop}_*(\text{MA}_{\tilde{\omega}, \text{NA}}(\tilde{\varphi}_{\text{NA}})) = \text{MA}_{\tilde{\omega}, \text{Trop}}(\tilde{\varphi})$ on $\tilde{X}$.

Notice that $\tilde{\mathbf{L}}_h = \pi^*\mathbf{L}$, and $\tilde{\omega}_{\text{NA}} = \pi^*\omega_{\text{NA}}$. Then, applying Proposition 12.3 to $\pi: \tilde{\mathbf{X}} \rightarrow \mathbf{X}$, there exists $\varphi_{\text{NA}} \in \text{CPSH}_{\omega, \text{NA}}(\mathbf{X}^{\text{an}})$ such that $\varphi_{\text{NA}} \circ \pi = \tilde{\varphi}_{\text{NA}}$. Since $\pi|_{\mathbf{Y}} = \text{id}_{\mathbf{Y}}$, and the open tropicalizations of both $\tilde{\mathbf{X}}$ and $\mathbf{X}$ coincides with $Y = \text{Trop}(\mathbf{Y}, \iota_o)$, we get $\varphi_o \circ \text{Trop} = \varphi_{\text{NA}}$. By [BJ22, Cor. 7.49], $\text{MA}_{\omega, \text{NA}}(\varphi_{\text{NA}})$ and $\text{MA}_{\omega, \text{NA}}(\tilde{\varphi}_{\text{NA}})$ has zero mass on $\mathbf{X}^{\text{an}} \setminus \mathbf{Y}^{\text{an}}$ and $\tilde{\mathbf{X}}^{\text{an}} \setminus \mathbf{Y}^{\text{an}}$ respectively. Since $\pi_*\text{MA}_{\omega, \text{NA}}(\tilde{\varphi}_{\text{NA}}) = \text{MA}_{\omega, \text{NA}}(\varphi_{\text{NA}})$, we have $\text{Trop}_*(\text{MA}_{\omega, \text{NA}}(\varphi_{\text{NA}})) = \text{MA}_{\omega, \text{Trop}}(\varphi_o)$ on Y . $\square$

12.4. Strongly faithful tropicalization. In the remaining part of this section, we say that a polyhedron is of *maximal dimension* if its dimension equals to $\dim_{\mathbb{K}}(\mathbf{X})$.

Definition 12.5. Let $\Delta_{\mathfrak{X}}$ be the dual complex of an SNC model $\mathfrak{X}$ of $\mathbf{X}$ with $\dim(\Delta_{\mathfrak{X}}) = \dim_{\mathbb{K}}(\mathbf{X})$. Let $\{f_1, \dots, f_k\} \subset \mathbb{K}(\mathbf{X})$ be a finite set of rational functions, and let

$$f = (f_1, \dots, f_k): \mathbf{X} \dashrightarrow \mathbf{G}_m^k$$

be the corresponding rational map. We say that the tropicalization

$$\mathrm{Trop}_f: \mathbf{X}^{\mathrm{val}} \rightarrow \mathrm{Trop}(\mathbf{X}, f) \subset \mathbb{R}^k$$

is *strongly faithful* with respect to $\Delta_{\mathfrak{X}}$ if $(\mathrm{Trop}_f)|_{\Delta_{\mathfrak{X}}}$ is injective, and $\varpi(\delta) = 1$ for any maximal dimension face $\delta \subset (\mathrm{Trop}_f)(\Delta_{\mathfrak{X}})$.

Remark 12.6. In the above definition, we do not require Trop_f be affine on each maximal dimensional face $\delta \subset \Delta_{\mathfrak{X}}$. In general, Trop_f is only piecewise affine on δ . $\diamond$

Remark 12.7. This definition of strongly faithful tropicalization is a refinement of the one considered in [GRW16]. However, these two notions are closely related. This is justified in the proof of the following proposition. $\diamond$

Theorem 12.8. *Let $\mathfrak{X}$ be an SNC model of $\mathbf{X}$, and $\Delta_{\mathfrak{X}}$ be the dual complex of $\mathfrak{X}$. Suppose that $\dim(\Delta_{\mathfrak{X}}) = \dim_{\mathbb{K}}(\mathbf{X})$. Then, there exists a finite subset $\{f_1, \dots, f_k\} \subset \mathbb{K}(\mathbf{X})$ such that Trop_f is a strongly faithful tropicalization with respect to $\Delta_{\mathfrak{X}}$ for $f = (f_1, \dots, f_k)$.*

Proof. Suppose that $\dim \mathbf{X} = d$. Let $\{s_0, s_1, \dots, s_r\}$ be a finite subset of $H^0(\mathfrak{X}, \mathcal{L}^{\otimes k})$ with $k \in \mathbb{N}$ such that $\iota = [(s_1)_{\mathbb{K}} : \dots : (s_r)_{\mathbb{K}}]: \mathbf{X} \rightarrow \mathbb{P}_{\mathbb{K}}^{r-1}$ is a closed embedding. We use $(\Delta_{\mathfrak{X}})_d$ to denote the set of d -dimensional faces of $\Delta_{\mathfrak{X}}$. For any $\delta \in (\Delta_{\mathfrak{X}})_d$ corresponding to a 0-dimensional stratum $\{x\} = \bigcap_{i \in I_{\delta}} E_i$ with I_{δ} a finite index set of cardinality $d+1$, we take $\{g_{\delta,i}\}_{i \in I_{\delta}}$ a finite subset of $\mathbb{K}(\mathbf{X}) = \mathbb{K}(\mathfrak{X})$ such that $g_{\delta,i}$ defines E_i in $\mathcal{O}_{\mathfrak{X},x}$ for each $i \in I_{\delta}$. Define the finite subset g of $\mathbb{K}(\mathbf{X})$ by

$$g := \{s_1/s_0, \dots, s_r/s_0, g_{\delta,i} \mid \delta \in (\Delta_{\mathfrak{X}})_d, i \in I_{\delta}\}.$$

Let $\{\mathcal{H}_i\}_{i \in I}$ be the set of irreducible components of $\bigcup_{g_k \in g} \mathrm{div}(g_k) \subset \mathfrak{X}$. Denote $\mathcal{H} := \sum_{i \in I} \mathcal{H}_i$ and $\mathbf{H} := \mathcal{H} \cap \mathbf{X}$. Slightly abuse of notation, we also denote by $g: \mathbf{X} \dashrightarrow \mathbf{T}$ the rational map defined by g , where $\mathbf{T}$ denotes a torus. Then, g is a closed embedding on $\mathbf{X} \setminus \mathrm{Supp}(\mathbf{H})$. Also, $\mathrm{Trop}_g|_{\delta}$ is injective for any $\delta \in (\Delta_{\mathfrak{X}})_d$.

By [Tem08], there exists a proper birational morphism $\pi: \mathfrak{X}' \rightarrow \mathfrak{X}$ such that $\pi^{-1}(\mathrm{Supp}(\mathcal{H}) \cup \mathfrak{X}_s) = \sum_{i \in I'} \mathcal{H}'_i + \mathfrak{X}'_s$ is SNC, and π is an isomorphism on $\pi^{-1}(\mathfrak{X} \setminus (\mathrm{Supp}(\mathcal{H}) \cup \mathfrak{X}_s))$. Let $\mathbf{X}' := \mathfrak{X}'_{\mathbb{K}}$, and $\mathbf{H}'_i := \mathcal{H}'_i \cap \mathbf{X}'$. Then, $g: \mathbf{X}' \setminus \mathrm{Supp}(\mathbf{H}') = \mathbf{X} \setminus \mathrm{Supp}(\mathbf{H}) \hookrightarrow \mathbf{T}$ is a closed embedding. On the other hand, $\Delta_{\mathfrak{X}} \subset \Delta_{\mathfrak{X}'}$ inside $\mathbf{X}^{\mathrm{val}} = \mathbf{X}'^{\mathrm{val}}$.

