

TROPICAL FUNCTION FIELDS, FINITE GENERATION, AND FAITHFUL TROPICALIZATION

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ABSTRACT. Given an algebraic variety defined over a discrete valuation field and a skeleton of its Berkovich analytification, the tropicalization process transforms function field of the variety, after base change to an algebraic closure, to a semifield of tropical functions on the skeleton. Our main result offers a complete characterization of this semifield: we show that a tropical function is in the image of the tropicalization map if and only if it takes the same slope near infinity along parallel half-lines of the skeleton. This extends a result of Baker and Rabinoff in dimension one to arbitrary dimensions. We use this characterization to establish that this semifield is finitely generated over the semifield of tropical rational numbers, providing a variant of a recent result by Ducros, Hrushovski, Loeser and Ye in the discrete valued field case. As a second application, we present a new proof of the faithful tropicalization theorem by Gubler, Rabinoff and Werner in the discrete valuation field case. The proof is constructive and provides explicit coordinate functions for the embedding of the skeleton, extending the existing results in dimension one to arbitrary dimensions.

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1. INTRODUCTION

Throughout this paper, let K be a complete discrete valuation field with valuation $\nu_K: K \rightarrow \mathbb{Z} \cup \{\infty\}$, valuation ring R with maximal ideal \mathfrak{m} , and residue field κ .

Let X be a smooth projective geometrically connected variety over K and let \mathcal{X} be a strictly semistable model of X , defined over the valuation ring of some finite extension of K . Let \mathcal{H} denote a horizontal divisor of \mathcal{X} such that $(\mathcal{X}, \mathcal{H})$ forms a strictly semistable pair. The special fiber of \mathcal{X} is denoted by \mathcal{X}_κ . After performing a possible finite base change, we assume that each connected component of the intersection of any collection of irreducible components of \mathcal{H} is geometrically connected.

Let $\Delta(\mathcal{X}, \mathcal{H})$ be the dual complex of $\mathcal{X}_\kappa + \mathcal{H}$, an abstract polyhedral complex with a recession fan defined as follows. Let Z_v for $v \in V_f$ be the irreducible components of \mathcal{X}_κ and W_ρ for $\rho \in V_\infty$

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the irreducible components of \mathcal{H} . To any connected component of an intersection of the form

$$\bigcap_{v \in A} Z_v \cap \bigcap_{\rho \in B} W_\rho$$

where $A \subseteq V_f$ is nonempty and $B \subseteq V_\infty$, we associate a copy of the polyhedron $\sigma^A \times \mathbb{R}_{\geq 0}^B$. Here, σ^A is the standard simplex in \mathbb{R}^A consisting of points with non-negative coordinates summing to one. The collection of these polyhedra forms $\Delta(\mathcal{X}, \mathcal{H})$, which glue together to define the polyhedral space $|\Delta(\mathcal{X}, \mathcal{H})|$ to which we refer as the support of $\Delta(\mathcal{X}, \mathcal{H})$. Moreover, the collection of cones consisting of a copy of $\mathbb{R}_{\geq 0}^B$ associated with each connected component of a nonempty intersection of the form $\bigcap_{\rho \in B} W_\rho$, as above, forms an abstract fan Υ referred to as the *recession fan* of $\Delta(\mathcal{X}, \mathcal{H})$. These cones glue together and form the support $|\Upsilon|$ of Υ . In particular, the rays of Υ correspond to the components W_ρ of \mathcal{H} . See Sections 2.2 and 3 for more details.

Two rays in $\Delta(\mathcal{X}, \mathcal{H})$ are said to be *parallel* if they are associated with the same component of \mathcal{H} , meaning they correspond to the same ray in Υ . More generally, two half-lines in $|\Delta(\mathcal{X}, \mathcal{H})|$ are *parallel* if the corresponding half-lines issued from the origin in $|\Upsilon|$ are the same.

Let X^{an} denote the Berkovich analytification of X (see [Ber90]). The dual complex $\Delta(\mathcal{X}, \mathcal{H})$ (more precisely, its support) naturally embeds in X^{an} forming a subspace called the *skeleton* $\text{Sk}(\mathcal{X}, \mathcal{H})$ of X^{an} associated with $(\mathcal{X}, \mathcal{H})$. This allows to identify $|\Delta(\mathcal{X}, \mathcal{H})|$ with $\text{Sk}(\mathcal{X}, \mathcal{H}) \hookrightarrow X^{\text{an}}$.

Let $\text{Rat}(X)$ denote the function field of X . Identifying $|\Delta(\mathcal{X}, \mathcal{H})|$ with the skeleton $\text{Sk}(\mathcal{X}, \mathcal{H}) \hookrightarrow X^{\text{an}}$, we associate to each point $x \in |\Delta(\mathcal{X}, \mathcal{H})|$ a valuation

$$\nu_x: \text{Rat}(X) \rightarrow \mathbb{R} \cup \{\infty\}.$$

For any $f \in \text{Rat}(X)$, the (*abstract*) *tropicalization* of f is defined as

$$\text{trop}(f): |\Delta(\mathcal{X}, \mathcal{H})| \rightarrow \mathbb{R} \cup \{\infty\}, \quad x \mapsto \nu_x(f).$$

More generally, consider the base change $X_{\bar{K}}$ of X to an algebraic closure \bar{K} of K . The tropicalization map naturally extends to $\text{Rat}(X_{\bar{K}})$, and for $f \in \text{Rat}(X_{\bar{K}})$, it yields a function $\text{trop}(f)$, as described above, on $|\Delta(\mathcal{X}, \mathcal{H})|$.

The set $\text{trop}(\text{Rat}(X_{\bar{K}}))$ of tropicalized rational functions on $X_{\bar{K}}$ is a semifield with the operation of addition corresponding to taking pointwise minimum of functions, and the operation of multiplication corresponding to the usual addition.

Our first main result gives a characterization of this semifield. This is done as follows.

We define $\text{Rat}(\Delta(\mathcal{X}, \mathcal{H}))$ as the union of $\{\infty\}$ and the space of facewise integral \mathbb{Q} -affine functions on $\Delta(\mathcal{X}, \mathcal{H})$ which take the same slope near infinity along parallel rays of $\Delta(\mathcal{X}, \mathcal{H})$. Similarly, we define $\text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|)$ as the union of $\{\infty\}$ and the space of piecewise integral \mathbb{Q} -affine functions on $|\Delta(\mathcal{X}, \mathcal{H})|$ which exhibit the same slope near infinity along parallel half-lines of $|\Delta(\mathcal{X}, \mathcal{H})|$. Equivalently, $\text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|)$ is the union of $\text{Rat}(\Delta')$ over integral \mathbb{Q} -affine refinements Δ' of $\Delta(\mathcal{X}, \mathcal{H})$ (see Section 2.2). We refer to $\text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|)$ as the *tropical function field* of $(\mathcal{X}, \mathcal{H})$.

Theorem 1.1 (Characterization Theorem). *For every $f \in \text{Rat}(X_{\bar{K}})$, the tropicalization $\text{trop}(f)$ belongs to $\text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|)$. The resulting map*

$$(1.1) \quad \text{trop}: \text{Rat}(X_{\bar{K}}) \rightarrow \text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|)$$

is surjective.

Consequently, the tropical function field coincides with the semifield of tropicalized rational functions on $X_{\bar{K}}$.

This result generalizes the work of Baker and Rabinoff [BR15] in dimension one to arbitrary dimensions, while also accounting for the presence of horizontal divisors. See also the work of Amini and Iriarte [AI22], which addresses the case of trivially valued fields of characteristic zero, and establishes the surjectivity of an analogous tropicalization map, defined relative to a compactification.

We note that our proof of the characterization theorem gives the surjectivity of tropicalization map from $\text{Rat}(X_L)$ to $\text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|)$ for an extension L of K with value group \mathbb{Q} .

To state our second main result, we recall that a semifield \mathbb{F} is said to be finitely generated over a semifield \mathbb{E} if there exist finitely many elements a_1, a_2, \dots, a_n of \mathbb{F} such that every element of \mathbb{F} can be expressed as $p(a_1, \dots, a_n)/q(a_1, \dots, a_n)$ with p and q n -variable polynomials over \mathbb{E} .

Let $\mathbb{TQ} := \mathbb{Q} \cup \{\infty\}$ denote the semifield of rational numbers with the operations of \min for addition and $+$ for multiplication. We observe that the definition of tropical function fields extends to any abstract polyhedral complex endowed with a recession fan (see Section 2.2).

Theorem 1.2 (Finite generation of tropical function fields). *Let Δ be an abstract polyhedral complex with a recession fan. The tropical function field $\text{Rat}(|\Delta|)$ is finitely generated over \mathbb{TQ} .*

In particular, $\text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|)$, as defined above, is finitely generated over \mathbb{TQ} . Motivated by the problem of characterizing functions on skeleta that arise from tropicalization—as in Theorem 1.1—Ducros, Hrushovski, Loeser, and Ye show in [DHL24], using model theory of valued fields, that $\text{trop}(\text{Rat}(X_{\bar{K}}))$ is finitely generated as a semifield over the value group (see §1.2 of *loc.cit.*). The combination of Theorems 1.1 and 1.2 provides a variant of their result, and moreover establishes the finite generation of $\text{trop}(\text{Rat}(X_L))$ for extensions L of K with value group \mathbb{Q} , without requiring L to be defectless. (As stated in [DHL24], the finite generation property does not hold in general for $\text{trop}(\text{Rat}(X))$.) We note that the notion of skeleton considered in *loc.cit.* differs slightly from ours. However, as we explain in Remark 3.1, in residue characteristic zero, the skeleta they consider are contained within those studied in this paper. In this setting, we thus obtain a new proof of their theorem.

As an application of Theorems 1.1 and 1.2, we provide an alternative proof of the faithful tropicalization theorem of Gubler, Rabinoff, and Werner [GRW16] in the discrete valued field case. See also [Duc12, Thm 5.1] and [DHL24, §1.2].

Theorem 1.3 (Gubler–Rabinoff–Werner [GRW16]). *Let X be a smooth projective variety over K with a strictly semistable pair $(\mathcal{X}, \mathcal{H})$, as above. Then, there exist $f_1, f_2, \dots, f_n \in \text{Rat}(X_{\bar{K}})$ such that, setting $U := X_{\bar{K}} \setminus \bigcup_{i=1}^n |\text{div}(f_i)|$, the map*

$$\psi: U^{\text{an}} \rightarrow \mathbb{R}^n, \quad x \mapsto (\nu_x(f_1), \dots, \nu_x(f_n))$$

provides a faithful tropicalization of the skeleton $\text{Sk}(\mathcal{X}, \mathcal{H})$ of $(\mathcal{X}, \mathcal{H})$, that is, the restriction of ψ to $\text{Sk}(\mathcal{X}, \mathcal{H})$ is a homeomorphism onto its image that preserves the integral structures.

Our proofs of Theorems 1.1, 1.2 and 1.3 yield additional insights. We explicitly construct $F_1, \dots, F_n \in \text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|)$ such that the map

$$(F_1, \dots, F_n): |\Delta(\mathcal{X}, \mathcal{H})| \rightarrow \mathbb{R}^n$$

is a homeomorphism onto its image preserving the integral structures. By Theorem 1.1, there exist $f_1, \dots, f_n \in \text{Rat}(X_{\bar{K}})$ satisfying $\text{trop}(f_1) = F_1, \dots, \text{trop}(f_n) = F_n$. Our proof shows that the embedding given by (f_1, \dots, f_n) provides a faithful tropicalization of the skeleton $\text{Sk}(\mathcal{X}, \mathcal{H})$ as stated in Theorem 1.3. Moreover, $\text{trop}(f_1), \dots, \text{trop}(f_n)$ generate $\text{trop}(\text{Rat}(X_{\bar{K}}))$, which is equal to $\text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|)$ by Theorem 1.1. In particular, the proof provides explicit coordinate functions for the embedding of the skeleton. This way, we extend the existing results on effective tropicalization in dimension one [BPR16, CM19, Hel19, KMM08, KY21, MRS23], and for special varieties of higher dimension [CHW14, DP16, Kut16, KY19, Ran17], to arbitrary varieties in all dimension.

Our proof of Theorem 1.2 gives a stronger result. Let Σ be an integral \mathbb{Q} -affine polyhedral complex in \mathbb{R}^n . We define $\text{Rat}(|\Sigma|)$ to be the set consisting of ∞ and piecewise integral \mathbb{Q} -affine functions on $|\Sigma|$ which take the same slope near infinity along any pair of parallel half-lines in $|\Sigma|$ (see Definition 2.2).

Let $\text{Rat}(\mathbb{R}^n)$ denote the subsemifield of the semifield of piecewise integral \mathbb{Q} -affine functions on \mathbb{R}^n generated by the coordinate functions x_1, \dots, x_n . Then, we have a natural restriction map

$$\pi: \text{Rat}(\mathbb{R}^n) \rightarrow \text{Rat}(|\Sigma|).$$

We denote by $\text{Ker}(\pi)$ the kernel of π as a congruence. For the notion of a congruence, which serves as the analogue of an ideal when forming a quotient in the context of semirings, see Section 3.6.

Theorem 1.4. *The restriction map π is surjective, meaning that the restrictions of the coordinate functions x_1, \dots, x_n to $|\Sigma|$ generate the semifield $\text{Rat}(|\Sigma|)$ over \mathbb{TQ} . Moreover, $\text{Ker}(\pi)$ is finitely generated as a congruence.*

This generalizes a result of Song [Son24] in dimension one to arbitrary dimensions when Σ is defined over \mathbb{TQ} .

As a final remark, we have restricted our attention to discretely valued fields to avoid technicalities and ensure a clearer presentation. Nevertheless, we are confident that the arguments developed here extend to more general valued fields. Likewise, the assumption that X is smooth, projective, and geometrically connected is not restrictive, as our tropicalization results pertain to the birational geometry of varieties.

The text is organized as follows. In Section 2, we provide preliminaries on polyhedral geometry, followed by background materials on tropical and non-Archimedean geometry in Section 3. Theorem 1.1 is proved in Section 4. The proofs of Theorems 1.2, 1.3, and 1.4 are given in Section 5.

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2. POLYHEDRAL COMPLEXES

We gather some definition and notation regarding embedded and abstract polyhedral complexes.

2.1. Polyhedral complexes in \mathbb{R}^n . We denote the points of \mathbb{R}^n with bold letters \mathbf{u} with coordinates u_1, \dots, u_n .

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product of \mathbb{R}^n . An *integral \mathbb{Q} -affine function* on \mathbb{R}^n is a function of the form

$$\mathbb{R}^n \rightarrow \mathbb{R}, \quad \mathbf{u} \mapsto \langle \mathbf{m}, \mathbf{u} \rangle + \gamma$$

for some $\mathbf{m} \in \mathbb{Z}^n$ and $\gamma \in \mathbb{Q}$. A function of the above form with $\mathbf{m} \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ is called an affine function.

An *integral \mathbb{Q} -affine polyhedron* in \mathbb{R}^n is a subset of the form

$$\sigma = \{ \mathbf{u} \in \mathbb{R}^n \mid L_i(\mathbf{u}) \geq 0 \text{ for } i = 1, \dots, r \}$$

for integral \mathbb{Q} -affine functions L_i on \mathbb{R}^n . Each face of an integral \mathbb{Q} -affine polyhedron is again integral \mathbb{Q} -affine. When σ is compact, it is called an *integral \mathbb{Q} -affine polytope*. If the L_i s in the definition of σ are integral linear, then σ is called a *rational cone*.