In [GRW16], the authors introduced an integral affine polyhedral complex denoted $S(\mathfrak{X}', \mathcal{H}')$ for the SNC pair $(\mathfrak{X}', \mathcal{H}')$. Furthermore, there is an inclusion map $\mathrm{emb}_{\mathfrak{X}', \mathcal{H}'}: S(\mathfrak{X}', \mathcal{H}') \hookrightarrow \mathbf{X}'^{\mathrm{val}}$, and a retraction map $r_{\mathfrak{X}', \mathcal{H}'}: \mathbf{X}'^{\mathrm{val}} \rightarrow (\mathfrak{X}', \mathcal{H}')$. By the construction, $\Delta_{\mathfrak{X}}$ is a subcomplex of $S(\mathfrak{X}', \mathcal{H}')$, and $(\mathrm{emb}_{\mathfrak{X}', \mathcal{H}'})|_{\Delta_{\mathfrak{X}}} = \mathrm{emb}_{\mathfrak{X}}$.

We say a tropicalization Trop_f is unimodular on $S(\mathfrak{X}', \mathcal{H}')$ if there exists a rational polyhedral subdivision $S''(\mathfrak{X}', \mathcal{H}')$ of $S(\mathfrak{X}', \mathcal{H}')$ such that Trop_f is a unimodular integral affine map on each maximal dimensional face in $S''(\mathfrak{X}', \mathcal{H}')$. By Theorem 9.5 in *loc. cit.*, there exists a finite subset $\{f_1, \dots, f_m\} \subset \mathbb{K}(\mathbf{X})$ containing g such that (Trop_f) is injective and unimodular on $S(\mathfrak{X}', \mathcal{H}')$.

By Proposition 8.2 in [GRW16], $\text{Trop}|_{S(\mathfrak{X}', \mathcal{H}')} : S(\mathfrak{X}', \mathcal{H}') \rightarrow \text{Trop}(\mathbf{X}, g)$ is surjective. Let $\pi_g : \text{Trop}(\mathbf{X}, f) \rightarrow \text{Trop}(\mathbf{X}, g)$ be the projection map to the components corresponding to g . Then we have the following commutative diagram

$$\begin{array}{ccc} & & \text{Trop}(\mathbf{X}, f) \\ & \nearrow \text{Trop}_f & \downarrow \pi_g \\ S(\mathfrak{X}', \mathcal{H}') & \xrightarrow{\text{Trop}_g} & \text{Trop}(\mathbf{X}, g) \end{array}$$

Now we fix a rational polyhedral subdivision $S'(\mathfrak{X}', \mathcal{H}')$ of $S(\mathfrak{X}', \mathcal{H}')$ such that $(\text{Trop}_f)|_{S'(\mathfrak{X}', \mathcal{H}')}$ is a facewise unimodular integral affine. We also fix integral polyhedral structures Π_f and Π_g on $\text{Trop}(\mathfrak{X}, f)$ and $\text{Trop}(\mathfrak{X}, g)$, respectively, such that for any $\delta \in \Pi_g$ and any $\delta' \in \Pi_f$, the preimage of δ in $\text{Trop}(\mathbf{X}, f)$ and the preimage of δ' in $S'(\mathfrak{X}', \mathcal{H}')$ are both unions of faces. Then, for any maximal dimensional face $\delta \in \text{Trop}(\mathbf{X}, g)$, we use $\pi_g^{-1}(\delta)$ to denote the set of maximal dimensional faces of $\text{Trop}(\mathbf{X}, f)$ contained in the preimage of δ , and $\text{Trop}_g^{-1}(\delta)$ to denote the set of maximal dimensional faces of $S'(\mathfrak{X}', \mathcal{H}')$ contained in the preimage of δ .

For any maximal dimensional face δ in Π , the skeletal Sturm–Tevelev multiplicity formula [GRW16, Thm. 8.4] implies that

$$\varpi_{\text{Trop}(\mathbf{X}, g)}(\delta) = \sum_{\delta_{S'} \in \text{Trop}_g^{-1}(\delta)} [N_\delta : N_{\delta_{S'}}].$$

On the other hand, applying the tropical Sturm–Tevelev multiplicity formula [ST08, Thm. 1.1] to π_g , we get

$$\begin{aligned} \varpi_{\text{Trop}(\mathbf{X}, g)}(\delta) &= \sum_{\delta' \in \pi_g^{-1}(\delta)} \varpi_{\text{Trop}(\mathbf{X}, f)}(\delta') [N_\delta : N_{\delta'}] \\ &= \sum_{\delta_{S'} \in \text{Trop}_g^{-1}(\delta)} \varpi_{\text{Trop}(\mathbf{X}, f)}(\text{Trop}_f(\delta_{S'})) [N_\delta : N_{\delta_{S'}}] \\ &\quad + \sum_{\delta'' \in \pi_g^{-1}(\delta) \setminus \text{Trop}_f(\text{Trop}_g^{-1}(\delta))} \varpi_{\text{Trop}(\mathbf{X}, f)}(\delta'') [N_\delta : N_{\delta''}]. \end{aligned}$$

By comparing the above two formulas, we get

$$\text{Trop}_f(\text{Trop}_g^{-1}(\delta)) = \pi_g^{-1}(\delta), \quad \text{and} \quad \varpi_{\text{Trop}(\mathbf{X}, f)}(\text{Trop}_f(\delta_{S'})) = 1 \text{ for any } \delta_{S'} \subset \text{Trop}_g^{-1}(\delta).$$

For any d -dimensional face $\delta_{S'} \in S'(\mathfrak{X}', \mathcal{H}')$ such that $\delta_{S'}$ is contained in $\Delta_{\mathfrak{X}}$, we have

$$\varpi_{\text{Trop}(\mathbf{X}, f)}(\text{Trop}_f(\delta_{S'})) = 1$$

since Trop_g is injective on $\delta_{S'}$. In summary, we have proven that $(\text{Trop}_f)|_{\Delta_{\mathfrak{X}}}$ is injective, and $\varpi_{\text{Trop}_f}(\delta') = 1$ for any maximal dimensional face $\delta' \subset \text{Trop}_f(\Delta_{\mathfrak{X}})$. Therefore, Trop_f is a strongly faithful tropicalization with respect to $\Delta_{\mathfrak{X}}$. $\square$

12.5. Tropical Monge–Ampère solutions and the comparison property. We recall the following definition given in [Li23a].

Definition 12.9. Let $\varphi \in C^0(\mathbf{X}^{\text{an}})$. We say that φ satisfies the *comparison property* if there exists an SNC model $\mathfrak{X}$ of $\mathbf{X}$ such that $\dim(\Delta_{\mathfrak{X}}) = \dim_{\mathbb{K}}(\mathbf{X})$, and $\varphi = \varphi \circ r_{\mathfrak{X}}$ on $r_{\mathfrak{X}}^{-1}(\text{relint}(\delta))$ for each maximal dimensional face δ in $\Delta_{\mathfrak{X}}$.

The comparison property was motivated by the study of the metric SYZ conjecture. A weaker version of this property was established in [Li26] for the generic fibres of maximally degenerating families of polarized Calabi–Yau manifolds, leading to a proof of the metric SYZ conjecture.

The following result shows that every continuous function on $\mathbf{X}^{\text{an}}$ which factors through a tropicalization satisfies the comparison property.

Proposition 12.10. *Let $\varphi \in C^0(\mathbf{X}^{\text{an}})$. Suppose that there exists a rational map*

$$f = (f_1, \dots, f_k): \mathbf{X} \dashrightarrow \mathbf{G}_m^k$$

such that $\varphi = \varphi \circ \text{Trop}_f$ on $\mathbf{X}^{\text{val}}$. Then, φ satisfies the retraction invariance property.