An *integral \mathbb{Q} -affine polyhedral complex* Σ in \mathbb{R}^n is a polyhedral complex whose cells are integral \mathbb{Q} -affine. Recall that this means Σ is a finite collection of integral \mathbb{Q} -affine polyhedra σ such that

- for each $\sigma \in \Sigma$, any face τ of σ is included in Σ .
- for each pair $\sigma, \eta \in \Sigma$ with nonempty intersection, $\sigma \cap \eta$ is a common face of σ and η .

Let Σ be an integral \mathbb{Q} -affine polyhedral complex in \mathbb{R}^n . For $\sigma, \tau \in \Sigma$, we write $\tau \preceq \sigma$ if τ is a face of σ . A *ray* of Σ is an unbounded one-dimensional element ρ in Σ . The *support* of Σ denoted by $|\Sigma|$ is the union of polyhedra $\sigma \in \Sigma$. We say that Σ is *complete* if $|\Sigma| = \mathbb{R}^n$.

Each integral \mathbb{Q} -affine polyhedron σ in \mathbb{R}^n can be expressed as a sum $P + \tau$, where P is an integral \mathbb{Q} -affine polytope and τ is a rational cone, uniquely determined by σ . We refer to τ as the *recession fan* of σ , and denote it by $\text{rec}(\sigma)$. When the collection $\text{rec}(\sigma)$, for $\sigma \in \Sigma$, forms a fan, we call it the *recession fan* of Σ , see [BGS11].

Let Σ, Σ' be two integral \mathbb{Q} -affine polyhedral complexes in \mathbb{R}^n . We say that Σ' is a *refinement* of Σ if $|\Sigma| = |\Sigma'|$ and for each $\sigma' \in \Sigma'$, there exists $\sigma \in \Sigma$ with $\sigma' \subseteq \sigma$.

Let $\sigma \subseteq \mathbb{R}^n$ be an integral \mathbb{Q} -affine polyhedron. An (*integral \mathbb{Q} -*)*affine function on σ* is by definition the restriction to σ of an (integral \mathbb{Q} -)affine function $\mathbb{R}^n \rightarrow \mathbb{R}$.

We define the space of facewise, resp. piecewise, (integral \mathbb{Q} -)affine functions on Σ , resp. $|\Sigma|$, as follows.

Definition 2.1 (Facewise (integral \mathbb{Q} -)affine functions and $\text{Rat}(\Sigma)$). Let Σ be an integral \mathbb{Q} -affine polyhedral complex in \mathbb{R}^n . We say that $F: |\Sigma| \rightarrow \mathbb{R}$ is *facewise* (integral \mathbb{Q} -)affine on Σ if $F|_{\sigma}: \sigma \rightarrow \mathbb{R}$ is (integral \mathbb{Q} -)affine for any $\sigma \in \Sigma$.

We denote by $\text{Rat}(\Sigma)$ the union of $\{\infty\}$ and the space of facewise integral \mathbb{Q} -affine functions on Σ which take the same slope at infinity along any pair of parallel rays in Σ . Similarly, we define $\text{Rat}(\Sigma, \mathbb{R})$ as the union of $\{\infty\}$ and the space of all facewise affine functions on Σ which take the same slope at infinity along any pair of parallel rays in Σ . \diamond

Definition 2.2 (Piecewise (integral \mathbb{Q} -)affine functions and $\text{Rat}(|\Sigma|)$). Let Σ be an integral \mathbb{Q} -affine polyhedral complex in \mathbb{R}^n . We say that $F: |\Sigma| \rightarrow \mathbb{R}$ is *piecewise* integral \mathbb{Q} -affine if there exists an integral \mathbb{Q} -affine refinement Σ' of Σ such that the restriction $F|_{\sigma'}: \sigma' \rightarrow \mathbb{R}$ is an integral \mathbb{Q} -affine function on σ' for each $\sigma' \in \Sigma'$. We denote by $\text{Rat}(|\Sigma|)$ the union of $\{\infty\}$ and the space of all piecewise integral \mathbb{Q} -affine functions F , as above, where F takes the same slopes at infinity along any two parallel half-lines of $|\Sigma|$.

Similarly, we define *piecewise* affine functions on $|\Sigma|$ and $\text{Rat}(|\Sigma|, \mathbb{R})$, by removing the integrality constraint. \diamond

Note that each element $F \in \text{Rat}(\Sigma)$ belongs to $\text{Rat}(\Sigma')$ for some integral \mathbb{Q} -affine refinement of Σ' . Moreover, a pair of integral \mathbb{Q} -affine refinements Σ' and Σ'' have a common integral \mathbb{Q} -affine refinement. Therefore, $\text{Rat}(|\Sigma|)$ is closed under addition and taking minimum. This turns $\text{Rat}(|\Sigma|)$ into a semifield (see Section 3.5). The same remark applies to $\text{Rat}(|\Sigma|, \mathbb{R})$.

For future use, we define convex functions on integral \mathbb{Q} -affine polyhedral complexes.

Definition 2.3 (Convex function). Let Σ be an integral \mathbb{Q} -affine polyhedral complex in \mathbb{R}^n and let $H: |\Sigma| \rightarrow \mathbb{R}$ be a facewise (integral \mathbb{Q} -)affine function on Σ . We say that H is *convex* if, for any $\sigma \in \Sigma$, there exists an (integral \mathbb{Q} -)affine function $L_{\sigma}: \mathbb{R}^n \rightarrow \mathbb{R}$ such that $H|_{\sigma} = L_{\sigma}|_{\sigma}$ and

$$(H - L_{\sigma})|_{\eta \setminus \sigma} \geq 0$$

on $\eta \setminus \sigma$ for each $\eta \in \Sigma$ with $\eta \succeq \sigma$. We say that H is *strictly convex* if the inequalities above are strict on $\eta \setminus \sigma$. \diamond

We say that a function $H: |\Sigma| \rightarrow \mathbb{R}$ is *concave* if $-H$ is convex.

Definition 2.4 (Convexity of a polyhedral complex). An integral \mathbb{Q} -affine polyhedral complex Σ is said to be *convex* if there exists a strictly convex function $H \in \text{Rat}(\Sigma)$. \diamond

2.2. Abstract polyhedral complex with a recession fan. For a set V , we denote by 2^V the power set of V .

An *abstract polyhedral complex with a recession fan* is defined as follows. Let V_f and V_∞ be two disjoint finite sets, with V_f nonempty. Let Π be a finite delta-complex on the set $V_f \cup V_\infty$. This means that Π is a finite set endowed with a partial order \preceq and there exists a map $\zeta: \Pi \rightarrow 2^{V_f \cup V_\infty}$ with the following properties:

- (1) There exists a minimum element in Π , denoted by $\widehat{0}$, with $\zeta(\widehat{0}) = \emptyset$.
- (2) For each $x \in V_f \cup V_\infty$, there exists a unique element τ of Π with $\zeta(\tau) = \{x\}$.
- (3) For each $\tau \in \Pi$, the restriction of ζ to the interval $[\widehat{0}, \tau] \subseteq \Pi$ is a bijection with the poset of subsets of $\zeta(\tau)$. Here, the partial order on subsets is given by inclusion.

In particular, for each pair of elements $\tau \preceq \eta$ in Π , we have $\zeta(\tau) \subseteq \zeta(\eta)$.

Denote by Δ the set of all elements of Π with nonempty intersection with V_f . We call Δ an *abstract polyhedral complex with vertex set V_f and recession rays V_∞* . By an abuse of notation, we denote the restriction of ζ to Δ by $\zeta: \Delta \rightarrow 2^{V_f \cup V_\infty}$. The complement $\Pi \setminus \Delta$ is itself a delta-complex. Let Υ be the abstract polyhedral complex with vertex set $\{0\}$ and recession rays V_∞ which as a set is equal to $\Pi \setminus \Delta$ endowed with the same partial order \preceq , with the map $\zeta_\infty: \Upsilon \rightarrow 2^{\{0\} \cup V_\infty}$ defined by $\zeta_\infty(\tau) = \{0\} \cup \zeta(\tau)$ for all $\tau \in \Pi \setminus \Delta$. Although the definition of Υ is dependent on Π , by an abuse of notation we refer to Υ as the recession fan of Δ , and call Δ an *abstract polyhedral complex with recession fan Υ* .

Given a nonempty subset $A \subseteq V_f$ and a subset $B \subseteq V_\infty$, we define the *extended simplex*

$$\Theta_{A \cup B} := \sigma^A \times \mathbb{R}_{\geq 0}^B \subseteq \mathbb{R}^{A \cup B} \subseteq \mathbb{R}^{V_f \cup V_\infty},$$

where $\mathbb{R}_{\geq 0}^B$ denotes the non-negative orthant in \mathbb{R}^B , and σ^A denotes the standard simplex in \mathbb{R}^A ,

$$(2.1) \quad \sigma^A := \left\{ (u_v)_{v \in A} \in \mathbb{R}_{\geq 0}^A \mid \sum_{v \in A} u_v = 1 \right\}.$$

We denote by ϵ_ρ , $\rho \in V_\infty$, the standard basis of \mathbb{R}^{V_∞} .

Given an abstract polyhedral complex Δ with recession fan Υ as defined above, we associate to each element $\tau \in \Delta$ a copy Θ_τ of the polyhedron $\Theta_{\zeta(\tau)} \subset \mathbb{R}^{V_f \cup V_\infty}$. The property (3) ensures that the polyhedra Θ_τ are nonempty and they glue together, along inclusions, for each pair of elements $\tau \preceq \eta$ in Δ , of Θ_τ as a face of Θ_η .

The resulting space is denoted by $|\Delta|$ and is called the *support* of Δ . When there is no risk of confusion, we simplify the notation and write τ in place of Θ_τ .

Using this simplified notation:

- The *finite part* of τ is denoted by τ_f and corresponds to the simplex $\sigma^{\zeta(\tau) \cap V_f}$.
- The *infinite part* of τ is denoted by τ_∞ , it corresponds to the cone $\mathbb{R}_{\geq 0}^{\zeta(\tau) \cap V_\infty}$ and naturally belongs to Υ .

Similarly, we define $|\Upsilon|$, the support of the recession fan Υ , by gluing the collection of Θ_τ , identified with $\mathbb{R}_{\geq 0}^{\zeta_\infty(\tau) \cap V_\infty}$, for all $\tau \in \Upsilon$. (This can be viewed as a multifan in \mathbb{R}^{V_∞} .)

A *ray* in Δ is an element ϱ with $\zeta(\varrho) \cap V_f$ and $\zeta(\varrho) \cap V_\infty$ each have size one. Two rays ϱ and ϱ' are *parallel* if they correspond to the same element of V_∞ .

More generally, two half-lines in $|\Delta|$ are *parallel* if they correspond to the same half-line issued from the origin in $|\Upsilon|$.

Example 2.5. (1) Let $V_f = \{v_1, v_2\}$ and $V_\infty = \{\rho, \rho'\}$. We set

$$\Delta = \{v_1, v_2, e, \varrho_1, \varrho'_1, \varrho_2, \varrho'_2, \sigma_1, \sigma_2\}, \quad \Upsilon = \{\widehat{0}, \rho, \rho', \tau_1, \tau_2\}, \quad \Pi = \Delta \cup \Upsilon$$

and set

$$\begin{aligned} \zeta(v_i) &= \{v_i\}, & \zeta(e) &= \{v_1, v_2\}, & \zeta(\varrho_i) &= \{v_i, \rho\}, & \zeta(\varrho'_i) &= \{v_i, \rho'\}, & \zeta(\sigma_i) &= \{v_i, \rho, \rho'\}, \\ \zeta(\rho) &= \{\rho\}, & \zeta(\rho') &= \{\rho'\}, & \zeta(\tau_1) &= \zeta(\tau_2) &= \{\rho, \rho'\}. \end{aligned}$$

Then, a half-line in σ_1 is parallel to a half-line in σ_2 if they are either parallel to ρ or they are parallel to ρ' .

(2) We take the same example as above, but we change Υ to

$$\Upsilon = \{\widehat{0}, \rho, \rho', \tau\} \quad \text{and} \quad \zeta(\tau) = \{\rho, \rho'\}.$$

In this second example, the recession cones of σ_1 and σ_2 are the same, and so a half-line in σ_1 is parallel to a half-line in σ_2 if they are the same half-line in τ . \diamond

An *integral \mathbb{Q} -affine refinement* Δ' of Δ consists in a collection of integral \mathbb{Q} -affine refinements of each τ , in the sense of the previous section when τ is viewed as $\Theta_{\zeta(\tau)}$ embedded in $\mathbb{R}^{V_f \cup V_\infty}$. These refinements are required to be compatible with each other along inclusion of faces. Two rays in Δ' are parallel if they are parallel half-lines in $|\Delta| = |\Delta'|$.

Given an abstract polyhedral complex Δ with recession fan Υ , we define $\text{Rat}(\Delta)$ as the union of $\{\infty\}$ and the space of facewise integral \mathbb{Q} -affine functions on Δ which take the same slope near infinity along parallel rays of Δ . We define $\text{Rat}(|\Delta|)$ as the union of $\{\infty\}$ and the collection of all facewise integral \mathbb{Q} -affine functions on some integral \mathbb{Q} -affine refinement Δ' which take the same slopes near infinity along parallel rays of Δ' . Equivalently, $\text{Rat}(|\Delta|)$ is the union of $\{\infty\}$ and the collection of all piecewise integral \mathbb{Q} -affine functions on $|\Delta|$ which take the same slope near infinity along two parallel half-lines in $|\Delta|$. The set $\text{Rat}(|\Delta|)$ has the structure of a semifield with addition corresponding to taking pointwise minimum of functions, and multiplication corresponding to the usual addition (see Section 3.5).

3. TROPICAL PRELIMINARIES

The purpose of this section is primarily to provide definitions and establish notation. For further details, we refer the reader to [Ber90, Ber99, Ber04, GRW16, BFJ16, dJ96, KY19, MN15].

3.1. Notation and conventions. As in the introduction, let K be a complete discrete valuation field. Let R be the valuation ring of K , \mathfrak{m} the maximal ideal of R , and $\kappa = R/\mathfrak{m}$ the residue field. We fix a uniformizer $\varpi \in \mathfrak{m}$ of R . Let $\nu_K: K^\times \rightarrow \mathbb{Z}$ be the valuation map with $\nu_K(\varpi) = 1$.

By a *variety X over K* we mean an irreducible, reduced, and separated scheme X of finite type over K . By a *variety \mathcal{X} over R* , we mean an irreducible, reduced, and separated scheme \mathcal{X} , flat and of finite type over R . The special fiber of \mathcal{X} is denoted by $\mathcal{X}_\kappa := \mathcal{X} \times_R \kappa$, and the generic fiber of \mathcal{X} is denoted by $\mathcal{X}_K := \mathcal{X} \times_R K$.