Proof. Let $\dim(\mathbf{X}) = d$. Fix an SNC model $\mathfrak{X}$ of $\mathbf{X}$ with $\dim(\Delta_{\mathfrak{X}}) = d$. Let δ be a d -dimensional face of $\Delta_{\mathfrak{X}}$ corresponding to a 0-dimensional strata $\{x\} = \bigcap_{0 \leq i \leq d} E_i$. Let $z_i \in \hat{\mathcal{O}}_{\mathfrak{X},x}$ such that E_i is defined locally by $z_i = 0$. Suppose that $\mathbf{t} = z_0^{b_0} z_1^{b_1} \cdots z_d^{b_d}$ for $b_0, \dots, b_d \in \mathbb{Z}_{>0}$. Cohen’s structure theorem yields an isomorphism

$$\hat{\mathcal{O}}_{\mathfrak{X},x} \cong \kappa[[z_0, z_1, \dots, z_d]] \cong \kappa[\mathbf{t}][[z_0, z_1, \dots, z_d]] / (\mathbf{t} - z_0^{b_0} z_1^{b_1} \cdots z_d^{b_d})$$

By construction, $r_{\mathfrak{X}}(v)$ equals to the tropicalization map $(v(z_0), \dots, v(z_d))$ for any v in the set $r_{\mathfrak{X}}^{-1}(\text{relint}(\delta))$, with $\sum_i b_i v(z_i) = v(\mathbf{t}) = 1$. Thus we can identify the restriction of $r_{\mathfrak{X}}$ over $\text{relint}(\delta)$ with the restriction of the tropicalization map $(\mathbb{G}_m^d)^{\text{an}} \rightarrow \mathbb{R}^d$ over the standard d -dimensional open simplex $\text{relint}(\delta_d^{\text{std}}) := \{u \in \mathbb{R}_{>0}^{d+1} \mid b_0 u_0 + \dots + b_d u_d = 1\}$.

Consider the tropicalizations $\text{Trop}_{(f,z)}: \mathbf{X}^{\text{val}} \rightarrow \text{Trop}(\mathbf{X}, (f, z))$, $\text{Trop}_z: \mathbf{X}^{\text{val}} \rightarrow \text{Trop}(\mathbf{X}, (z))$ and the projection map $\pi_z: \text{Trop}(\mathbf{X}, (f, z)) \rightarrow \text{Trop}(\mathbf{X}, (z))$. Then, by assumption, we have $\varphi = \varphi \circ \text{Trop}_{(f,z)}$ on $r_{\mathfrak{X}}^{-1}(\delta) \cap \mathbf{X}^{\text{val}}$. Take a simplicial rational polyhedral structure Π_{δ} on δ such that for any $\delta' \in (\Pi_{\delta})_d$, $\pi_z^{-1}(\delta')$ is a union of polyhedra, and the restriction of π_z on the relative interior each of these polyhedra is injective.

Now, we view $f = (f_1, \dots, f_k)$ as elements in $\hat{\mathcal{O}}_{\mathfrak{X},x}$, and we identify $\text{relint}(\delta)$ with $\text{relint}(\delta_d^{\text{std}})$. Note that for a sufficiently irrational point⁹ $y \in \text{relint}(\delta_d^{\text{std}})$, the cardinality of $\pi^{-1}(y) \subset \text{Trop}((\mathbb{G}_m^d)^{\text{an}}, (f, z))$ equals to 1. This implies that for any d -dimensional polyhedron δ' in $(\Pi_{\delta})_d$, the restriction of π_z on $\pi_z^{-1}(\text{relint}(\delta'))$ is a bijection. Therefore, $\varphi = \varphi \circ \text{Trop}_z$ on $r_{\mathfrak{X}}^{-1}(\text{relint}(\delta')) \cap \mathbf{X}^{\text{val}}$.

There exists a successive weighted blowup $\mathfrak{X}'$ of the strata in $\mathfrak{X}_s$ such that the support of the dual complex $\Delta_{\mathfrak{X}'} = \Delta_{\mathfrak{X}}$, and the restriction of $\Delta_{\mathfrak{X}'}$ on δ equals to Π_{δ} for each d -dimensional face δ in $\Delta_{\mathfrak{X}}$. Moreover, we have $r_{\mathfrak{X}} = r_{\mathfrak{X}'}$. Therefore, $\varphi = \varphi \circ r_{\mathfrak{X}'}$ on $r_{\mathfrak{X}'}^{-1}(\text{relint}(\delta'))$ for each d -dimensional face δ' in $\Delta_{\mathfrak{X}'}$. Since $\varphi, r_{\mathfrak{X}'}$ are continuous and $\mathbf{X}^{\text{val}}$ is dense in $\mathbf{X}^{\text{an}}$, this implies that φ satisfies the partial retraction invariance property with respect to $\mathfrak{X}'$. $\square$

We now relate the tropical Monge–Ampère solution on a strongly faithful tropicalization to a non-Archimedean Monge–Ampère solution on $\mathbf{X}^{\text{an}}$ which satisfies the comparison property.

Theorem 12.11. *Under the assumptions of Theorem 12.4, suppose further that Y be a strongly faithful open tropicalization with respect to $\Delta_{\mathfrak{X}}$, where $\Delta_{\mathfrak{X}}$ is the dual complex of an SNC model $\mathfrak{X}$ of $\mathbf{X}$ with $\dim(\Delta_{\mathfrak{X}}) = \dim_{\mathbb{K}}(\mathbf{X})$. Let μ be any probability measure which is supported on $\Delta_{\mathfrak{X}}$ and absolute continuous with respect to the Lebesgue measure. Suppose further that the bounded function $\varphi_o: Y \rightarrow \mathbb{R}$ satisfies $\varphi_o + h \in \text{LConv}(Y)$, and $\text{MA}_{\text{Trop}}(\varphi_o + h) = \deg(\mathbf{L}^d) \text{Trop}_*(\mu)$*

⁹Sufficiently irrational here means that the coordinates $y_1, \dots, y_d$ of y are linearly independent over $\mathbb{Q}$.

on Y . Then $\varphi_o \circ \text{Trop}$ can be extended to $\varphi_{\text{NA}} \in \text{CPSH}_{\omega, \text{NA}}(\mathbf{X}^{\text{an}})$ satisfying $\text{MA}_{\omega, \text{NA}}(\varphi_{\text{NA}}) = \deg(\mathbf{L}^d) \mu$. Moreover, φ_{NA} satisfies the comparison property.

Proof. Write $Y = \text{Trop}(\mathbf{X}, g)$ for $g = (g_1, \dots, g_k)$. By Theorem 12.4, $\varphi_o \circ \text{Trop}$ can be extended to $\varphi_{\text{NA}} \in \text{CPSH}_{\omega, \text{NA}}$ satisfying $(\text{Trop}_f)_*(\text{MA}_{\omega, \text{NA}}(\varphi_{\text{NA}})) = \text{MA}_{\text{Trop}}(\varphi_o + h)$. That φ_{NA} satisfies the comparison property follows from Proposition 12.10. By [Pay09], $\mathbf{X}^{\text{val}}$ is homeomorphic to $\varprojlim_f \text{Trop}(\mathbf{X}, f) = \varprojlim_f \text{Trop}(\mathbf{X}, (g, f))$, where the inverse limit is over the set of rational maps of $\mathbf{X}$ to tori.

Denote $\nu := \text{MA}_{\omega, \text{NA}}(\varphi_{\text{NA}})$ on $\mathbf{X}^{\text{an}}$. Denote $\nu_g := \text{MA}_{\text{Trop}}(\varphi_o + h)$ on $\text{Trop}(\mathbf{X}, g)$, and denote $\nu_{g,f} := \text{MA}_{\text{Trop}}((\varphi_o + h) \circ \pi_{g,f})$ on $\text{Trop}(\mathbf{X}, (g, f))$. Here $\pi_{g,f}: \text{Trop}(\mathbf{X}, (g, f)) \rightarrow \text{Trop}(\mathbf{X}, f)$ denotes the projection map. Then, [APW26, Prop. 5.20] and Theorem 12.4 imply that $(\pi_{g,f})_* \nu_{g,f} = \nu_g$ and $(\text{Trop}_{g,f})_* \nu = \nu_{g,f}$.

Since $\varpi(\delta) = 1$ for any maximal dimensional face δ in $\text{Trop}_g(\Delta_{\mathfrak{X}})$, the tropical Strumfels–Tevelev multiplicity formula implies that $|\pi_{g,f}^{-1}(y)| = 1$ for a sufficiently irrational point $y \in \delta$. This implies that there exists a rational simplicial polyhedral structure Π on $\Delta_{\mathfrak{X}}$ such that the restriction of $\pi_{g,f}$ on $\pi_{g,f}^{-1}(\bigcup_{\delta' \in \Pi_d} \text{relint}(\delta'))$ is a bijection. Therefore, $\pi_{g,f}^{-1}(\bigcup_{\delta' \in (\Pi_d)} \text{relint}(\delta')) \subset \text{Trop}_{(g,f)}(\Delta_{\mathfrak{X}})$. On the other hand, since μ_g is absolute continuous with respect to the Lebesgue measure, $\nu_g(\Delta_{\mathfrak{X}} \setminus \bigcup_{\delta' \in (\Pi_d)} \text{relint}(\delta')) = 0$. This implies that $\text{Supp}(\nu_{g,f}) \subset \text{Trop}_{(g,f)}(\Delta_{\mathfrak{X}})$. Passing to the limit over all rational maps f , we obtain that $\text{Supp}(\nu) \subset \Delta_{\mathfrak{X}}$. Since Trop_f is injective on $\Delta_{\mathfrak{X}}$ and $(\text{Trop}_f)_*(\mu) = (\text{Trop}_f)_*(\nu)$, we obtain $\mu = \nu$ on $\mathbf{X}^{\text{an}}$. $\square$

APPENDIX A. UNIFORM LIPSCHITZ PROPERTY OF TROPICAL PSH FUNCTIONS

Let X be a compact tropical variety with open part X_o . Let ω be a semipositive $(1,1)$ -form on X . We prove the following theorem.

Theorem A.1. *Let K be a compact subset of X_o . There exists a constant C_K such that each $\varphi \in \text{PSH}_{\omega}(X)$ is C_K -Lipschitz.*

Here, the notion of Lipschitz function is defined as follows. We fix a finite open cover $X_o = \bigcup_{i \in I} W_i$ by tropical charts $(W_i, \phi_i, W_i \rightarrow U_i)$. For each $i \in I$, we also fix an Euclidean metric on U_i , and denote by d_i the induced metric on W_i via ϕ_i . Then $\varphi: S \rightarrow \mathbb{R}$, for $S \subset X_o$ is called C -Lipschitz if $\varphi|_{S \cap W_i}$ is C -Lipschitz with respect to d_i , for each $i \in I$.

Let $\bar{\Pi}$ be a polyhedral structure on X , with induced polyhedral structure Π on X_o (see Section 2.1.6). For each point $x \in X_o$, let $\vec{T}_x X$ be the *pointed tangent space of X at x* defined by

$$\vec{T}_x X := \bigcup_{\substack{\sigma \ni x \\ \sigma \in \Pi}} \vec{T}_x \sigma,$$

where $\vec{T}_x \sigma$ is the set of all ϵ in the tangent space T_{σ} such that $x + t\epsilon \in \sigma$ for an small positive real t . The pointed tangent space $\vec{T}_x X$ is thus a subset of the tangent space of x in X (defined by taking the linear span of the faces $\sigma \in \Pi$ than contain x). Note that the pointed tangent space does not depend on the choice of the polyhedral structure.

For a point x and $f \in C^0(U)$ defined on a neighborhood U of x , we set

$$\partial_{\epsilon} f(x) := \lim_{t \rightarrow 0+} \frac{f(x + t\epsilon) - f(x)}{t}, \quad \epsilon \in \vec{T}_x X,$$

if this limit exists.

Note that the 1-skeleton $\Pi_{(1)} = \Pi_0 \cup \Pi_1$ of Π has vertex set Π_0 , a set of finite edges given by $\Pi_{f,1}$, and a set of infinite edges called *legs* corresponding to nonempty intersections $L \cap X_o$ for $L \in \bar{\Pi}_1$. On each edge of the 1-skeleton, we assign length one to the primitive vector of that edge. In this way, we obtain a metric graph structure on the 1-skeleton that we denote by Γ . Topologically, Γ is the support $|\Pi_{(1)}|$, and is endowed with the shortest path metric. Note that any function φ on X yields a function on Γ that we denote by $\varphi|_\Gamma$.

We first prove the theorem for the restriction of ω -psh functions to Γ .

Proposition A.2. *Let X be a tropical variety of dimension d , and let $\bar{\Pi}$ be a polyhedral structure on X , with induced polyhedral structure Π on X_o . Let Γ be the corresponding metric graph obtained from the one-skeleton $\Pi_{(1)}$ by assigning length one to the primitive vector of each edge in Π_1 . Then, there exists a positive constant C such that $\varphi|_\Gamma$ is C -Lipschitz for any $\varphi \in \text{PSH}_\omega(X)$.*

For two vertices v and u in Π_0 , we write $v \sim u$ if v and u are adjacent. In this case, we denote by vu the corresponding oriented finite edge with source v and tail u , and let $\epsilon_{vu} \in \vec{T}_v \Gamma$ be the unit tangent vector (corresponding to the primitive vector of $\vec{T}_v X$ tangent to the edge vu in the direction of u) based at v and pointing toward u . By an abuse of notation, for each point x in the interval edge vu , $x \neq u$, we denote by ϵ_{vu} the unit tangent vector in $\vec{T}_x X$ in the direction of v . We denote by $\text{rec}(\Gamma) = \text{rec}(\Pi)$ the recession fan of Π and Γ . For $\rho \in \text{rec}(\Gamma)$, we write $v \sim \rho$ if there exists an infinite edge of Γ based at v and pointing in the direction of ρ . In this case, we denote this oriented infinite edge by $v\rho$, and the corresponding unit tangent vector by $\epsilon_{v\rho}$ (corresponding to the primitive vector of $\vec{T}_v X$ in the direction of ρ). To simplify the notation, set $\partial_{vu} := \partial_{\epsilon_{vu}}$, and $\partial_{v\rho} := \partial_{\epsilon_{v\rho}}$.

Lemma A.3. *Notation as in Proposition A.2, there exist constants $A, B > 0$ such that for any $\varphi \in \text{PSH}_\omega(X)$, the following inequalities hold.*

- For any vertex v of Γ ,

$$(A.1) \quad \max_{u \sim v} \partial_{vu} \varphi(v) \geq -A \min_{u \sim v} \partial_{vu} \varphi(v) - B.$$

- For any infinite edge $v\rho$ of Γ ,

$$(A.2) \quad \partial_{v\rho} \varphi(v) \geq -A \min_{\rho \sim v} \partial_{v\rho} \varphi(v) - B$$

- For any oriented finite edge vu of Γ and any point x on the segment vu , $x \neq u$,

$$(A.3) \quad \partial_{vu} \varphi(v) \leq \partial_{vu} \varphi(x) + B/2 \leq -\partial_{uv} \varphi(u) + B.$$

- For any oriented infinite edge $v\rho$ of Γ and x on the leg $v\rho$ of Γ ,

$$(A.4) \quad \partial_{v\rho} \varphi(v) \leq \partial_{v\rho} \varphi(x) + B/2 \leq B.$$

Proof. Fix a vertex v of Γ . Let h be a local potential of ω in a neighborhood of v . Then, Proposition 3.6 implies that

$$\sum_{u \sim v} \partial_{vu}(\varphi + h)(v) + \sum_{v \sim \rho} \partial_{v\rho}(\varphi + h)(v) \geq 0$$

Note that $\varphi + h$ is a convex function bounded above on each infinite edge $v\rho$, Lemma 3.11 implies that $\partial_{v\rho}(\varphi + h)(v) \leq 0$. On the other hand, $|\partial_{vu} h(x)|$ (resp. $|\partial_{v\rho} h(x)|$) is uniformly bounded on vu (resp. $v\rho$), and the number of edges in Γ is finite. Therefore, there exist constants $A, B > 0$ such that inequality A.1 and inequality A.2 hold.