3.2. Semistable models and skeleta. Let \mathcal{X} be a projective variety over R of dimension $d + 1$. Let Z_v , $v \in V_f$, denote the irreducible components of the special fiber \mathcal{X}_κ . For a subset $A \subseteq V_f$, we put $Z_A = \bigcap_{v \in A} Z_v$. We say that \mathcal{X} is *strictly semistable* if \mathcal{X} is covered by open subsets \mathcal{U} that admit an étale morphism

$$\psi: \mathcal{U} \rightarrow \mathcal{S} := \text{Spec}(R[x_0, \dots, x_d]/\langle x_0 \cdots x_r - \varpi \rangle) \quad \text{for } 0 \leq r \leq d.$$

Under these conditions, \mathcal{X} is regular, the special fiber \mathcal{X}_κ is reduced, and for each subset $A \subseteq V_f$, each connected component of Z_A is smooth over κ . For further details, see [GRW16, Definition 3.1] and [dJ96, 2.16].

Let X be a smooth projective variety of dimension d over K . A *strictly semistable model* of X over R is a strictly semistable projective variety \mathcal{X} over R endowed with an isomorphism $\mathcal{X}_K \rightarrow X$ over K .

Let \mathcal{X} be a projective variety over R of dimension $d + 1$. Let $\mathcal{H} = \sum_{\rho \in V_\infty} W_\rho$ be a sum of finitely many hypersurfaces W_ρ on \mathcal{X} , where the W_ρ are irreducible and pairwise distinct. We set $\mathcal{Y} := \mathcal{X}_\kappa + \mathcal{H}$.

We say that the pair $(\mathcal{X}, \mathcal{H})$ is a *strictly semistable pair* if \mathcal{X} is covered by open subsets \mathcal{U} which admit an étale morphism

$$\psi: \mathcal{U} \rightarrow \mathcal{S} := \text{Spec}(R[x_0, \dots, x_d] / \langle x_0 \cdots x_r - \varpi \rangle)$$

for $0 \leq r \leq d$ such that the restriction of W_ρ to \mathcal{U} is either trivial or defined by $\psi^*(x_j)$ for some $r + 1 \leq j \leq d$. In this case, \mathcal{X} is strictly semistable, \mathcal{Y} is a divisor with strict normal crossings on \mathcal{X} , and for each $B \subseteq V_\infty$, $W_B = \bigcap_{\rho \in B} W_\rho$ is a disjoint union of varieties over R , which are themselves strictly semistable over R , cf. [GRW16, Definition 3.1] and [dJ96, 6.3, 6.4]. We refer to the W_ρ , $\rho \in V_\infty$, as the *horizontal components* and to the Z_v , $v \in V_f$, as the *vertical components* of \mathcal{Y} .

Given a smooth projective variety X defined over K , we say that $(\mathcal{X}, \mathcal{H})$ is a *strictly semistable pair* of X if \mathcal{X} is a strictly semistable model of X .

From a strictly semistable pair $(\mathcal{X}, \mathcal{H})$ of X , we obtain a delta-complex $\Pi(\mathcal{X}, \mathcal{H})$ giving rise to an abstract polyhedral complex $\Delta(\mathcal{X}, \mathcal{H})$ and a recession fan $\Upsilon(\mathcal{X}, \mathcal{H})$, and the corresponding skeleton $\text{Sk}(\mathcal{X}, \mathcal{H})$ in X^{an} , with its integral polyhedral complex structure, as follows.

Stratification. Let $(\mathcal{X}, \mathcal{H})$ be a strictly semistable pair of X . By the definition of a strictly semistable pair, $\mathcal{Y} = \mathcal{X}_\kappa + \mathcal{H}$ is a reduced divisor on \mathcal{X} , and each irreducible component of \mathcal{Y} is either a connected smooth projective variety over κ (for the Z_v , $v \in V_f$), or a connected regular projective variety over R (for the W_ρ , $\rho \in V_\infty$).

We define a stratification of \mathcal{Y} as follows. We put $\mathcal{Y}^{(0)} := \mathcal{Y}$, and for each $n \in \mathbb{Z}_{>0}$, we let $\mathcal{Y}^{(n)}$ be the complement of the set of regular points in $\mathcal{Y}^{(n-1)}$. Thus, we obtain a chain of closed subsets:

$$\mathcal{Y} = \mathcal{Y}^{(0)} \supseteq \mathcal{Y}^{(1)} \supseteq \dots \supseteq \mathcal{Y}^{(t)} \supseteq \mathcal{Y}^{(t+1)} = \emptyset,$$

where $t \leq d$.

We say that S is a *stratum* of \mathcal{Y} if there exist subsets $A \subseteq V_f$ and $B \subseteq V_\infty$ such that S is a nonempty connected component of $(Z_A \cap Z_B) \setminus \mathcal{Y}^{(|A|+|B|)}$. We say that S is *special* if $S \subseteq \mathcal{X}_\kappa$. Note that S is a special stratum if and only if $S \neq \emptyset$ and $A \neq \emptyset$.

Dual complex and (extended) simplices associated with special strata. Let $\Pi(\mathcal{X}, \mathcal{H})$ be the set consisting of an element $\widehat{0}$ and other elements in bijection with the strata of \mathcal{Y} . We endow this set with the partial order corresponding to the reverse inclusion of strata, and declaring that $\widehat{0}$ is the minimum element. We define the map $\zeta: \Pi(\mathcal{X}, \mathcal{H}) \rightarrow 2^{V_f \cup V_\infty}$, by setting $\zeta(\widehat{0}) = \emptyset$ and sending each stratum to the union of the corresponding index sets $A \subseteq V_f$ and $B \subseteq V_\infty$. This turns $\Pi(\mathcal{X}, \mathcal{H})$ into a delta-complex on the set $V_f \cup V_\infty$.

Let $\Delta(\mathcal{X}, \mathcal{H})$ be the corresponding abstract polyhedral complex and $\Upsilon(\mathcal{X}, \mathcal{H})$ the corresponding recession fan in the sense of Section 2.2. Specifically, $\Delta(\mathcal{X}, \mathcal{H})$ is the set of elements τ in bijection with the special strata S_τ of \mathcal{Y} . We refer to $\Delta(\mathcal{X}, \mathcal{H})$ as the *dual complex* of $(\mathcal{X}, \mathcal{H})$.

Let S_τ , $\tau \in \Delta(\mathcal{X}, \mathcal{H})$, be a special stratum of \mathcal{Y} with the corresponding index sets $A \subseteq V_f$ and $B \subseteq V_\infty$ as above. We associate to τ the extended simplex $\Theta_\tau = \Theta_{A \cup B} = \sigma^A \times \mathbb{R}_{>0}^B$.

There is a canonical embedding of Θ_τ in X^{an} , defined as follows. To each $\mathbf{u} \in \Theta_\tau$, we associate an absolute value $|\cdot|_{\tau, \mathbf{u}}$ on the function field $\text{Rat}(X)$ which extends $|\cdot|_K$. Let $\xi = \xi_\tau$ be the generic point of S_τ . For each $v \in A$ and $\rho \in B$, we choose a local equation $z_v = 0$ of Z_v and $w_\rho = 0$ of W_ρ in \mathcal{X} at ξ .

Let $\widehat{\mathcal{O}}_{\mathcal{X},\xi}$ be the completion of $\mathcal{O}_{\mathcal{X},\xi}$. By [MN15, §2.4], each element $f \in \widehat{\mathcal{O}}_{\mathcal{X},\xi}$ has an expansion of the form

$$f = \sum_{\mathbf{m}=(m_v)_{v \in A} \times (m_\rho)_{\rho \in B} \in \mathbb{Z}_{\geq 0}^{A \cup B}} a_{\mathbf{m}} \prod_{v \in A} z_v^{m_v} \cdot \prod_{\rho \in B} w_\rho^{m_\rho},$$

where each coefficient $a_{\mathbf{m}}$ is either zero or a unit in $\widehat{\mathcal{O}}_{\mathcal{X},\xi}$. Such an expansion is called an *admissible expansion* of f .

We write $\mathbf{u} = (u_v)_{v \in A} \times (u_\rho)_{\rho \in B} \in \sigma^A \times \mathbb{R}_{\geq 0}^B$, and set

$$(3.1) \quad |f|_{\tau, \mathbf{u}} := \max_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{A \cup B}, a_{\mathbf{m}} \neq 0} \left\{ \exp\left(-\sum_{v \in A} u_v m_v - \sum_{\rho \in B} u_\rho m_\rho\right) \right\}.$$

By [MN15, Prop 2.4.4], $|\cdot|_{\tau, \mathbf{u}}$ is a well-defined absolute value on $\mathcal{O}_{\mathcal{X},\xi}$. This absolute value extends to an absolute value $|\cdot|_{\tau, \mathbf{u}}$ on $\text{Rat}(X)$. Furthermore, since $\varpi = \lambda \prod_{v \in A} z_v$ for some unit λ in $\mathcal{O}_{\mathcal{X},\xi}$, we have $|\varpi|_{\tau, \mathbf{u}} = \exp(-1)$, which shows that $|\cdot|_{\tau, \mathbf{u}}$ agrees with $|\cdot|_K$ for elements in K . Thus, for each $\mathbf{u} \in \Theta_\tau$, we have a point $|\cdot|_{\tau, \mathbf{u}} \in X^{\text{an}}$ lying on the generic point of X .

With the above notation, the assignment

$$(3.2) \quad \Theta_\tau = \sigma^A \times \mathbb{R}_{\geq 0}^B \rightarrow \{|\cdot|_{\tau, \mathbf{u}} \in X^{\text{an}} \mid \mathbf{u} \in \Theta_\tau\} \subset X^{\text{an}}, \quad \mathbf{u} \mapsto |\cdot|_{\tau, \mathbf{u}}$$

is a homeomorphism, cf. [MN15, Prop 3.4], [GRW16, 4.2, 4.3]. We thus get an identification

$$(3.3) \quad \Theta_\tau \simeq \{|\cdot|_{\tau, \mathbf{u}} \in X^{\text{an}} \mid \mathbf{u} \in \sigma^A \times \mathbb{R}_{\geq 0}^B\} \subset X^{\text{an}}.$$

The polyhedron Θ_τ is endowed with a natural integral affine structure.

Note that by the construction, $|\cdot|_{\tau, \mathbf{u}}$ is independent of the choice of local equations $z_v = 0$ and $w_\rho = 0$ for $\mathcal{X}_{\kappa, v}$ and H_ρ . Thus, the embedding of Θ_τ in the Berkovich space given in (3.2) is intrinsic, cf. [GRW16, p. 169].

Let S_σ be another special stratum such that the Zariski closure $\overline{S_\sigma}$ of S_σ contains S_τ . Then, Θ_σ is a face of Θ_τ . Moreover, the embedding of Θ_σ and Θ_τ in X^{an} are compatible, as we now explain. Since $\overline{S_\sigma} \supseteq S_\tau$ and S_σ is a special stratum, there exist a subset A' of A and a subset B' of B such that S_σ is defined in \mathcal{X} at its generic point by $z_v = 0$ for all $v \in A'$ and $w_\rho = 0$ for all $\rho \in B'$. The extended simplex $\Theta_\sigma = \sigma^{A'} \times \mathbb{R}_{\geq 0}^{B'}$ is the face of $\Theta_\tau = \sigma^A \times \mathbb{R}_{\geq 0}^B$ consisting of those points with $u_v = 0$ for $v \in A \setminus A'$ and $u_\rho = 0$ for $\rho \in B \setminus B'$.

Then, by the very definition of $|\cdot|_{\tau, \mathbf{u}}$, we see that $|\cdot|_{\tau, \mathbf{u}} = |\cdot|_{\sigma, \mathbf{u}}$ for any $\mathbf{u} \in \Theta_\sigma = \sigma^{A'} \times \mathbb{R}_{\geq 0}^{B'}$. It follows that the embedding of Θ_σ in X^{an} coincides with the restriction to the face $\Theta_\sigma \subseteq \Theta_\tau$ of the embedding of Θ_τ in X^{an} .

Skeleton $\text{Sk}(\mathcal{X}, \mathcal{H})$. The *skeleton* $\text{Sk}(\mathcal{X}, \mathcal{H}) \subseteq X^{\text{an}}$ associated with the strictly semistable pair $(\mathcal{X}, \mathcal{H})$ is defined by

$$\text{Sk}(\mathcal{X}, \mathcal{H}) := \bigcup_{\tau \in \Delta(\mathcal{X}, \mathcal{H})} \Theta_\tau.$$

The skeleton can be identified with the support of $\Delta(\mathcal{X}, \mathcal{H})$, embedded in X^{an} .

In the following, we often denote by σ, τ, δ the elements of $\Delta(\mathcal{X}, \mathcal{H})$. For two elements τ, δ in $\Delta(\mathcal{X}, \mathcal{H})$, we write

$$(3.4) \quad \tau \succeq \delta$$

if Θ_δ is a face of Θ_τ , equivalently, if $\overline{S_\delta} \supseteq S_\tau$.

Parallel rays in $\Delta(\mathcal{X}, \mathcal{H})$ and parallel half-lines in $\text{Sk}(\mathcal{X}, \mathcal{H})$. We define parallel rays of $\Delta(\mathcal{X}, \mathcal{H})$ as in Section 2.2. Identifying $\text{Sk}(\mathcal{X}, \mathcal{H})$ with the support of $\Delta(\mathcal{X}, \mathcal{H})$, we define the notion of parallel half-lines for the skeleton.

Remark 3.1. A slightly different notion of a skeleton is considered in the work of [Duc12] and [DHL24]. We compare their notion with ours.

Let \mathbb{G}_m^d be the algebraic torus of dimension d . The analytification $(\mathbb{G}_m^d)^{\text{an}}$ has a canonical skeleton denoted by S_d defined as follows. Let $\mathbb{P}_R^d = \text{Proj } R[T_0, \dots, T_d]$ be the projective d -space over R . We take the horizontal divisor $\mathcal{H} = \bigcup_{i=0}^d W_i$, where W_i is given by $T_i = 0$ for $i = 0, \dots, d$. Then $(\mathbb{P}_R^d, \mathcal{H})$ is a strictly semistable pair, and S_d is equal to the skeleton $\text{Sk}(\mathbb{P}_R^d, \mathcal{H})$. Note that S_d can be canonically identified with $\mathbb{R}^{d+1}/\mathbb{R}\mathbf{1}$, where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^{d+1}$.

Suppose now that U is an open subset of X and consider a finite collection of quasi-finite maps $\varphi_i: U \rightarrow \mathbb{G}_m^d$. By a slight abuse of notation, we still denote by φ_i the corresponding map between the analytifications. A subset S of $\bigcup_i \varphi_i^{-1}(S_d)$ defined by a Boolean combination of inequalities between log-rational functions, i.e., tropicalizations of rational functions, is called a skeleton in *loc. cit.* Let $\text{PL}(S)$ be the semifield of tropical functions consisting of ∞ and the set of piecewise affine functions on S .

We claim that for discretely valued fields of residue characteristic zero, S is contained in a skeleton $\text{Sk}(\mathcal{X}', \mathcal{H}')$ for some strictly semistable pair of X' , where X' is a compactification of U after a finite extension of K . (We will use resolution of singularities below, but it might be possible to use alterations to get the same result without restriction on characteristics, as done by Gubler [Gub07, Prop 5.2]. See also Berkovich [Ber99].)