Now fix an oriented finite edge vu of Γ . Let h be a local potential of ω in a neighborhood of vu . The convexity of $\varphi + h$ on the segment vu implies that

$$\partial_{vu}(\varphi + h)(v) \leq \partial_{vu}(\varphi + h)(x) \leq -\partial_{uv}(\varphi + h)(u).$$

Applying the above inequality to the case when $\varphi = 0$, we get $\partial_{vu}h(v) \leq \partial_{vu}h(x) \leq \partial_{vu}h(u)$. Therefore, there exists $B_{vu} > 0$ such that for any $x \in vu$, $x \neq u$, we have

$$0 \leq \partial_{vu}h(x) - \partial_{vu}h(v) \leq B_{vu}/2, \quad \text{and} \quad 0 \leq -\partial_{uv}h(u) - \partial_{vu}h(x) \leq B_{vu}/2.$$

Since the number of edges in Γ is finite, there exists $B > 0$ such that inequality A.3 follows.

Now fix an oriented infinite edge $v\rho$ of Γ . Let h be a local potential of ω in a neighborhood of $v\rho$. Since $\varphi + h$ is a convex function bounded above on $v\rho$, Lemma 3.11 implies that

$$\partial_{v\rho}(\varphi + h)(v) \leq \partial_{v\rho}(\varphi + h)(x) \leq 0$$

Inequality A.4 follows from an argument similar to the one in the previous paragraph. $\square$

Proof of Proposition A.2. Let N be the number of vertices of Γ . Define the sequence $s_k, k = 0, 1, \dots, N+1$ recursively by setting $s_0 := B+1$, and $s_{k+1} := B + (s_k + B)/A$. Note that $s_k > B$ for all k .

By inequality A.2 and inequality A.4, it is sufficient to prove that $\text{PSH}_\omega(X)$ is uniformly Lipschitz on finite edges of Γ . We claim that for any $\varphi \in \text{PSH}_\omega(X)$, φ is $(s_{N+1} - B/2)$ -Lipschitz on each finite edge of Γ . Suppose this is not the case. Then, there exist $\varphi \in \text{PSH}_\omega(X)$, an oriented finite edge v_0v_1 , and a point x on the segment v_0v_1 , $x \neq v_1$, such that $\partial_{v_0v_1}\varphi(x) \geq s_{N+1} - B/2$. By inequality A.3, we have

$$-\partial_{v_1v_0}\varphi(v_1) \geq s_{N+1} - B > 0.$$

Then, inequality A.1 ensures the existence of a vertex v_2 adjacent to v_1 such that

$$\partial_{v_1v_2}\varphi(v_1) \geq -A\partial_{v_1v_0}\varphi(v_1) - B \geq A(s_{N+1} - B) - B = s_N > B.$$

One can pursue the argument and find vertices $v_3, \dots, v_{N+1}$ such that $\partial_{v_kv_{k+1}}\varphi(v_k) \geq s_{N+1-k} > B$. Therefore, inequality A.3 implies that for any $y \in v_kv_{k+1}$,

$$\partial_{v_kv_{k+1}}\varphi(y) > \partial_{v_kv_{k+1}}\varphi(v_k) - B/2 > B/2 > 0$$

Therefore, we have $\varphi(v_{k+1}) > \varphi(v_k)$ for each $1 \leq k \leq N$. On the other hand, the pigeonhole principle implies that there exist $1 \leq i < j \leq N+1$ such that $v_i = v_j$. This is clearly a contradiction and the proposition follows. $\square$

We are prepared to prove Theorem A.1.

Proof of Theorem A.1. Enlarging K if necessary, and fixing a point $x \in K$, we can choose a simplicial polyhedral structure $\bar{\Pi}$ on X , with induced polyhedral structure Π on X_o , such that $x \in \Pi_0$, $K \subset |\Pi_f|$, and such that for any $\tau \in \Pi$, there exists $i \in I$ such that τ is contained in the tropical chart W_i . Here, $(W_i, \phi_i)_i$ refers to the finite cover by charts with respect to which the Lipschitz property is defined.

Let $\text{PSH}_{\omega,x}(X) = \{\varphi \in \text{PSH}_\omega(X) \mid \varphi(x) = 0\}$. We only need to prove that $\varphi|_K$ is uniformly Lipschitz for $\varphi \in \text{PSH}_{\omega,x}(X)$. By Proposition 3.9, it suffices to show that for any $\tau \in \Pi_f$, $\|\varphi\|_{C^0(\tau)}$ is uniformly bounded for all $\varphi \in \text{PSH}_{\omega,x}(X)$.

Proposition A.2 implies that there exists $M > 0$ such that $|\varphi(v)| \leq M$ for any $\varphi \in \text{PSH}_{\omega,x}(X)$ and any vertex $v \in \Pi_0$. Since $\varphi|_\tau$ is convex, we have $\varphi|_\tau \leq M$ for any $\varphi \in \text{PSH}_{\omega,x}(X)$.

It remains to prove a uniform lower bound for $\varphi|_\tau$, for any $\tau \in \Pi_f$. Fix such a face τ and let $i \in I$ be such that $\tau \subset W_i$. We use $\tau_{(0)}$ to denote the set of vertices of τ .

Fix $v \in \tau_{(0)}$. Let J be an index set, and let $\{e_j\}_{j \in J}$ be the set of edges in the graph Π_1 adjacent to v . For $e_j = vu$ (resp. $e_j = v\rho$) with $j \in J$, denote by $\mathbf{e}_j := \mathbf{e}_{vu}$ (resp. $\mathbf{e}_j := \mathbf{e}_{v\rho}$), i.e. the primitive tangent vector to the edge e_j based at v . The balancing condition ensures the existence of $\{a_j\}_{j \in J}$ with $a_j > 0$, such that $\sum_{j \in J} a_j \mathbf{e}_j = 0$ (in the ambient vector space containing $\phi_i(W_i)$).

Denote by $J_1 := \{j \in J \mid e_j \subset \tau\} \subset J$, and denote $J_2 := J \setminus J_1$. Note that $\{\mathbf{e}_j\}_{j \in J_1}$ is linearly independent since τ is a simplex. Let T_τ denote the tangent space of τ , and $\|\cdot\|$ denote the Euclidean norm on T_τ induced by the Euclidean metric on W_i . Set

$$S := \{\mathbf{e} \in T_\tau \mid \|\mathbf{e}\| = 1, v + t\mathbf{e} \in \tau \text{ for some } t > 0\}.$$

Then, there exists $\underline{B} > 0$ such that for any $\mathbf{e} \in S$, we have $\mathbf{e} = \sum_{j \in J_1} b_j \mathbf{e}_j$ for some $0 \leq b_j < \underline{B}$. Set $\underline{A} := \max_{j \in J_1} \underline{B}/a_j$.