Note that the tropicalization map $\text{trop}: \text{Rat}(X'_K) \rightarrow \text{PL}(S)$ factorizes through $\text{Rat}(\text{Sk}(\mathcal{X}', \mathcal{H}'))$, as a composition of the restriction map $\text{Rat}(\text{Sk}(\mathcal{X}', \mathcal{H}')) \rightarrow \text{PL}(S)$ and the tropicalization map $\text{trop}: \text{Rat}(X'_K) \rightarrow \text{Rat}(\text{Sk}(\mathcal{X}', \mathcal{H}'))$. Thus, Theorems 1.1 and 1.2 imply that the image of $\text{trop}: \text{Rat}(X'_K) \rightarrow \text{PL}(S)$ is finitely generated, giving a new proof of [DHL24, Main Theorem (Berkovich setting)] over discrete valued fields of residue characteristic zero.

To prove the claim, we find first a compactification X_i of U and an extension $\psi_i: X_i \rightarrow \mathbb{P}_K^d$ of φ_i for each i . Then we find a compactification X' of U which dominates all the X_i . Denote by $\tilde{\psi}_i$ the composition of $X' \rightarrow X_i$ and ψ_i . We take a model \mathcal{X}' of X' such that $\tilde{\psi}_i: X' \rightarrow \mathbb{P}_K^d$ extends to $\tilde{\psi}_i: \mathcal{X}' \rightarrow \mathbb{P}_R^d$. By the semistable reduction theorem (see [KKMSD73, p. 198]), after a finite extension K'/K , we may assume that the special fiber is strictly semistable. Using Hironaka's theorem on resolution of singularities, we may assume that the union of the special fiber of \mathcal{X}' and the union of the pull-backs of \mathcal{H} along $\tilde{\psi}_i$ for all i are normal crossing divisors. Let $\mathcal{H}' = \bigcup_i \tilde{\psi}_i^*(\mathcal{H})_{\text{red}}$. Then the pair $(\mathcal{X}', \mathcal{H}')$ is a strictly semistable pair.

We show that $\bigcup_i \varphi_i^{-1}(S_d)$ is contained in $\text{Sk}(\mathcal{X}', \mathcal{H}')$, which implies the result for S . Take a point $c \in S_d = \mathbb{R}^{d+1}/\mathbb{R}\mathbf{1}$, and choose a representative of c in \mathbb{R}^{d+1} such that the minimum of its coordinates is zero. Without loss of generality, assume that $c_0 = 0$. Let $t_j = T_j/T_0$, and denote by $J \subseteq [d+1]$ the nonzero coordinates of c . The center of the valuation defined by c is the generic point of the intersection of \mathbb{P}_κ^d and the hyperplanes $(T_j = 0)$ in \mathbb{P}_R^d for $j \in J$. Since $\tilde{\psi}_i: \mathcal{X}' \rightarrow \mathbb{P}_R^d$ is proper, a point $x \in \varphi_i^{-1}(c)$ has a center in the special fiber of \mathcal{X}' , which is the generic point of the intersection S_τ of an irreducible component of the special fiber of \mathcal{X}' and horizontal components W_j in $\psi^*(\mathcal{H})_{\text{red}}$, for $j \in J$, dominating the hyperplanes $(T_j = 0)$. For such a stratum in the special fiber of \mathcal{X}' , there is a unique element in $\varphi_i^{-1}(c)$ (the point x) with center the generic point η_τ of that stratum. Denote by e_j the order of vanishing of t_j along W_j at η_τ . Then x is the point of the cone $\Theta_\tau \cong \mathbb{R}_{\geq 0}^d \subseteq \text{Sk}(\mathcal{X}', \mathcal{H}')$ with coordinates c_j/e_j for $j \in J$. This shows that $\bigcup_i \varphi_i^{-1}(S_d)$ is contained in $\text{Sk}(\mathcal{X}', \mathcal{H}')$. \diamond

3.3. Tropicalization and the tropical function field of $(\mathcal{X}, \mathcal{H})$. We retain the notation and assumptions introduced in the previous subsection.

We define $\text{Rat}(\Delta(\mathcal{X}, \mathcal{H}))$ and $\text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|)$ as in Section 2.2. The latter, $\text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|)$, is referred to as the *tropical function field of $(\mathcal{X}, \mathcal{H})$* .

Let $f \in \text{Rat}(\mathcal{X}) = \text{Rat}(X)$ be a nonzero element. Let $x \in \text{Sk}(\mathcal{X}, \mathcal{H})$, and let $\tau \in \Delta(\mathcal{X}, \mathcal{H})$ and $\mathbf{u} \in \sigma^A \times \mathbb{R}_{\geq 0}^B$ such that $x = |\cdot|_{\tau, \mathbf{u}}$, see (3.3). We set

$$\text{trop}(f)(x) := -\log |f|_{\tau, \mathbf{u}} \in \mathbb{R}.$$

This yields a function

$$\text{trop}(f): \text{Sk}(\mathcal{X}, \mathcal{H}) \rightarrow \mathbb{R}.$$

We extend the definition by setting $\text{trop}(0) = \infty$.

Proposition 3.2. *For each $f \in \text{Rat}(\mathcal{X}) = \text{Rat}(X)$, we have $\text{trop}(f) \in \text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|)$.*

Proof. We may assume that $f \neq 0$. Take a special stratum S_τ , $\tau \in \Delta(\mathcal{X}, \mathcal{H})$. Let $A \subseteq V_f$ and $B \subseteq V_\infty$ be the subsets such that S_τ is a connected component of $Z_A \cap W_B$. We write $f = f_1/f_2$ with $f_1, f_2 \in \widehat{\mathcal{O}}_{\mathcal{X}, \xi_\tau}$, where ξ_τ is the generic point of S_τ .

Using the notation from (3.1), we represent f_1 and f_2 by admissible expansions

$$f_1 = \sum_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{A \cup B}} a_{\mathbf{m}} \prod_{v \in A} z_v^{m_v} \prod_{\rho \in B} w_\rho^{m_\rho}, \quad f_2 = \sum_{\mathbf{n} \in \mathbb{Z}_{\geq 0}^{A \cup B}} a_{\mathbf{n}} \prod_{v \in A} z_v^{n_v} \prod_{\rho \in B} w_\rho^{n_\rho}.$$

For each $\mathbf{u} \in \Theta_\tau = \sigma^A \times \mathbb{R}_{\geq 0}^B$, we have

$$\text{trop}(f)(\mathbf{u}) = \min_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{A \cup B}, a_{\mathbf{m}} \neq 0} \left\{ \sum_{v \in A} m_v u_v + \sum_{\rho \in B} m_\rho u_\rho \right\} - \min_{\mathbf{n} \in \mathbb{Z}_{\geq 0}^{A \cup B}, a_{\mathbf{n}} \neq 0} \left\{ \sum_{v \in V} n_v u_v + \sum_{\rho \in B} n_\rho u_\rho \right\}.$$

It follows that $\text{trop}(f)$ is a continuous piecewise integral \mathbb{Q} -affine function.

Furthermore, let $\varrho = \mathbf{u} + \mathbb{R}_{\geq 0} \mathbf{e}_\rho$ be a ray in Θ_τ parallel to $\rho \in B$. Let W_ρ be the horizontal irreducible component of \mathcal{H} corresponding to ρ . The slope of $\text{trop}(f)$ near infinity along ϱ equals to the order of vanishing of f along W_ρ , which remains constant for rays parallel to ρ .

More generally, on each Θ_τ , $\text{trop}(f)$ is the difference of the two functions H_1, H_2 , where each H_i is the tropicalization of f_i . Since H_1 is the minimum of affine functions consisting of a bounded term ($\sum_{v \in A} m_v u_v$) and an unbounded term ($\sum_{\rho \in B} m_\rho u_\rho$), and similarly for H_2 , the asymptotic behavior near infinity of H_i is captured by the minimum of their unbounded term

$$H_{1, \infty} := \min_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{A \cup B}} \left\{ \sum_{\rho \in B} m_\rho u_\rho \right\}, \quad H_{2, \infty} := \min_{\mathbf{n} \in \mathbb{Z}_{\geq 0}^{A \cup B}} \left\{ \sum_{\rho \in B} n_\rho u_\rho \right\}.$$

Let S_η be the stratum of \mathcal{Y} which contains S_τ and is a connected component of the intersection $\bigcap_{\rho \in B} W_\rho$. Denote by ξ_η the generic point of η . The admissible expansion for f_i at ξ_τ given above provides an admissible expansion of f_i at ξ_η of the form

$$f_1 = \sum_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{A \cup B}} \widehat{a}_{\mathbf{m}} \prod_{\rho \in B} w_\rho^{m_\rho}, \quad f_2 = \sum_{\mathbf{n} \in \mathbb{Z}_{\geq 0}^{A \cup B}} \widehat{a}_{\mathbf{n}} \prod_{\rho \in B} w_\rho^{n_\rho},$$

where $\widehat{a}_{\mathbf{m}} = a_{\mathbf{m}} \prod_{v \in A} z_v^{m_v}$ and $\widehat{a}_{\mathbf{n}} = a_{\mathbf{n}} \prod_{v \in A} z_v^{n_v}$. The associated cone Θ_η of \mathcal{Y} lives in the Berkovich analytification of X , when K is viewed as a trivially valued field. The tropicalization of f_i on Θ_η is given by the admissible expansions in the form $\text{trop}(f_i) = \min_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{A \cup B}} \left\{ \sum_{\rho \in B} m_\rho u_\rho \right\}$ and $\min_{\mathbf{n} \in \mathbb{Z}_{\geq 0}^{A \cup B}} \left\{ \sum_{\rho \in B} n_\rho u_\rho \right\}$, and does not depend on the choice of admissible expansions (see [JM12, Prop 3.1]). This is equal to $H_{i, \infty}$.

By the definition of parallel half-lines in $|\Delta(\mathcal{X}, \mathcal{H})|$, we infer that $\text{trop}(f)$ has the same slope near infinity along two parallel half-lines in $|\Delta(\mathcal{X}, \mathcal{H})|$.

We infer that $\text{trop}(f)$ belongs to $\text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|)$. \square

Let \bar{K} be an algebraic closure of K . Let $X_{\bar{K}} = X \times_K \bar{K}$. Recall that we assume that X and each connected component of the intersection of any collection of irreducible components of \mathcal{H} are geometrically connected.

Proposition 3.3. *The tropicalization map extends to $\text{trop}: \text{Rat}(X_{\bar{K}}) \rightarrow \text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|)$.*

Proof. Let K' be a finite field extension of K , and let $f \in \text{Rat}(X_{K'})$, where $X_{K'} := X \times_K K'$. Denote by R' the integral closure of R in K' . Then, R' is a discrete valuation ring (see [Mat80, Chap. 12, Th. 72]). Let ϖ' be a uniformizer of R' , and recall that ϖ denotes a uniformizer of R . We can write

$$\varpi = \varpi'^e u' \quad \text{with } e \geq 1 \text{ and } u' \in R'^{\times}.$$

The base change $\mathcal{X} \times_R R'$ is expressed étale locally as

$$\text{Spec}(R'[x_0, \dots, x_d]/(x_0 \cdots x_r - \varpi'^e u')).$$

By Theorem 4.1, there exists a regular strictly semistable model $\mathcal{X}' \rightarrow \text{Spec}(R')$ along with a surjective morphism $\mu: \mathcal{X}' \rightarrow \mathcal{X} \times_R R'$. Let $\pi': \mathcal{X} \times_R R' \rightarrow \mathcal{X}$ be the natural morphism and set $\pi := \pi' \circ \mu: \mathcal{X}' \rightarrow \mathcal{X}$. Let \mathcal{H}' be the strict transform of \mathcal{H} by π . Then, $(\mathcal{X}', \mathcal{H}')$ is a strictly semistable pair on $X_{K'}$, and $\Delta(\mathcal{X}', \mathcal{H}')$ forms an integral \mathbb{Q} -affine refinement of $\Delta(\mathcal{X}, \mathcal{H})$. In particular, the skeleton $\text{Sk}(\mathcal{X}, \mathcal{H})$ is naturally identified with $|\Delta(\mathcal{X}', \mathcal{H}')|$. By our assumption, the recession fan of $\Delta(\mathcal{X}', \mathcal{H}')$ is the same as that of $\Delta(\mathcal{X}, \mathcal{H})$.

By applying Proposition 3.2 to $(\mathcal{X}', \mathcal{H}')$, we obtain the tropicalization map

$$\text{trop}: \text{Rat}(X_{K'}) \rightarrow \text{Rat}(|\Delta(\mathcal{X}', \mathcal{H}')|) = \text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|).$$

For $f \in \text{Rat}(X)$, viewed in $\text{Rat}(X_{K'})$, the tropicalization $\text{trop}(f)$ obtained above coincides with the previously defined tropicalization.

Hence, we obtain the tropicalization map $\text{trop}: \text{Rat}(X_{\bar{K}}) \rightarrow \text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|)$. \square

3.4. Faithful tropicalization. We recall the definition of faithful tropicalization; see [GRW16, §9] for further details.

Let $(\mathcal{X}, \mathcal{H})$ be a strictly semistable pair, where \mathcal{X} is a strictly semistable model of X defined over K .

Given nonzero rational functions $f_1, \dots, f_n \in \text{Rat}(X_{\bar{K}})$, let U be the open subset of $X_{\bar{K}}$ obtained by removing the support of the divisors $\text{div}(f_j)$, $j = 1, \dots, n$. Consider the map $\phi = (f_1, \dots, f_n): U \rightarrow \mathbf{G}_{m, \bar{K}}^n$ from U to the algebraic torus $\mathbf{G}_{m, \bar{K}}^n$. The tropicalization of ϕ induces a map $\phi^{\text{trop}}: U^{\text{an}} \rightarrow \mathbb{R}^n$.

Since the skeleton $\text{Sk}(\mathcal{X}, \mathcal{H})$ is included in U^{an} , the restriction of ϕ^{trop} gives a map

$$\phi^{\text{trop}}: \text{Sk}(\mathcal{X}, \mathcal{H}) \rightarrow \mathbb{R}^n.$$

We say that ϕ^{trop} is *faithful* if the following conditions are satisfied:

- The restriction of ϕ^{trop} to the skeleton is injective.
- On each Θ_τ , $\tau \in \Delta(\mathcal{X}, \mathcal{H})$, the restriction $\phi^{\text{trop}}|_{\Theta_\tau}$ is piecewise integral \mathbb{Q} -affine.
- The inverse of $\phi^{\text{trop}}|_{\Theta_\tau}$ from $\phi^{\text{trop}}(\Theta_\tau)$ to Θ_τ is also piecewise integral \mathbb{Q} -affine.

Equivalently, ϕ^{trop} is faithful if it is injective, its image is an integral \mathbb{Q} -affine polyhedral complex in \mathbb{R}^n , and the inverse map from the image of ϕ^{trop} to $\text{Sk}(\mathcal{X}, \mathcal{H})$ is piecewise integral \mathbb{Q} -affine.

3.5. Semifields and finite generation. We set

$$\mathbb{T} := \mathbb{R} \cup \{\infty\}.$$

For $a, b \in \mathbb{T}$, we set

$$a \oplus b := \min\{a, b\}, \quad a \odot b := a + b.$$

With these operations, \mathbb{T} forms a semifield, where ∞ is the identity element for \oplus and 0 is the identity element for \odot . We call \mathbb{T} the *tropical semifield*.