Fix $\mathbf{e} = \sum_{j \in J_1} b_j \mathbf{e}_j \in S$. Then, we have

$$\mathbf{e} + \sum_{j \in J_1} (\underline{A}a_j - b_j) \mathbf{e}_j + \sum_{j \in J_2} (\underline{A}a_j) \mathbf{e}_j = 0,$$

with $(\underline{A}a_j - b_j) > 0$ for each $j \in J_1$, and $\underline{A}a_j > 0$ for each $j \in J_2$. Let h be a local potential of ω on W_i . Then, by Proposition 3.6, we have

$$\partial_{\mathbf{e}}(\varphi + h)(v) + \sum_{j \in J_1} (\underline{A}a_j - b_j) \partial_{\mathbf{e}_j}(\varphi + h)(v) + \sum_{j \in J_2} (\underline{A}a_j) \partial_{\mathbf{e}_j}(\varphi + h)(v) \geq 0$$

for every $\varphi \in \text{PSH}_{\omega,x}(X)$. On the other hand, Proposition A.2 implies that there exists $C > 0$ such that $|\partial_{\mathbf{e}_j} \varphi(v)| \leq C$ for each $j \in J$. Since h is Lipschitz on τ (Proposition 3.9), there exists $C' > 0$ such that $|\partial_{\mathbf{e}} h(v)|, |\partial_{\mathbf{e}_j} h(v)| \leq C'$ for each $\mathbf{e} \in S$ and each $j \in J$. Thus, $\partial_{\mathbf{e}} \varphi(v) \geq M'$ for every $\varphi \in \text{PSH}_{\omega,x}(X)$ and every $\mathbf{e} \in S$, with the constant M' defined by

$$M' = -\underline{A}(C + C') \left(\sum_{j \in J} a_j \right) - (\underline{B} + 1)C'.$$

Therefore, $\varphi(y) \geq \varphi(v) + \partial_{(y-v)} \varphi(v) \geq -M + \text{diam}(\tau) \cdot M'$ for every $\varphi \in \text{PSH}_{\omega,x}(X)$ and $y \in \tau$. Here, $y - v \in T_\tau$, and $\text{diam}(\tau)$ denotes the diameter of τ . This proves that $\text{PSH}_{\omega,x}(X)$ has a uniform lower bound on τ . $\square$

APPENDIX B. POLYHEDRAL GEOMETRY TOOLKIT

This appendix is devoted to stating and proving four polyhedral-geometric lemmas used in the proof of Proposition 11.6.

Lemma B.1. *Let Δ be any simplicial polyhedral complex in $\mathbb{R}^n$, and let $\{x_i\}_{i \in I}$ be a finite set of points in the support of Δ . Then, there exists a refinement Δ' of Δ such that Δ' is simplicial and $\Delta'_0 = \Delta_0 \cup \{x_i\}_{i \in I}$.*

Proof. The refinement can be realized by performing stellar subdivisions successively at x_i , see [Ewa96, § III.2]. $\square$

We say that a one-dimensional fan is *balanceable* if the set of its rays is positively dependent (see Definition 11.5). In particular (if the fan is rational), there exists a tropical weight turning it into a tropical fan.

Lemma B.2. *Let $\mathcal{L}$ be a balanceable 1-dimensional fan in $E = \mathbb{R}^n$ with $\text{span}(\mathcal{L}) = E$. Let $\mathcal{F}'$ be a simplicial (not necessarily complete) fan supported on E such that*

- (1) $\mathcal{F}'_1 \subset \mathcal{L}_1$.
- (2) $|\mathcal{F}'| = \text{conv}(\mathcal{F}'_1)$, where conv denotes the convex hull of $\mathcal{F}'_1$.
- (3) $\rho \not\subseteq |\mathcal{F}'|$ for any $\rho \in \mathcal{L}_1 \setminus \mathcal{F}'_1$.

Then, there is a simplicial fan $\mathcal{F}$ supported on E satisfying

- (1) $\mathcal{F}'$ is a subfan of $\mathcal{F}$.
- (2) $\mathcal{F}_1 = \mathcal{L}_1$.
- (3) $\mathcal{F}$ is complete.

Proof. We will prove inductively that for any subset of rays $\mathcal{F}'_1 \subset \mathcal{R} \subset \mathcal{L}_1$, there exists a simplicial fan $\mathcal{F}^{\mathcal{R}}$ supported on E such that

- (i) $\mathcal{F}'$ is a subfan of $\mathcal{F}^{\mathcal{R}}$.
- (ii) $(\mathcal{F}^{\mathcal{R}})_1 = \mathcal{R}$.
- (iii) $|\mathcal{F}^{\mathcal{R}}| = \text{conv}(\mathcal{R})$.

Granted this, we take $\mathcal{F} := \mathcal{F}^{\mathcal{L}_1}$. Then, $\mathcal{F}$ satisfies the first two conditions listed in the statement of the lemma. Since $\mathcal{L}$ is balanceable, we have $|\mathcal{F}^{\mathcal{L}_1}| = \text{conv}(\mathcal{L}_1) = E$, as desired.

When $\mathcal{R} = \mathcal{F}'_1$, we can simply take $\mathcal{F}^{\mathcal{R}} = \mathcal{F}'$. Suppose we have constructed $\mathcal{F}^{\mathcal{R}-}$ for $\mathcal{F}'_1 \subset \mathcal{R}_- \subset \mathcal{L}_1$ satisfying Conditions (i)-(iii). Let $\mathcal{R} = \mathcal{R}_- \sqcup \{\rho\} \subset \mathcal{L}_1$, we wish to construct $\mathcal{F}^{\mathcal{R}}$ with the desired properties.

If $\rho \subset |\mathcal{F}^{\mathcal{R}-}|$, we take $\mathcal{F}^{\mathcal{R}}$ to be the stellar subdivision of $\mathcal{F}^{\mathcal{R}-}$ at ρ . Then, $\mathcal{F}^{\mathcal{R}}$ is a simplicial fan satisfying $(\mathcal{F}^{\mathcal{R}})_1 = \mathcal{R}$ and $|\mathcal{F}^{\mathcal{R}}| = |\mathcal{F}^{\mathcal{R}-}| = \text{conv}(\mathcal{R})$. Since $\rho \not\subseteq |\mathcal{F}'|$ by assumption, the stellar subdivision does not change any cone in $\mathcal{F}'$. Therefore, $\mathcal{F}' \subset \mathcal{F}^{\mathcal{R}}$, as desired.

It remains to treat the case when $\rho \not\subseteq |\mathcal{F}^{\mathcal{R}-}|$. Let $\mathcal{K} := |\mathcal{F}^{\mathcal{R}-}| \subset E$. Denote by l the 1-dimensional subspace of E generated by ρ . We take an $(n-1)$ -dimensional subspace $E_1 \subset E$ such that $E = E_1 + l$. Fix an Euclidean metric on E , and denote by $\mathbb{S}_{E_1}$ the unit sphere in E_1 .

Fix $e_0 \in \rho \setminus \{0\}$. We define a function $\mathfrak{t}: \mathbb{S}_{E_1} \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$\mathfrak{t}(y) := \inf\{\lambda \in \mathbb{R} \mid y - \lambda e_0 \in \mathcal{K}\}$$

Set $\mathcal{C} := \{y \mid \mathfrak{t}(y) < +\infty\} \subset \mathbb{S}_{E_1}$, and define $\pi: \mathcal{C} \rightarrow \partial\mathcal{K}$ by

$$\pi(y) := y - \mathfrak{t}(y)e_0 \quad \text{for } y \in \mathcal{C}.$$

Claim B.3. *There is a subfan $\mathcal{E}$ of $\mathcal{F}^{\mathcal{R}-}$ such that $\mathcal{C}(\pi(\mathcal{C})) = |\mathcal{E}|$, where $\mathcal{C}(\pi(\mathcal{C}))$ denotes the cone generated by $\pi(\mathcal{C})$ ¹⁰. Furthermore, for any $\sigma \in \mathcal{E}$, $\text{Cone}(\sigma, \rho)$ is a simplicial cone, and $\text{Cone}(\sigma, \rho) \cap \mathcal{K} = \sigma$.*