Similarly, we define the tropical subsemifield \mathbb{TQ} as

$$\mathbb{TQ} := \mathbb{Q} \cup \{\infty\} \subseteq \mathbb{T}.$$

Let S be a semiring with binary operations \oplus and \odot . We say that S is a *semiring over* \mathbb{TQ} if there exists an injective semiring homomorphism $\mathbb{TQ} \rightarrow S$. In this case, for any $a \in \mathbb{TQ}$ and $s \in S$, the operation $a \odot s$ is well-defined via this homomorphism. Moreover, if S is a semifield, we call it a semifield *over* \mathbb{TQ} .

Example 3.4 ($\text{Rat}(|\Sigma|)$). Let $\text{Rat}(|\Sigma|)$ be defined as in Definition 2.2. For $F, G \in \text{Rat}(|\Sigma|)$ and $a \in \mathbb{TQ}$, we define

$$F \oplus G := \min\{F, G\}, \quad F \odot G := F + G, \quad a \odot F := a + F.$$

Under these operations, $F \oplus G$, $F \odot G$, and $a \odot F$ all belong to $\text{Rat}(|\Sigma|)$. Moreover, $\text{Rat}(|\Sigma|)$ forms a semiring over \mathbb{TQ} . Furthermore, for any $F \in \text{Rat}(|\Sigma|)$ other than ∞ , we have $F \odot (-F) = 0$. This shows that $\text{Rat}(|\Sigma|)$ is a semifield over \mathbb{TQ} . \diamond

Let S be a semiring over \mathbb{TQ} , and $s_1, \dots, s_r \in S$. Define the subsemiring of S generated by s_1, \dots, s_r as

$$\mathbb{TQ}[s_1, \dots, s_r] := \left\{ \bigoplus_{(i_1, \dots, i_r) \in A \subseteq \mathbb{Z}_{\geq 0}^r} a_{i_1, \dots, i_r} \odot s_1^{\odot i_1} \odot \dots \odot s_r^{\odot i_r} \mid A \text{ finite and } a_{i_1, \dots, i_r} \in \mathbb{TQ} \right\}$$

The semifield of fractions of $\mathbb{TQ}[s_1, \dots, s_r]$ denoted $\mathbb{TQ}(s_1, \dots, s_r)$ is defined by localizing at its nonzero elements. If S is a semifield, then $\mathbb{TQ}(s_1, \dots, s_r) \subseteq S$.

Definition 3.5 (Finitely generated semifields). Let S be a semifield over \mathbb{TQ} . We say that S is *finitely generated as a semifield over* \mathbb{TQ} if there exist $s_1, \dots, s_r \in S$ such that $S = \mathbb{TQ}(s_1, \dots, s_r)$. \diamond

3.6. Congruences. Let S be a semiring with binary operations $+$ for addition and \cdot for multiplication. An equivalence relation \sim on S is a *congruence* if it satisfies the following property: For any $a, b, a', b' \in S$, if $a \sim b$ and $a' \sim b'$, then

$$a + a' \sim b + b' \quad \text{and} \quad a \cdot a' \sim b \cdot b'.$$

We refer to [HW93, §I.7] for further details.

If \sim is a congruence on S , then the quotient $\bar{S} := S/\sim$ is itself a semiring. Conversely, if $\varphi: S \rightarrow S'$ is a semiring homomorphism, then the kernel

$$\text{Ker}(\varphi) := \{(s_1, s_2) \in S^2 \mid \varphi(s_1) = \varphi(s_2)\}$$

is a congruence on S . In the context of semirings, the operation of taking the quotient of a ring by an ideal is replaced by taking the quotient by a congruence.

Definition 3.6 (Finite generation property for congruences). Let \sim be a congruence on S . We say that \sim is *finitely generated* if there exist elements $F_1, \dots, F_r, G_1, \dots, G_r \in S$ with $F_i \sim G_i$ for all i such that the following holds:

If \sim' denotes the minimal congruence on S generated by the relations $F_1 \sim' G_1, \dots, F_r \sim' G_r$, then \sim' coincides with \sim . \diamond

4. CHARACTERIZATION THEOREM

As before, let K be a complete discrete valuation field. Given a finite extension K' of K , we denote by R' the integral closure of R in K' , and by κ' the residue field of R' .

Let X be a smooth, geometrically connected, projective variety over K , and suppose that X admits a strictly semistable model \mathcal{X} over R . Let $(\mathcal{X}, \mathcal{H})$ be a strictly semistable pair, and assume that each connected component of the intersection of any collection of irreducible components of \mathcal{H} is geometrically connected. Let $\Delta(\mathcal{X}, \mathcal{H})$ be the dual complex of $(\mathcal{X}, \mathcal{H})$, and let $\text{Sk}(\mathcal{X}, \mathcal{H}) \subseteq X^{\text{an}}$ be the associated skeleton. Denote by Υ the recession fan of $\Delta(\mathcal{X}, \mathcal{H})$.

In this section, we prove Theorem 1.1. Specifically, we show that for any Φ in the tropical function field $\text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|)$, there exists an element $\phi \in \text{Rat}(X_{\bar{K}})$ such that $\text{trop}(\phi) = \Phi$.

To establish this assertion, we need the following result, see [Har01, Car21].

Theorem 4.1. *Notation as above:*

- (C1) *There exist a finite extension field K'/K , a strictly semistable model $\mathcal{X}' \rightarrow \text{Spec}(R')$ of $X' := X \times_K K'$ and a morphism $\pi: \mathcal{X}' \rightarrow \mathcal{X}$ with the following properties:*
 - (i) *Let \mathcal{H}' be the strict transform of \mathcal{H} by π . Then, $(\mathcal{X}', \mathcal{H}')$ is a strictly semistable pair, and $\Delta(\mathcal{X}', \mathcal{H}')$ is a subdivision of $\Delta(\mathcal{X}, \mathcal{H})$ with the same recession fan.*
 - (ii) *Let Z'_v , $v \in V'_f$, be the irreducible components of $\mathcal{X}'_{\kappa'}$, and W'_ρ , $\rho \in V_\infty$ be the irreducible components of \mathcal{H}' . Then, for any subset $\emptyset \neq A \subseteq V'_f$ and $B \subseteq V_\infty$, the intersection $\bigcap_{v \in A} Z'_v \cap \bigcap_{\rho \in B} W'_\rho$ is either empty or irreducible.*
- (C2) *Let $\Phi \in \text{Rat}(\text{Sk}(\mathcal{X}, \mathcal{H}))$. Then, there exists a finite extension field K'/K and a strictly semistable model $\mathcal{X}' \rightarrow \text{Spec}(R')$ of $X' := X \times_K K'$ satisfying Properties (i) and (ii) of (C1) such that $\Phi \in \text{Rat}(\Delta(\mathcal{X}', \mathcal{H}'))$.*

Proof. We follow the desingularization procedure described in [Har01], which involves performing successive below-ups, adapted to the strata obtained from intersections of the components of \mathcal{H} and those of the special fiber. This process results in a semistable model $\mathcal{X}' \rightarrow \text{Spec}(R')$ with a map $\pi: \mathcal{X}' \rightarrow \mathcal{X}$. The corresponding dual complex yields a triangulation of the dual complex of $(\mathcal{X}, \mathcal{H})$. More precisely, it forms an order m subdivision of $\Delta(\mathcal{X}, \mathcal{H})$, in the sense of [Car21], where m is the ramification index of K'/K . For $m > 1$, this subdivision has no multiple faces. Moreover, by our assumption on \mathcal{H} , the recession fan does not change, from which (C1) follows.

For each $\Phi \in \text{Rat}(\text{Sk}(\mathcal{X}, \mathcal{H}))$, there exists a large m and an m -subdivision of $\Delta(\mathcal{X}, \mathcal{H})$ as above over which Φ is facewise integral \mathbb{Q} -affine. The second statement follows. \square

Our strategy for proving Theorem 1.1 goes as follows. By replacing K with K' and \mathcal{X} with \mathcal{X}' , we may and do assume that our model \mathcal{X} satisfies the properties stated in (C1) and (C2) of Theorem 4.1, and that Φ is facewise integral \mathbb{Q} -affine on each (extended simplex) Θ_τ associated with a special stratum S_τ of \mathcal{X} . Moreover, passing to a large field extension K' of K if necessary, we can assume that the constant terms determined by the restriction of Φ to Θ_τ (see proof of Proposition 4.8) belong to the value group of K , for all τ , and that the residue field κ has sufficiently large cardinality (if it is finite).

Recall that $\text{Sk}(\mathcal{X}, \mathcal{H}) = \bigcup_{\tau \in \Delta(\mathcal{X}, \mathcal{H})} \Theta_\tau$ (see Section 3.2). Given a function $F \in \text{Rat}(\text{Sk}(\mathcal{X}, \mathcal{H}))$ and a pair $\tau \preceq \eta$ in $\Delta(\mathcal{X}, \mathcal{H})$, we denote by $F|_\tau$ and $F|_{\eta \setminus \tau}$ the restrictions of F to Θ_τ and $\Theta_\eta \setminus \Theta_\tau$, respectively.

For each $\tau \in \Delta(\mathcal{X}, \mathcal{H})$, we will construct two functions with *opposite* behaviors. The first function denoted by $f_\tau \in \text{Rat}(\mathcal{X}) = \text{Rat}(X)$ satisfies

- (i) $\text{trop}(f_\tau)|_\tau \equiv 0$;
- (ii) $\text{trop}(f_\tau)|_{\eta \setminus \tau} > 0$ for any $\eta \in \Delta(\mathcal{X}, \mathcal{H})$. Furthermore, if $\eta \in \Delta(\mathcal{X}, \mathcal{H})$, then the slope of $\text{trop}(f_\tau)|_{\eta \setminus \tau}$ near infinity along any half-line of $\eta \setminus \tau$ that is not parallel to τ is positive.

We will show from the construction that $\text{trop}(f_\tau)|_{\eta \setminus \tau}$ is concave and the slope of $\text{trop}(f_\tau)|_{\eta \setminus \tau}$ near infinity along any ray of $\eta \setminus \tau$ that is not parallel to a ray of τ is positive, from which we deduce the more general statement about half-lines.

The second function denoted by $g_\tau \in \text{Rat}(\mathcal{X}) = \text{Rat}(X)$ satisfies

$$\text{trop}(g_\tau)|_\tau = \Phi|_\tau.$$

With these two functions $f_\tau, g_\tau \in \text{Rat}(\mathcal{X}) = \text{Rat}(X)$, we set

$$(4.1) \quad \phi := \sum_{\tau \in \Delta(\mathcal{X}, \mathcal{H})} \beta_\tau f_\tau^{N_\tau} g_\tau \in \text{Rat}(\mathcal{X}) = \text{Rat}(X),$$

where the coefficients $\beta_\tau \in R^\times$ and the integers $N_\tau \gg 1$ will be suitably chosen. We then show that $\text{trop}(\phi) = \Phi$, as desired.

4.1. Construction of f_τ . By the definition of a strictly semistable pair, \mathcal{X} is covered by open subsets that admit an étale morphism $\mathcal{U} \rightarrow \text{Spec}(R[x_0, \dots, x_d]/\langle x_0 \cdots x_r - \varpi \rangle)$.

Let τ be an element of $\Delta(\mathcal{X}, \mathcal{H})$, and let S_τ be the special stratum of $(\mathcal{X}, \mathcal{H})$ corresponding to τ defined by $A_\tau \subseteq V_f$ and $B_\tau \subseteq V_\infty$. Let ξ_τ denote the generic point of S .

By (C1) in Theorem 4.1, we have

$$S_\tau = \bigcap_{v \in A_\tau} Z_v \cap \bigcap_{\rho \in B_\tau} W_\rho.$$

Recall that $\mathcal{Y} = \mathcal{X}_\kappa + \mathcal{H}$. To simplify the notation, we set

$$D_\tau := \sum_{v \in A_\tau} Z_v + \sum_{\rho \in B_\tau} W_\rho, \quad E_\tau := \mathcal{Y} - D_\tau = \sum_{v \in V_f \setminus A_\tau} Z_v + \sum_{\rho \in V_\infty \setminus B_\tau} W_\rho.$$

We pick an ample line bundle \mathcal{L} over \mathcal{X} .

Lemma 4.2. *For a sufficiently large positive integer N , there exists a section*

$$s \in H^0(D_\tau, \mathcal{L}^{\otimes N}(-E_\tau)|_{D_\tau})$$

such that $s(\xi_\tau) \neq 0$.

Proof. Since \mathcal{L} is ample, the restriction $\mathcal{L}|_{D_\tau}$ is also ample. Therefore, for sufficiently large N , $\mathcal{L}^{\otimes N}(-E_\tau)|_{D_\tau}$ is globally generated. It follows that there exists $s \in H^0(D_\tau, \mathcal{L}^{\otimes N}(-E_\tau)|_{D_\tau})$ such that $s(\xi_\tau) \neq 0$. \square

Lemma 4.3. *Let s be as in Lemma 4.2. Then, there exists a section*

$$s' \in H^0(\mathcal{Y}, \mathcal{L}^{\otimes N}|_{\mathcal{Y}})$$

such that $s'|_{D_\tau} = s$ and $s'|_{E_\tau} = 0$.

Proof. The restriction map $\mathcal{O}_{\mathcal{Y}} \twoheadrightarrow \mathcal{O}_{E_\tau}$ induces, after tensoring by $\mathcal{L}^{\otimes N}$, the exact sequence

$$0 \rightarrow \mathcal{L}^{\otimes N}(-E_\tau)|_{D_\tau} \rightarrow \mathcal{L}^{\otimes N}|_{\mathcal{Y}} \rightarrow \mathcal{L}^{\otimes N}|_{E_\tau} \rightarrow 0.$$

In the corresponding long exact sequence of cohomologies, let $s' \in H^0(\mathcal{Y}, \mathcal{L}^{\otimes N}|_{\mathcal{Y}})$ be the push-out of $s \in H^0(D_\tau, \mathcal{L}^{\otimes N}(-E_\tau)|_{D_\tau})$. In other words, s' is the zero extension of s to \mathcal{Y} . Then, s' has the desired properties. \square

Lemma 4.4. *If N is a sufficiently large positive integer, then the restriction map*

$$H^0(\mathcal{X}, \mathcal{L}^{\otimes N}) \rightarrow H^0(\mathcal{Y}, \mathcal{L}^{\otimes N}|_{\mathcal{Y}})$$

is surjective. For such an $N \gg 1$, we take s as in Lemma 4.2 and s' as in Lemma 4.3. Then, there exists

$$\tilde{s} \in H^0(\mathcal{X}, \mathcal{L}^{\otimes N})$$

such that $\tilde{s}|_{\mathcal{Y}} = s'$.

Proof. The first assertion follows from the vanishing $H^1(\mathcal{X}, \mathcal{L}^{\otimes N}(-\mathcal{Y})) = 0$ for sufficiently large N . The second statement follows directly from the first. \square

Proposition 4.5. *Notation as above, for each $\tau \in \Delta(\mathcal{X}, \mathcal{H})$, there exists $f_\tau \in \text{Rat}(\mathcal{X}) = \text{Rat}(X)$ such that*

$$\text{trop}(f_\tau)|_\tau \equiv 0 \quad \text{and} \quad \text{trop}(f_\tau)|_{\eta \setminus \tau} > 0 \quad \text{for each } \eta \in \Delta(\mathcal{X}, \mathcal{H}).$$

Furthermore, $\text{trop}(f_\tau)|_\eta$ is concave for any $\eta \in \Delta(\mathcal{X}, \mathcal{H})$, and the slope of $\text{trop}(f_\tau)|_{\eta \setminus \tau}$ near infinity along any ray of η that is not parallel to a ray of τ is positive.