Proof of Claim B.3. Fix $x_0 \in \pi(\mathcal{C})$. Suppose that $x_0 \in \text{relint}(\sigma)$ for some $\sigma \in \mathcal{F}^{\mathcal{R}-}$ with

$$\sigma = \text{Cone}(x_1, \dots, x_k), \quad \text{for } x_1, \dots, x_k \in \sigma.$$

We first show that $\{e_0, x_1, \dots, x_k\}$ is linearly independent in E . Suppose not. Then $e_0 \in T_\sigma$, where T_σ denotes the tangent space of σ . Write $x_0 = y_0 - \mathfrak{t}(y_0)e_0$ for $y_0 \in \mathbb{S}_{E_1}$. Let $\{c_i\}_{i \geq 1}$

¹⁰For a subset $S \subset E$, the cone $\mathcal{C}(S)$ generated by S is defined to be $\mathcal{C}(S) := \{\lambda s \mid \lambda \in \mathbb{R}_{\geq 0}, s \in S\}$. It should be distinguished from the convex cone generated by a subset S , which equals the cone generated by the convex hull of S , and is denoted by $\text{Cone}(S)$.

be an increasing sequence of real numbers such that $\lim_{i \rightarrow +\infty} c_i = \mathfrak{t}(y_0)$. Then, we have $y_0 - c_i e_0 \rightarrow x_0$, and $y_0 - c_i e_0 \in T_\sigma$ for each $i \geq 1$. On the other hand, $y_0 - c_i e_0 \notin \mathcal{K}$ and hence $y_0 - c_i e_0 \notin \sigma$ for each $i \geq 1$. This contradicts the assumption that $x_0 \in \text{relint}(\sigma)$. Therefore, we have shown that $\{e_0, x_1, \dots, x_k\}$ is linearly independent in E .

Fix $x \in \sigma \setminus \{0\}$. Since $E = E_1 + l$ and $e_0 \notin T_\sigma$, one can always write $x = b(y - \lambda e_0)$ with $y \in \mathbb{S}_{E_1}$, $b \in \mathbb{R}_{>0}$ and $\lambda \in \mathbb{R}$. We would like to show that $\mathfrak{t}(y) = \lambda$. Suppose not. Then, there exists $\lambda' < \lambda$ such that $y - \lambda' e_0 \in \mathcal{K}$. Since $\mathcal{K}$ is a convex cone, we have $\text{Cone}(y - \lambda' e_0, x_1, \dots, x_k) \subset \mathcal{K}$. Note that $\sigma = \bigcup_{1 \leq i \leq k} \text{Cone}(x, x_1, \dots, \hat{x}_i, \dots, x_k)$. Hence, we may assume that $x_0 \in \text{Cone}(x, x_2, \dots, x_k)$ without loss of generality. Furthermore, $x_0 \in \text{relint}(\sigma)$ implies that $x_0 \notin \text{Cone}(x_2, \dots, x_k)$. Then, we can write $x_0 = ax + \sum_{j=2}^k a_j x_j$, with $a > 0$ and $a_j \geq 0$ for each $2 \leq j \leq k$, and we have

$$\begin{aligned} ab(y - \lambda' e_0) + \sum_{j=2}^k a_j x_j &= ax + \sum_{j=2}^k a_j x_j + ab(\lambda - \lambda') e_0 \\ &= x_0 + ab(\lambda - \lambda') e_0 = y_0 - \mathfrak{t}(y_0) e_0 + ab(\lambda - \lambda') e_0 \in \mathcal{K} \end{aligned}$$

with $ab(\lambda - \lambda') > 0$. This contradicts the definition of $\mathfrak{t}(y_0)$. Therefore, we conclude that $\mathfrak{t}(y) = \lambda$ for any $b(y - \lambda e_0) \in \sigma \setminus \{0\}$.

Now, we define the set of cones $\mathcal{E}$ by

$$\mathcal{E} := \{\sigma \in \mathcal{F}^{\mathcal{R}-} \mid \text{relint}(\sigma) \cap \pi(\mathcal{C}) \text{ is nonempty}\}.$$

By construction, we have $\mathcal{C}(\pi(\mathcal{C})) \subset |\mathcal{E}|$. On the other hand, the previous paragraph implies that $\sigma \subset \mathcal{C}(\pi(\mathcal{C}))$ for any $\sigma \in \mathcal{E}$. In particular, this implies that $|\mathcal{E}| \subset \mathcal{C}(\pi(\mathcal{C}))$, and we have $\tau \in \mathcal{E}$ for each element $\sigma \in \mathcal{E}$ and each face $\tau < \sigma$. Therefore, $\mathcal{E}$ is a fan with support $\mathcal{C}(\pi(\mathcal{C}))$.

Since $\{e_0, x_1, \dots, x_k\}$ is linearly independent in E , $\text{Cone}(\sigma, \rho)$ is simplicial. On the other hand, for any $x = b(y - \lambda e_0) \in \sigma \setminus \{0\}$, the equality $\mathfrak{t}(y) = \lambda$ implies $\text{Cone}(x, \rho) \cap \mathcal{K} = \text{Cone}(x)$. Hence,

$$\text{Cone}(\sigma, \rho) \cap \mathcal{K} = \bigcup_{x \in \sigma \setminus \{0\}} (\text{Cone}(x, \rho) \cap \mathcal{K}) = \bigcup_{x \in \sigma \setminus \{0\}} \text{Cone}(x) = \sigma \quad \square$$

We take $\mathcal{F}^{\mathcal{R}} := \mathcal{F}^{\mathcal{R}-} \cup \{\text{Cone}(\sigma, \rho) \mid \sigma \in \mathcal{E}\}$. We first verify that $\mathcal{F}^{\mathcal{R}}$ is a simplicial fan. It is enough to show that $\tau_1 \cap \tau_2 \in \mathcal{F}^{\mathcal{R}}$ for any $\tau_1, \tau_2 \in \mathcal{F}^{\mathcal{R}}$. There are three possible cases to consider:

- (1) If $\tau_1, \tau_2 \in \mathcal{F}^{\mathcal{R}-}$, then $\tau_1 \cap \tau_2 \in \mathcal{F}^{\mathcal{R}-} \subset \mathcal{F}^{\mathcal{R}}$ holds trivially.
- (2) If $\tau_i = \text{Cone}(\sigma_i, \rho)$ for $\sigma_i \in \mathcal{E}$ for each $i \in \{1, 2\}$, we show that $\tau_1 \cap \tau_2 = \text{Cone}(\sigma_1 \cap \sigma_2, \rho) \in \mathcal{F}^{\mathcal{R}}$. The inclusion $\text{Cone}(\sigma_1 \cap \sigma_2, \rho) \subset \tau_1 \cap \tau_2$ is evident. Thus, it is sufficient to prove that $\tau_1 \cap \tau_2 \subset \text{Cone}(\sigma_1 \cap \sigma_2, \rho)$. Let $z \in \tau_1 \cap \tau_2$. Then, one can write $z = x_1 + \mu_1 e_0 = x_2 + \mu_2 e_0$ with $x_i \in \sigma_i$ and $\mu_i \geq 0$ for each $i \in \{1, 2\}$. Without loss of generality, we assume $\mu_1 \leq \mu_2$. Then, one has $x_1 = x_2 + (\mu_2 - \mu_1) e_0 \in \sigma_1 \cap \text{Cone}(\sigma_2, \rho)$. By Claim B.3, $\sigma_1 \cap \text{Cone}(\sigma_2, \rho) = \sigma_1 \cap (\text{Cone}(\sigma_2, \rho) \cap \mathcal{K}) = \sigma_1 \cap \sigma_2$. Therefore, $z \in \text{Cone}(\sigma_1 \cap \sigma_2, \rho)$, which implies that $\tau_1 \cap \tau_2 \subset \text{Cone}(\sigma_1 \cap \sigma_2, \rho)$.
- (3) Otherwise, we may assume that $\tau_1 \in \mathcal{F}^{\mathcal{R}-}$ and $\tau_2 = \text{Cone}(\sigma, \rho)$ for $\sigma \in \mathcal{E}$. Then $\tau_1 \cap \tau_2 = \tau_1 \cap (\tau_2 \cap \mathcal{K}) = \tau_1 \cap \sigma \in \mathcal{F}^{\mathcal{R}-}$ by Claim B.3.

We now check that $\mathcal{F}^{\mathcal{R}}$ satisfies the three properties (i)-(iii). We have $\mathcal{F}' \subset \mathcal{F}^{\mathcal{R}-} \subset \mathcal{F}^{\mathcal{R}}$, the first inclusion coming from the induction hypothesis, and the second from the construction.

Also, $(\mathcal{F}^{\mathcal{R}})_1 = \mathcal{R}$ by construction. As for the third property, it is clear that $|\mathcal{F}^{\mathcal{R}}| \subset \text{conv}(\mathcal{R})$. To prove $\text{conv}(\mathcal{R}) \subset |\mathcal{F}^{\mathcal{R}}|$, fix $z = x + \mu e_0 \in \text{conv}(\mathcal{R})$ with $x \in \mathcal{K}$ and $\mu \geq 0$. Since $E = E_1 + l$, one can write $x = by - ae_0$ for $y \in \mathbb{S}_{E_1}$, $b \in \mathbb{R}_{\geq 0}$, and $a \in \mathbb{R}$.

- (1) If $b = a = 0$, then we have $x = 0$ and $z = \mu e_0 \in \rho \subset |\mathcal{F}^{\mathcal{R}}|$.
- (2) If $b = 0$ and $a \neq 0$, then the assumption that $\rho \not\subset |\mathcal{F}^{\mathcal{R}-}|$ and $x \in \mathcal{K} = |\mathcal{F}^{\mathcal{R}-}|$ imply that $a > 0$. Therefore, we have $-e_0 = a^{-1}x \in \mathcal{K}$, and hence $z = (\mu - a)e_0 \in l \subset \mathcal{K} \cup \rho \subset |\mathcal{F}^{\mathcal{R}}|$.
- (3) If $b > 0$, then $\mathfrak{t}(y) \leq \frac{a}{b} < +\infty$. Since $\mathcal{K}$ is convex, $\text{Cone}(x, y - \mathfrak{t}(y)e_0) \subset \mathcal{K}$. On the other hand, $\text{Cone}(e_0, y - \mathfrak{t}(y)e_0) \subset \text{Cone}(\sigma, \rho) \subset |\mathcal{F}^{\mathcal{R}}|$ for some $\sigma \in \mathcal{E}$. Therefore, $z \in \text{Cone}(e_0, x) = \text{Cone}(e_0, y - \mathfrak{t}(y)e_0) \cup \text{Cone}(x, y - \mathfrak{t}(y)e_0) \subset |\mathcal{F}^{\mathcal{R}}|$. $\square$

Lemma B.4. *Let E be a Euclidean space of dimension n . Let T be a finite set, with subsets $T_1, T_2 \subset T$, and $S = T_1 \cap T_2$. We use $F^{\mathfrak{S}}(T)$ to denote the set of injective maps $g: T \rightarrow E$ such that $\sigma(g) := \text{conv}(g(S))$, $\tau_1(g) := \text{conv}(g(T_1))$, and $\tau_2(g) := \text{conv}(g(T_2))$ are simplices in E . Let $f \in F^{\mathfrak{S}}(T)$ satisfy $\tau_1(f) \cap \tau_2(f) = \sigma(f)$. Then, there exists $\epsilon > 0$ such that for any $f' \in F^{\mathfrak{S}}(T)$ with $d(f, f') < \epsilon$, one has $\tau_1(f') \cap \tau_2(f') = \sigma(f')$. Here $d(f, f') := \sup_{t \in T} d(f(t), f'(t))$.*

Proof. Choose $r \gg 0$ such that $\tau_1(f'), \tau_2(f') \subset B(0, r)$ for any $f' \in F^{\mathfrak{S}}(T)$ with $d(f, f') < 1$. For any $0 < \epsilon_1 \ll 1$, there exists $0 < \epsilon_2 < \epsilon_1/2$ such that for any $f' \in F^{\mathfrak{S}}(T)$ with $d(f, f') < \epsilon_2$, there exists an isometry $\phi_{f'}: E \rightarrow E$ satisfying

- (1) $d(\phi_{f'}(x), x) < \epsilon_1/2$ for any $x \in B(0, r)$.
- (2) $\phi_{f'}(\langle \sigma(f') \rangle_{\text{aff}}) = \langle \sigma(f) \rangle_{\text{aff}}$, where $\langle \cdot \rangle_{\text{aff}}$ denotes the affine subspace generated by a polyhedron.

Note that $d(f, \phi_{f'} \circ f') < \epsilon_2 + \epsilon_1/2 < \epsilon_1$ under the above conditions. Therefore, it suffices to prove the lemma in the case when $\langle \sigma(f') \rangle_{\text{aff}} = \langle \sigma(f) \rangle_{\text{aff}}$.

After translation, we may assume that $E_{\sigma(f)} := \langle \sigma(f) \rangle_{\text{aff}}$ is a vector subspace of E . Write $E = E_{\sigma(f)} \oplus E_{\sigma(f)}^{\perp}$, and denote by $p: E \rightarrow E_{\sigma(f)}^{\perp}$ the projection map. Then by assumption, $p(\tau_1(f)) \cap p(\tau_2(f)) = \{0\}$. Observe that for any $f' \in F^{\mathfrak{S}}(T)$ satisfying $\langle \sigma(f') \rangle_{\text{aff}} = \langle \sigma(f) \rangle_{\text{aff}}$, $p(\tau_i(f'))$ are simplices in $E_{\sigma(f)}^{\perp}$, and further we have $p^{-1}(0) \cap \tau_1(f') \cap \tau_2(f') = \sigma(f')$. Hence, showing $p(\tau_1(f')) \cap p(\tau_2(f')) = \{0\}$ would imply $\tau_1(f') \cap \tau_2(f') = \sigma(f')$. Since $d(p(x), p(x')) \leq d(x, x')$ for any $x, x' \in E$, it is enough to prove the lemma for $\tau_1(p \circ f), \tau_2(p \circ f) \subset E_{\sigma(f)}^{\perp}$ with $\tau_1(p \circ f) \cap \tau_2(p \circ f) = \{0\}$, where we view $p \circ f$ as an injective map from $T/\{S \rightarrow *\}$ to $E_{\sigma(f)}^{\perp}$. Therefore, we may henceforth assume that $T_1 \cap T_2 = \{s\}$, and $\tau_1(f) \cap \tau_2(f) = \{0\}$.

Denote by $\mathbb{S}_E$ the unit sphere in E . For each $f' \in F^{\mathfrak{S}}(T)$ with $f'(s) = 0$ and for each $i \in \{1, 2\}$, define $P_i(f') := \mathbb{S}_E \cap \text{Cone}(f'(T_i))$. Then, $\tau_1(f') \cap \tau_2(f') = \{0\}$ is equivalent to $P_1(f') \cap P_2(f') = \emptyset$. Thus, by assumption $d(P_1(f), P_2(f)) =: \epsilon_3 > 0$. Take $\epsilon := \epsilon_3 M/6$, where $M := \min_{t \in T \setminus \{s\}} \|f(t)\|$. Then, for any $f' \in F^{\mathfrak{S}}(T)$ with $f'(s) = 0$ and $d(f, f') < \epsilon$, we have $d(P_1(f'), P_2(f')) > \epsilon_3/3$. This proves the lemma. $\square$

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