Proof. Let $\tilde{s} \in H^0(\mathcal{X}, \mathcal{L}^{\otimes N})$ be as in Lemma 4.4. For each $\eta \in \Delta(\mathcal{X}, \mathcal{H})$, let ξ_η denote the generic point of stratum S_η in \mathcal{Y} . Since \mathcal{L} is ample, for sufficiently large N , there exists $\tilde{s}_0 \in H^0(\mathcal{X}, \mathcal{L}^{\otimes N})$ such that $\tilde{s}_0(\xi_\eta) \neq 0$ for all $\eta \in \Delta(\mathcal{X}, \mathcal{H})$. Taking N larger if necessary, we may and do assume that the integer N is the same for both \tilde{s} and \tilde{s}_0 .

We define

$$f_\tau := \tilde{s}/\tilde{s}_0 \in \text{Rat}(\mathcal{X}) = \text{Rat}(X)$$

and we aim to show that f_τ has desired properties.

First, we verify that $\text{trop}(f_\tau)|_\tau \equiv 0$. Recall that $A_\tau \subseteq V_f$ and $B_\tau \subseteq V_\infty$ are the subsets with $S_\tau = \bigcap_{v \in A_\tau} Z_v \cap \bigcap_{\rho \in B_\tau} W_\rho$. We choose local equations $z_v = 0$ for Z_v and $w_\rho = 0$ for W_ρ in \mathcal{X} at the generic point ξ_τ of S_τ .

Since $f_\tau(\xi_\tau) = \frac{\tilde{s}(\xi_\tau)}{\tilde{s}_0(\xi_\tau)} \neq 0$ by Lemma 4.2, for an admissible expansion of f_τ in $\widehat{\mathcal{O}}_{\mathcal{X}, \xi_\tau}$

$$f_\tau = \sum_{\mathbf{m}=(m_v)_v \times (m_\rho)_\rho \in \mathbb{Z}_{\geq 0}^{A_\tau \cup B_\tau}} a_{\mathbf{m}} \prod_{v \in A_\tau} z_v^{m_v} \prod_{\rho \in B_\tau} w_\rho^{m_\rho},$$

with each coefficient $a_{\mathbf{m}}$ is either zero or a unit in $\widehat{\mathcal{O}}_{\mathcal{X}, \xi_\tau}$, we have $a_{\mathbf{0}} \neq 0$, and so we get

$$|f_\tau|_{\tau, \mathbf{u}} = 1 \quad \text{for any } \mathbf{u} \in \sigma^{A_\tau} \times \mathbb{R}_{\geq 0}^{B_\tau}.$$

It follows that $\text{trop}(f_\tau)|_\tau \equiv 0$, proving the first claim.

Next, we verify that $\text{trop}(f_\tau)|_{\eta \setminus \tau} > 0$ for each element η of $\Delta(\mathcal{X}, \mathcal{H})$. We may assume that $\eta \setminus \tau \neq \emptyset$, otherwise, there is nothing to prove. This implies that $\eta \not\leq \tau$.

For the special stratum S_η of \mathcal{Y} corresponding to η with generic point ξ_η , consider the corresponding subsets $A_\eta \subseteq V_f$ and $B_\eta \subseteq V_\infty$ with

$$S_\eta = \bigcap_{v \in A_\eta} Z_v \cap \bigcap_{\rho \in B_\eta} W_\rho.$$

Let $\delta = \eta \cap \tau$, with $A_\delta = A_\eta \cap A_\tau$ and $B_\delta = B_\eta \cap B_\tau$. Since $\eta \not\leq \tau$, we have $|A_\delta| + |B_\delta| < |A_\eta| + |B_\eta|$.

We choose local equations $z_v = 0$ and $w_\rho = 0$ at ξ_η in \mathcal{X} for Z_v and W_ρ , respectively, for $v \in A_\eta, \rho \in B_\eta$.

Let $v \in A_\eta \setminus A_\tau$. Since $v \notin A_\tau$, Z_v does not appear in D_τ . By the construction of \tilde{s} , we see that $\tilde{s}|_{Z_v} \equiv 0$. It follows that $\tilde{s}|_{(z_v=0)} \equiv 0$ for each $v \in A_\eta \setminus A_\tau$.

Let now $\rho \in B_\eta \setminus B_\tau$. Since $\rho \notin B_\tau$, W_ρ does not appear in D_τ . By the construction of \tilde{s} , we see that $\tilde{s}|_{W_\rho} \equiv 0$. It follows that $\tilde{s}|_{(w_\rho=0)} \equiv 0$ for each $\rho \notin B_\tau$

Identifying $\Theta_\eta = \sigma^{A_\eta} \times \mathbb{R}_{\geq 0}^{B_\eta}$ and $\Theta_\delta = \sigma^{A_\delta} \times \mathbb{R}_{\geq 0}^{B_\delta}$, the inclusion $\Theta_\delta \subseteq \Theta_\eta$ is given by

$$\Theta_\delta = \left\{ \mathbf{u} = (u_v)_{v \in A_\eta} \times (u_\rho)_{\rho \in B_\eta} \in \Theta_\eta \mid u_v = u_\rho = 0 \text{ for all } v \in A_\eta \setminus A_\delta, \rho \in B_\eta \setminus B_\delta \right\}.$$

Since $\tilde{s}_0(\xi_\eta) \neq 0$ and $\tilde{s}|_{(z_v=0)}$ and $\tilde{s}|_{(w_\rho=0)}$ both vanish for $v \in A_\eta \setminus A_\delta$ and $\rho \in B_\eta \setminus B_\delta$, the admissible expansion of f_τ in $\widehat{\mathcal{O}}_{\mathcal{X}, \xi_\eta}$

$$(4.2) \quad f_\tau = \sum_{\mathbf{m}=(m_v)_v \times (m_\rho)_\rho \in \mathbb{Z}_{\geq 0}^{A_\eta \cup B_\eta}} a_{\mathbf{m}} \prod_{v \in A_\eta} z_v^{m_v} \prod_{\rho \in B_\eta} w_\rho^{m_\rho}$$

verifies $m_v > 0$ and $m_\rho > 0$ for all $v \in A_\eta \setminus A_\delta$ and $\rho \in B_\eta \setminus B_\delta$. It follows that $|f_\tau|_{\eta, \mathbf{u}} < 1$ if $\mathbf{u} \in \Theta_\eta \setminus \Theta_\delta$. We conclude that $\text{trop}(f_\tau)|_{\eta \setminus \tau} > 0$, giving the second claim.

It remains to show the last assertion. It follows from (4.2) that $\text{trop}(f_\tau)|_\eta$ is concave, being the minimum of a collection of affine functions.

The rays of $\eta \setminus \tau$ that are not parallel to rays of τ correspond precisely to the elements of $B_\eta \setminus B_\tau$. Let ρ be such an element. Since $\tilde{s}|_{(w_\rho=0)} \equiv 0$, we see that $m_\rho > 0$ for all \mathbf{m} with $a_{\mathbf{m}} \neq 0$. Therefore, the slope of $\text{trop}(f_\tau)|_\eta$ is positive along ρ , as required. \square

Corollary 4.6. *For any τ and η in $\Delta(\mathcal{X}, \mathcal{H})$, the slope of $\text{trop}(f_\tau)|_\eta$ near infinity along any half-line of η that is not parallel to τ is positive.*

Proof. By Proposition 3.2, $\text{trop}(f_\tau) \in \text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})|)$. The statement now follows from the concavity of $\text{trop}(f_\tau)|_\eta$ and positivity of the slope of $\text{trop}(f_\tau)|_\eta$ near infinity along any ray of η that is not parallel to a ray of τ . \square

4.2. Construction of g_τ . In this subsection, we construct g_τ . We need the following lemma.

Lemma 4.7. *Let $E \subseteq \mathcal{X}$ be an irreducible divisor. Given points x_1, \dots, x_n on E and points y_1, \dots, y_m in $\mathcal{X} \setminus E$, there is a rational function u which vanishes on E with order of vanishing one, belongs to each local ring $\mathcal{O}_{\mathcal{X}, x_i}$, $i = 1, \dots, n$, and is a unit in $\mathcal{O}_{\mathcal{X}, y_j}$ for each $j = 1, \dots, m$.*

Proof. We take an ample line bundle \mathcal{L} on \mathcal{X} . There is a global section $s \in H^0(\mathcal{X}, \mathcal{L}^{\otimes N})$ for sufficiently large N such that $s(x_i) \neq 0$ for any $i = 1, \dots, n$ and $s(y_j) \neq 0$ for any $j = 1, \dots, m$. Let $H = \text{div}(s)$. Then, $\mathcal{L} \cong \mathcal{O}_{\mathcal{X}}(H)$, and H does not pass through $x_1, \dots, x_n, y_1, \dots, y_m$ and does not contain E . We choose another large integer N' such that $N'H - E$ is very ample. Taking a generic section $u \in H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(N'H - E))$, the corresponding rational function satisfies all the required properties. \square

We will apply this lemma to $E = Z_v$ for $v \in V_f$ (resp. $E = W_\rho$ for $\rho \in V_\infty$). The points x_i will be the generic points ξ_τ of the special strata S_τ with $v \in \zeta(\tau)$ (resp. $\rho \in \zeta(\tau)$), while the points y_j will be the generic points of the special strata S_η with $v \notin \zeta(\eta)$ (resp. $\rho \notin \zeta(\eta)$).

Proposition 4.8. *Let $\Phi \in \text{Rat}(\Delta(\mathcal{X}, \mathcal{H}))$. Then, for each $\tau \in \Delta(\mathcal{X}, \mathcal{H})$, there exists $g_\tau \in \text{Rat}(\mathcal{X}) = \text{Rat}(X)$ such that $\text{trop}(g_\tau)$ belongs to $\text{Rat}(\Delta(\mathcal{X}, \mathcal{H}))$ and $\text{trop}(g_\tau)|_\tau = \Phi|_\tau$.*

Note that, in particular, $\text{trop}(g_\tau)$ is integral \mathbb{Q} -affine on each cell of $\Delta(\mathcal{X}, \mathcal{H})$.

Proof. As before, let S_τ be the special stratum corresponding to τ , and $A_\tau \subseteq V_f$ and $B_\tau \subseteq V_\infty$ be such that $S_\tau = \bigcap_{v \in A_\tau} Z_v \cap \bigcap_{\rho \in B_\tau} W_\rho$. Applying Lemma 4.7, we choose local equations $z_v = 0$ for Z_v and $w_\rho = 0$ for W_ρ in \mathcal{X} at the generic point ξ_τ of S_τ , with the property that z_v and w_ρ belong to all the local rings $\mathcal{O}_{\mathcal{X}, \xi_\eta}$ for $\eta \in \Delta(\mathcal{X}, \mathcal{H})$, and they are unit if v is not a vertex and ρ is not a ray of η .

Identifying Θ_τ with $\sigma^{A_\tau} \times \mathbb{R}_{\geq 0}^{B_\tau}$, the restriction $\Phi|_\tau$ can be written in the form

$$(4.3) \quad \Phi|_\tau(\mathbf{u}) = c_\tau + \sum_{v \in A_\tau} a_v u_v + \sum_{\rho \in B_\tau} a_\rho u_\rho \quad \forall \mathbf{u} \in \sigma^{A_\tau} \times \mathbb{R}_{\geq 0}^{B_\tau},$$

for rational c_τ and integers a_v, a_ρ . Since the c_τ belongs to the valuation group by assumption, we can find $\gamma_\tau \in K$ with $\nu_K(\gamma_\tau) = c_\tau$. If we set

$$g_\tau := \gamma_\tau \prod_{v \in A} z_v^{a_v} \prod_{\rho \in B} w_\rho^{a_\rho} \in \text{Rat}(X) = \text{Rat}(\mathcal{X}),$$

then $\text{trop}(g_\tau)|_\tau = \Phi|_\tau$, which gives the second half of the proposition.

To get the first half, we write

$$g_\tau = \gamma_\tau \frac{\prod_{v \in A} z_v^{\max\{a_v, 0\}} \prod_{\rho \in B} w_\rho^{\max\{a_\rho, 0\}}}{\prod_{v \in A} z_v^{-\min\{a_v, 0\}} \prod_{\rho \in B} w_\rho^{-\min\{a_\rho, 0\}}}.$$

Let η be a cell of $\Delta(\mathcal{X}, \mathcal{H})$. We need to show that $\text{trop}(g_\tau)$ is integral \mathbb{Q} -affine on η . By the choice of the parameters z_v and w_ρ , if $v \notin \zeta(\eta)$ (resp. $\rho \notin \zeta(\eta)$), then the parameter z_v (resp. w_ρ) is a unit in $\mathcal{O}_{\mathcal{X}, \xi_\eta}$. Moreover, if $v \in \zeta(\eta)$ (resp. $\rho \in \zeta(\eta)$), the parameter z_v (resp. w_ρ) is in $\mathcal{O}_{\mathcal{X}, \xi_\eta}$ and has order of vanishing one along the component Z_v (resp. W_ρ). It follows that both the numerator and the denominator are admissible expansions in the local ring $\mathcal{O}_{\mathcal{X}, \xi_\eta}$ for each $\eta \in \Delta(\mathcal{X}, \mathcal{H})$. This implies that the tropicalization of g_τ is integral \mathbb{Q} -affine on η . \square

4.3. Proof of Theorem 1.1. Before giving the proof of Theorem 1.1, we state useful lemmas.

Lemma 4.9. *Let $x_{ij} \in K$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Assume that the residue field κ has cardinality larger than n . Then, there exist $\beta_i \in R^\times$ for $1 \leq i \leq m$ such that, for any $1 \leq j \leq n$, we have $|\sum_{i=1}^m \beta_i x_{ij}| = \max\{|x_{1j}|, \dots, |x_{mj}|\}$.*

Proof. We can assume without loss of generality that $\max\{|x_{1j}|, \dots, |x_{mj}|\} = 1$ for all $j = 1, \dots, n$. Let \bar{x}_{ij} be the reduction of x_{ij} in the residue field. Consider the m by n matrix $M = (\bar{x}_{ij})$. It will be enough to show the existence of a vector $y = (y_1, \dots, y_m)$ in κ^m such that the coordinates of yM are all nonzero. Indeed, for such y , we will choose $\beta_j \in R^\times$ with reduction in κ equal to y_j . It will follow that the reduction of $\sum_{i=1}^m \beta_i x_{ij}$ in κ will be nonzero, and we conclude that $|\sum_{i=1}^m \beta_i x_{ij}| = 1 = \max\{|x_{1j}|, \dots, |x_{mj}|\}$, as required.

It remains to show the existence of y . For each j , the equation $\sum_{i=1}^m y_i \bar{x}_{ij} = 0$ defines a hyperplane in κ^m . Since the cardinality of κ is larger than n , the union of these n hyperplanes does not cover the full space, and therefore, there exists $y \in \kappa^m$ which does not belong to any of these hyperplanes. For this element, the coordinates of yM are all nonzero. \square

Let $\Theta_{A \cup B} := \sigma^A \times \mathbb{R}_{\geq 0}^B$ and $\Theta_{A' \cup B'} := \sigma^{A'} \times \mathbb{R}_{\geq 0}^{B'}$ be extended simplices with $A \subseteq A'$ and $B \subseteq B'$. We allow $A = B = \emptyset$ in which case we set $\Theta_{A \cup B} = \Theta_{\emptyset \cup \emptyset} = \emptyset$, but we assume that $A' \neq \emptyset$ and so $\Theta_{A' \cup B'} \neq \emptyset$. We regard $\Theta_{A \cup B}$ as the subset of $\Theta_{A' \cup B'}$ given by

$$\Theta_{A \cup B} = \{(u) \in \Theta_{A' \cup B'} \mid u_v = u_\rho = 0 \text{ for all } v \in A' \setminus A, \rho \in B' \setminus B\}.$$

Lemma 4.10. *Let F be a piecewise integral \mathbb{Q} -affine function and G an integral \mathbb{Q} -affine function on $\Theta_{A' \cup B'}$ such that*

$$F|_{\Theta_{A \cup B}} \equiv 0, \quad G|_{\Theta_{A \cup B}} \equiv 0, \quad \text{and} \quad F|_{\Theta_{A' \cup B'} \setminus \Theta_{A \cup B}} > 0.$$

Assume that F is concave. For $\rho \in B' \setminus B$, consider the vector $\mathbf{e}_\rho \in \mathbb{R}^{B'}$. We assume that

- (i) *Either, the slope of F near infinity in the direction of \mathbf{e}_ρ is positive,*
- (ii) *Or, if the slope of F near infinity along \mathbf{e}_ρ is zero, then the slope of G near infinity along \mathbf{e}_ρ is also zero.*

Then, for any sufficiently large $N \gg 1$, we have $(NF - G)|_{\Theta_{A' \cup B'} \setminus \Theta_{A \cup B}} > 0$.

Proof. We choose a refinement of $\Theta_{A' \cup B'}$ such that on any polyhedron of the refinement F and G are integral \mathbb{Q} -affine. Since F is strictly positive on $\Theta_{A' \cup B'} \setminus \Theta_{A \cup B}$, for any sufficiently large N , $NF - G$ is strictly positive on compact cells of the refinement. We claim that for any half-line on an unbounded cell of the refinement, the slope of $NF - G$ is non-negative for sufficiently large N . Since $NF - G$ is concave on $\Theta_{A' \cup B'}$, it is enough to prove the statement for the slopes of $NF - G$ near infinity along the rays of $\Theta_{A' \cup B'}$. This comes from our assumptions (i) and (ii). To conclude, for any unbounded cell of the refinement, since $NF - G$ is affine on the cell, takes positive values on vertices away from $\Theta_{A \cup B}$, and has non-negative slopes along rays, it remains positive on the cell everywhere outside $\Theta_{A \cup B}$. \square

We begin the proof of Theorem 1.1. With the notation of the previous sections, we set

$$(4.4) \quad \phi := \sum_{\tau \in \Delta(\mathcal{X}, \mathcal{H})} \beta_\tau f_\tau^{N_\tau} g_\tau \in \text{Rat}(\mathcal{X}) = \text{Rat}(X)$$

as in (4.1), where the coefficients $\beta_\tau \in R^\times$ and the integers $N_\tau \gg 1$ will be suitably chosen later. We claim that $\text{trop}(\phi) = \Phi$ on $\text{Sk}(\mathcal{X}, \mathcal{H})$.

Fix an element $\tau \in \Delta(\mathcal{X}, \mathcal{H})$. Then, we write

$$\phi = \beta_\tau f_\tau^{N_\tau} g_\tau + \sum_{\substack{\eta \in \Delta(\mathcal{X}, \mathcal{H}) \\ \eta \neq \tau}} \beta_\eta f_\eta^{N_\eta} g_\eta.$$

We will show that $\text{trop}(\phi)|_\tau = \Phi|_\tau$.

First, note that since $\beta_\tau \in R^\times$, $\text{trop}(f_\tau)|_\tau \equiv 0$ and $\text{trop}(g_\tau)|_\tau = \Phi|_\tau$, we have

$$\text{trop}(\beta_\tau f_\tau^{N_\tau} g_\tau)|_\tau = -\log|\beta_\tau| + N_\tau \text{trop}(f_\tau)|_\tau + \text{trop}(g_\tau)|_\tau = \Phi|_\tau.$$

We now evaluate $\text{trop}(\beta_\eta f_\eta^{N_\eta} g_\eta)|_\tau$ for any $\eta \neq \tau$, dividing into three cases.

Case 1. Suppose that $\eta \prec \tau$ ($\eta \neq \tau$). Then, by the property of f_η , we have $\text{trop}(f_\eta)|_\eta \equiv 0$ and $\text{trop}(f_\eta)|_{\tau \setminus \eta} > 0$. We also have $\text{trop}(g_\eta)|_\eta = \Phi|_\eta$.

Restricting to η , we have

$$\text{trop}(\beta_\eta f_\eta^{N_\eta} g_\eta)|_\eta = \Phi|_\eta = (\Phi|_\tau)|_\eta = (\text{trop}(\beta_\tau f_\tau^{N_\tau} g_\tau)|_\tau)|_\eta = \text{trop}(\beta_\tau f_\tau^{N_\tau} g_\tau)|_\eta.$$

And, on $\tau \setminus \eta$, we have

$$\text{trop}(\beta_\eta f_\eta^{N_\eta} g_\eta)|_{\tau \setminus \eta} - \text{trop}(\beta_\tau f_\tau^{N_\tau} g_\tau)|_{\tau \setminus \eta} = (N_\eta \text{trop}(f_\eta)|_{\tau \setminus \eta} + \text{trop}(g_\eta)|_{\tau \setminus \eta}) - \Phi|_{\tau \setminus \eta}.$$

We apply Lemma 4.10 to $\Theta_{A_\tau \cup B_\tau}$ and $\Theta_{A_\eta \cup B_\eta}$, and $F = \text{trop}(f_\eta)|_\tau$ and $G = \Phi|_\tau - \text{trop}(g_\eta)|_\tau$. Then, F is concave, piecewise integral \mathbb{Q} -affine and G is integral \mathbb{Q} -affine. We have $F|_\eta \equiv 0$, $G|_\eta \equiv 0$, and $F|_{\tau \setminus \eta} > 0$. Furthermore, by the last statement of Proposition 4.5, the slope of $F = \text{trop}(f_\eta)|_\tau$ near infinity along any ray of $\tau \setminus \eta$ is positive.

From Lemma 4.10, we conclude that

$$\text{trop}(\beta_\eta f_\eta^{N_\eta} g_\eta)|_{\tau \setminus \eta} > \text{trop}(\beta_\tau f_\tau^{N_\tau} g_\tau)|_{\tau \setminus \eta} \quad \text{if } N_\eta \gg 1.$$

Here, N_η is independent of the choice of N_τ , because $\text{trop}(f_\tau)|_\tau \equiv 0$.

Case 2. Suppose that $\eta \succ \tau$ ($\eta \neq \tau$). Since $\text{trop}(f_\eta)|_\eta \equiv 0$, we have $\text{trop}(f_\eta)|_\tau \equiv 0$. Also, we have $\text{trop}(g_\eta)|_\tau = (\Phi|_\eta)|_\tau = \Phi|_\tau$. Thus,

$$\text{trop}(\beta_\eta f_\eta^{N_\eta} g_\eta)|_\tau = \Phi|_\tau \quad \text{for any positive integer } N_\eta.$$

Case 3. In the remaining case, we have $\eta \not\succeq \tau$ and $\eta \not\preceq \tau$. We divide this into two subcases.

3.1. Suppose first that $\eta \cap \tau \neq \emptyset$. By the properties of f_η , we have

$$\text{trop}(f_\eta)|_{\tau \setminus \eta} > 0 \quad \text{and} \quad \text{trop}(f_\eta)|_{\tau \cap \eta} \equiv 0.$$

Additionally, we have

$$\text{trop}(g_\eta)|_{\tau \cap \eta} = (\Phi|_\eta)|_{\tau \cap \eta} = \Phi|_{\tau \cap \eta}.$$

Thus, we get

$$\text{trop}(\beta_\eta f_\eta^{N_\eta} g_\eta)|_{\tau \cap \eta} = \Phi|_{\tau \cap \eta} = \text{trop}(\beta_\tau f_\tau^{N_\tau} g_\tau)|_{\tau \cap \eta}.$$

Next, as in Case 1, apply Lemma 4.10 to $\Theta_{A_\tau \cup B_\tau}$ and $\Theta_{A_\tau \cap \eta \cup B_\tau \cap \eta}$, and two piecewise integral \mathbb{Q} -affine functions $F = \text{trop}(f_\eta)|_\tau$ and $G = \Phi|_\tau - \text{trop}(g_\eta)|_\tau$. Again, F is concave and G is affine. Moreover, we have $F|_{\tau \cap \eta} \equiv 0$, $G|_{\tau \cap \eta} \equiv 0$, and $F|_{\tau \setminus (\tau \cap \eta)} > 0$. Furthermore, by the last statement of Proposition 4.5, the slope of $F = \text{trop}(f_\eta)|_\tau$ near infinity along each ray of $\tau \setminus (\tau \cap \eta)$ is positive. We conclude that

$$\text{trop}(\beta_\eta f_\eta^{N_\eta} g_\eta)|_{\tau \setminus \eta} > \text{trop}(\beta_\tau f_\tau^{N_\tau} g_\tau)|_{\tau \setminus \eta} \quad \text{if } N_\eta \gg 1.$$

Note that N_η is again independent of the choice of N_τ , because $\text{trop}(f_\tau)|_\tau \equiv 0$.

3.2. Suppose that $\eta \cap \tau = \emptyset$. In this case, we have $\text{trop}(f_\eta)|_\tau = \text{trop}(f_\eta)|_{\tau \setminus \eta} > 0$.

We are going to apply Lemma 4.10 to $\Theta_{A_\tau \cup B_\tau}$ and $\Theta_{\emptyset \cup \emptyset} = \emptyset$, and two piecewise integral \mathbb{Q} -affine functions $F = \text{trop}(f_\eta)|_\tau$ and $G = \Phi|_\tau - \text{trop}(g_\eta)|_\tau$. Note that F is concave and G is affine.

We observe that $F > 0$. Next, we check the slope conditions for F and G . Let $S_\tau, S_\eta \in \Delta(\mathcal{X}, \mathcal{H})$ be the special strata corresponding to τ, η , with generic points ξ_τ, ξ_η , respectively. Let $A_\tau, A_\eta \subseteq V_f$ and $B_\tau, B_\eta \subseteq V_\infty$ be the corresponding subsets. Since $\eta \cap \tau = \emptyset$, we have $A_\tau \cap A_\eta = \emptyset$.

Consider a ray $\rho \in B_\tau$. Two cases can occur depending on whether ρ belongs to B_η or not. In the latter case, W_ρ does not appear in D_η . By the choice of f_η , since f_η vanishes on the complement E_η of D_η , the slope of F along the ray ρ given by ϵ_ρ will be positive.

In the former case, let ρ' be the ray of η defined by ρ . Then, ρ and ρ' are parallel. By the definition of $\Phi \in \text{Rat}(\Delta(\mathcal{X}, \mathcal{H}))$, the slope of $\Phi|_\tau$ along ρ and that of $\Phi|_\eta$ along ρ' are the same. Now, in the analogue equation to (4.3) for η ,

$$\Phi|_\eta(\mathbf{u}) = c_\eta + \sum_{v \in A_\eta} a'_v u_v + \sum_{\rho \in B_\eta} a'_\rho u_\rho \quad \forall \mathbf{u} \in \sigma^{A_\eta} \times \mathbb{R}_{\geq 0}^{B_\eta},$$

consider the coefficient a'_ρ of u_ρ . Let $w_\rho = 0$ be the local equation for W_ρ at the generic point ξ_η . By the definition of g_η , the exponent of w_ρ in g_η is equal to a'_ρ , and so the order of g_η along the horizontal divisor W_ρ is equal to a'_ρ . It follows that the slope of $\text{trop}(g_\eta)|_\tau$ along ρ and that of $\text{trop}(g_\eta)|_\eta$ along ρ' are the same integer a'_ρ . Thus, for $\rho \in B_\eta \cap B_\tau$, the slope of G along ϵ_ρ is zero.

We have confirmed that F and G satisfy the slope conditions in Lemma 4.10. We apply that lemma to conclude that

$$\text{trop}(\beta_\eta f_\eta^{N_\eta} g_\eta)|_{\tau \setminus \eta} > \text{trop}(\beta_\tau f_\tau^{N_\tau} g_\tau)|_{\tau \setminus \eta} \quad \text{if } N_\eta \gg 1.$$

Note that, once again, N_η is independent of the choice of N_τ , because $\text{trop}(f_\tau)|_\tau \equiv 0$.

Proof of Theorem 1.1. In the definition of ϕ given in (4.4), for each $\tau \in \Delta(\mathcal{X}, \mathcal{H})$, we choose $N_\tau \gg 1$ sufficiently large (which depends on $\text{trop}(g_\tau)$ and Φ , but not on cells of $\Delta(\mathcal{X}, \mathcal{H})$ other than τ).

By Lemma 4.9, since κ has large enough cardinality, we can choose $\beta_\tau \in R^\times$ suitably so that by what we have seen in Cases 1, 2, 3, we get, for any τ ,

$$(4.5) \quad \begin{aligned} \text{trop}(\phi)|_\tau = \text{trop} & \left(\beta_\tau f_\tau^{N_\tau} g_\tau + \sum_{\substack{\eta \in \Delta(\mathcal{X}, \mathcal{H}) \\ \eta \prec \tau}} \beta_\eta f_\eta^{N_\eta} g_\eta + \sum_{\substack{\eta \in \Delta(\mathcal{X}, \mathcal{H}) \\ \eta \succ \tau}} \beta_\eta f_\eta^{N_\eta} g_\eta \right. \\ & \left. + \sum_{\substack{\eta \in \Delta(\mathcal{X}, \mathcal{H}) \\ \eta \not\prec \tau, \eta \not\succ \tau, \eta \cap \tau \neq \emptyset}} \beta_\eta f_\eta^{N_\eta} g_\eta + \sum_{\substack{\eta \in \Delta(\mathcal{X}, \mathcal{H}) \\ \eta \succ \tau, \eta \not\prec \tau, \eta \cap \tau = \emptyset}} \beta_\eta f_\eta^{N_\eta} g_\eta \right) \Big|_\tau = \text{trop}(\beta_\tau f_\tau^{N_\tau} g_\tau)|_\tau = \Phi|_\tau. \end{aligned}$$

Indeed, recall from Case 2 that g_τ and g_η 's with $\eta \succ \tau$ are monomials whose tropicalizations give the same function $\Phi|_\tau$ on τ , and the constant term of f_τ and f_η 's with $\eta \succ \tau$ are all units in $\widehat{\mathcal{O}}_{\mathcal{X}, \xi_\tau}$, where ξ_τ is the generic point of the special stratum corresponding to τ . Thus, we can choose $\beta_\tau \in R^\times$ ($\tau \in \Delta(\mathcal{X}, \mathcal{H})$) suitably so that $\text{trop} \left(\beta_\tau f_\tau^{N_\tau} g_\tau + \sum_{\substack{\eta \in \Delta(\mathcal{X}, \mathcal{H}) \\ \eta \succ \tau}} \beta_\eta f_\eta^{N_\eta} g_\eta \right) \Big|_\tau = \text{trop}(\beta_\tau f_\tau^{N_\tau} g_\tau)|_\tau$ holds for any τ . Since $-\log|x| > -\log|y|$ implies $-\log|x+y| = -\log|y|$ for any nonzero elements x, y in a non-Archimedean field, Cases 1 and 3 then give (4.5).

We infer that $\text{trop}(\phi) = \Phi$ on $\text{Sk}(\mathcal{X}, \mathcal{H})$, completing the proof of Theorem 1.1. \square

5. FINITE GENERATION AND FAITHFUL TROPICALIZATION

In this section, we prove Theorems 1.2, 1.3, and 1.4.

5.1. Preliminary results. Let Δ be an abstract polyhedral complex with recession fan Υ , set of vertices V_f and set of rays V_∞ . For a finite set C , we denote by \mathbf{e}_C the point of \mathbb{R}^C with coordinates all equal to one.

For a nonempty subset $A \subseteq V_f$ and a subset $B \subseteq V_\infty$, let $\Theta_{A \cup B} = \sigma^A \times \mathbb{R}_{\geq 0}^B$ be the corresponding extended simplex in $\mathbb{R}^{A \cup B} \subseteq \mathbb{R}^{V_f \cup V_\infty}$. The point $p = \frac{1}{|A|} \mathbf{e}_{A \cup B} = \frac{1}{|A|} (\mathbf{e}_A, \mathbf{e}_B)$ belongs to the extended simplex $\Theta_{A \cup B}$, and we refer to it as the *barycenter* of $\Theta_{A \cup B}$. For each $\tau \in \Delta$ with $\zeta(\tau) = A \cup B$, we denote by p_τ the barycenter of τ .

The *barycentric subdivision* Δ' of Δ is defined as follows. The vertex set V'_f of Δ' is the collection of the barycenters p_τ one for each τ in Δ . A chain $\tau_\bullet = (\tau_i)_{i=1}^k$ in Δ of positive length $k \in \mathbb{Z}_{>0}$ is a collection of elements $\tau_1 \prec \tau_2 \prec \cdots \prec \tau_k$. Given a subset $B \subseteq V_\infty$, a chain S_\bullet in B of non-negative length $l \in \mathbb{Z}_{\geq 0}$ is a chain of nonempty subsets $S_1 \subset \cdots \subset S_l$ of B . For $l = 0$, we get the empty chain with no element. To each pair $(\tau_\bullet, S_\bullet)$ consisting of a chain τ_\bullet in Δ of some length $k \in \mathbb{Z}_{>0}$ and a chain S_\bullet in $\zeta(\tau_1) \cap V_\infty$ of some length $l \in \mathbb{Z}_{\geq 0}$, we associate the polyhedron $\eta(\tau_\bullet, S_\bullet) \subseteq \tau_k$ defined by taking the convex-hull in τ_k of the points $p_{\tau_1}, \dots, p_{\tau_k}$ and all the rays $\mathbb{R}_{\geq 0} \mathbf{e}_{S_1}, \dots, \mathbb{R}_{\geq 0} \mathbf{e}_{S_l}$. Then, Δ' is the collection of all the polyhedra $\eta(\tau_\bullet, S_\bullet)$, see Figures 1 and 2. If there is no recession ray, this gives the usual notion of barycentric subdivision.

Note that for each vertex $v \in V_f$, we have $p_v = v$, so that we get an inclusion $V_f \subseteq V'_f$. It is easy to check that the recession fan Υ' of Δ' is the barycentric subdivision in the classical sense of the recession fan Υ of Δ . Denote by V'_∞ the set of recession rays of Δ' . We have $V_\infty \subseteq V'_\infty$.

Denote by \preceq the partial order on Δ' defined by the inclusion of faces. It is described as follows. Given elements $(\tau_\bullet, S_\bullet)$ and $(\omega_\bullet, T_\bullet)$, we have $\eta(\tau_\bullet, S_\bullet) \preceq \eta(\omega_\bullet, T_\bullet)$ provided that

- τ_\bullet is included in ω_\bullet , that is, each element τ_i coincides with an element $\omega_{n(i)}$ in ω_\bullet , and

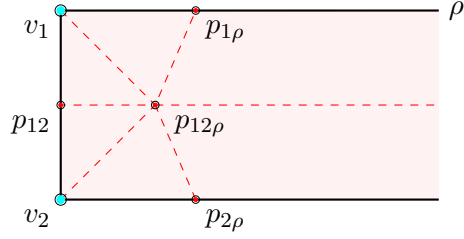


FIGURE 1. Barycentric subdivision of an extended simplex of dimension two with two vertices and a single ray.

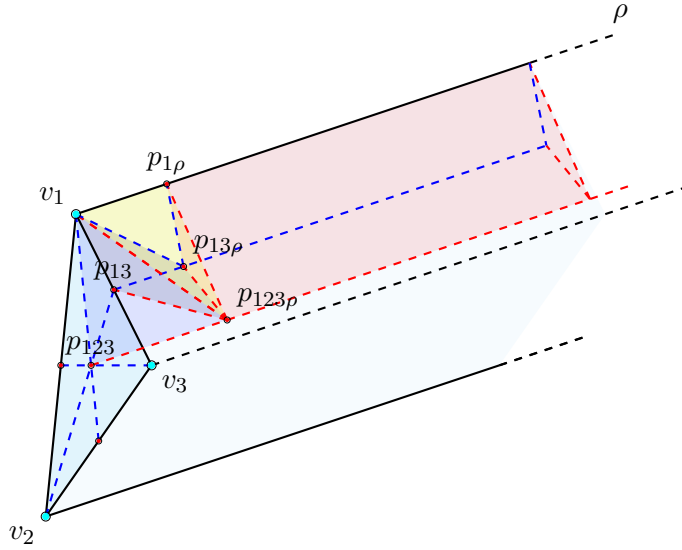


FIGURE 2. Barycentric subdivision of an extended simplex with three vertices and a single ray. Only a portion of the subdivision is given, with four solids depicted in color: three tetrahedra with vertices $\{v_1, p_{13}, p_{123}, p_{123\rho}\}$, $\{v_1, p_{13}, p_{13\rho}, p_{123\rho}\}$, and $\{v_1, p_{1\rho}, p_{13\rho}, p_{123\rho}\}$, and a prism with vertices $\{p_{1\rho}, p_{13\rho}, p_{123\rho}\}$ and a ray ρ .

- we have $S_\bullet \subseteq T_\bullet$, that is, each element S_j coincides with an element $T_{m(j)}$ in T_\bullet .

We next embed Δ' , preserving its integral structure. Let V_f'' denote a collection of distinct symbols \mathfrak{p}_τ , one for each $\tau \in \Delta$. Let V_∞'' denote a collection of distinct symbols S , one for each nonempty subset S of V_∞ included in $\zeta(\tau)$ for some $\tau \in \Delta$. The embedding of Δ' will be defined in $\mathbb{R}^{V_f'' \cup V_\infty''}$, with integral structure given by the lattice $\mathbb{Z}^{V_f'' \cup V_\infty''}$. (We distinguish p_τ and \mathfrak{p}_τ , where the former is a point of $|\Delta|$ and the latter will become an axis of the space which contains the embedding of $|\Delta|$. Similarly, we distinguish \mathfrak{e}_S and \mathbf{e}_S , where the former is a point in \mathbb{R}^S and the latter will become an axis of the space which contains the embedding of $|\Delta|$.)

First, we define the map

$$\varphi: V_f' \rightarrow \mathbb{R}^{V_f'' \cup V_\infty''}, \quad \varphi(p_\tau) := \frac{1}{|\zeta(\tau) \cap V_f|} \mathfrak{e}_{\mathfrak{p}_\tau}.$$

Specifically, if $\zeta(\tau) = A \cup B$ with $A \subseteq V_f$ and $B \subseteq V_\infty$, then we set $\varphi(p_\tau) = \frac{1}{|A|} \mathfrak{e}_{\mathfrak{p}_\tau}$.

For each pair $(\tau_\bullet, S_\bullet)$ with τ_\bullet a chain of positive length k in Δ and S_\bullet a chain of non-negative length l in $\zeta(\tau_1) \cap V_\infty$, let $\sigma(\tau_\bullet, S_\bullet)$ denote the convex hull in $\mathbb{R}^{V_f'' \cup V_\infty''}$ of the points $\varphi(p_{\tau_1}), \dots, \varphi(p_{\tau_k})$ and the rays $\mathbb{R}_{\geq} \mathbf{e}_{S_1}, \dots, \mathbb{R}_{\geq} \mathbf{e}_{S_l}$. Using linear interpolation, we extend φ to a bijective map

$$\varphi: \eta(\tau_\bullet, S_\bullet) \rightarrow \sigma(\tau_\bullet, S_\bullet),$$

for each pair $(\tau_\bullet, S_\bullet)$ as above.

Let Σ be the collection of the polyhedra $\sigma(\tau_\bullet, S_\bullet)$, where τ_\bullet is a nonempty chain in Δ and S_\bullet is a chain in $\zeta(\tau_1) \cap V_\infty$. Note that Σ forms a subcomplex of the product $P \times \mathbb{R}^{V_\infty''}$ where P is the simplex in $\mathbb{R}^{V_f''}$ obtained as the convex hull of the points $\varphi(p_\tau)$ for $\tau \in \Delta$. It follows directly from this observation that Σ is an integral \mathbb{Q} -affine polyhedral complex in $\mathbb{R}^{V_f'' \cup V_\infty''}$.

Proposition 5.1. *Notation as above, the maps φ glue together to define a bijective map $\varphi: |\Delta'| \rightarrow |\Sigma|$. Moreover, under this bijection, the recession fan of Δ' is identified with the recession fan of Σ .*

The maps φ and its inverse $\varphi^{-1}: |\Sigma| \rightarrow |\Delta'|$ preserve the integral affine structures. In particular, we have $\text{Rat}(|\Delta'|) = \text{Rat}(|\Sigma|)$.

Proof. The first assertion is straightforward. To show that φ and its inverse preserve the integral structure, it suffices to prove that each map

$$\varphi: \eta(\tau_\bullet, S_\bullet) \rightarrow \sigma(\tau_\bullet, S_\bullet)$$

is induced from an isomorphism between the lattice of $\eta(\tau_\bullet, S_\bullet)$ and the lattice of $\sigma(\tau_\bullet, S_\bullet)$. Let N and N' denote the former and the latter lattices, respectively.

Write $\tau_\bullet = \tau_1 \prec \dots \prec \tau_k$, with $\zeta(\tau_i) = A_i \cup B_i$, where $A_i \subseteq V_f$ and $B_i \subseteq V_\infty$. Write $S_\bullet = S_1 \subset \dots \subset S_l \subseteq B_1 = \zeta(\tau_1) \cap V_\infty$.

Let $m_{i,j}$ be the least common multiple of $|A_i|$ and $|A_j|$ for each pair $i, j = 1, \dots, k$. The lattice N' is generated by the vectors

$$m_{i,j} \left(\frac{1}{|A_i|} \mathbf{e}_{p_{\tau_i}} - \frac{1}{|A_j|} \mathbf{e}_{p_{\tau_j}} \right), \quad i, j = 1, \dots, k, \quad \text{and} \quad \mathbf{e}_{S_j}, \quad j = 1, \dots, l.$$

Similarly, the lattice N is generated by the vectors

$$m_{i,j} \left(\frac{1}{|A_i|} \mathbf{e}_{A_i \cup B_i} - \frac{1}{|A_j|} \mathbf{e}_{A_j \cup B_j} \right), \quad i, j = 1, \dots, k, \quad \text{and} \quad \mathbf{e}_{S_j}, \quad j = 1, \dots, l.$$

The map φ is the restriction of a linear map ψ from the vector space spanned by $\mathbf{e}_{A_i \cup B_i}$ and \mathbf{e}_{S_j} to the vector space spanned by $\mathbf{e}_{p_{\tau_i}}$ and \mathbf{e}_{S_j} defined by $\psi(\mathbf{e}_{A_i \cup B_i}) = \mathbf{e}_{p_{\tau_i}}$ and $\psi(\mathbf{e}_{S_j}) = \mathbf{e}_{S_j}$. It is straightforward to verify that ψ and its inverse are both integral maps. This establishes the preservation of the integral structure and proves the second statement in the proposition. The final assertion follows immediately as a consequence of the first two statements. \square

Note that the embedding φ of Δ' induces an embedding of the recession fan of Δ' as a subfan of the fan corresponding to the positive orthant $\mathbb{R}_{\geq 0}^{V_\infty''}$. Denote by Υ the recession fan of Σ .

Proposition 5.2. *There exists an integral \mathbb{Q} -affine polyhedral complex structure $\widehat{\Sigma}$ on $\mathbb{R}^{V_f'' \cup V_\infty''}$ that contains Σ as a polyhedral subcomplex, and has a recession fan $\widehat{\Upsilon}$ which is a completion of Υ to a complete fan in $\mathbb{R}^{V_f'' \cup V_\infty''}$. Additionally, we can choose $\widehat{\Sigma}$ to be convex with a convex recession fan $\widehat{\Upsilon}$.*

Proof. As observed, Σ is a polyhedral subcomplex of the product $P \times \mathbb{R}_{\geq 0}^{V_\infty''}$, where P is the simplex with vertices $\varphi(p_\tau)$ for $\tau \in \Delta$. We extend P to a convex, integral \mathbb{Q} -affine polyhedral complex structure on the entire space $\mathbb{R}^{V_f''}$, ensuring that it has a convex recession fan.

Similarly, the positive orthant $\mathbb{R}_{\geq 0}^{V_\infty''}$ is a polyhedral subcomplex of the polyhedral complex formed by all the orthants, which is also convex. The product of these polyhedral complex structures on

$\mathbb{R}^{V_f''}$ and $\mathbb{R}^{V_\infty''}$ yields a polyhedral complex structure $\widehat{\Sigma}$ with the desired properties stated in the proposition. \square

We fix a polyhedral tubular neighborhood U of $|\Sigma|$ in $\mathbb{R}^{V_f'' \cup V_\infty''}$, which has a deformation retraction to $|\Sigma|$ via a facewise affine map.

Next, we select a family of integral \mathbb{Q} -affine polyhedral complexes $(\widehat{\Sigma}_\alpha)_\alpha$ with support $\mathbb{R}^{V_f'' \cup V_\infty''}$, such that:

- Each $\widehat{\Sigma}_\alpha$ is a simplicial refinement of $\widehat{\Sigma}$;
- For each $\alpha \in \mathbb{Z}_{>0}$, there exists a strictly convex function $\widehat{H}_\alpha \in \text{Rat}(\widehat{\Sigma}_\alpha)$; and
- For any refinement $\widehat{\Sigma}'$ of $\widehat{\Sigma}$, there exists an α such that $\widehat{\Sigma}_\alpha$ is a refinement of $\widehat{\Sigma}'$.

Furthermore, by restricting to a suitable subfamily if necessary, we can assume without loss of generality that every face τ of $\widehat{\Sigma}_\alpha$ intersecting $|\Sigma|$ lies entirely within U .

For each α , let Σ_α denote the induced subdivision of $|\Sigma|$ from $\widehat{\Sigma}_\alpha$. Thus, we have $|\Sigma_\alpha| = |\Sigma|$ and

$$\bigcup_{\alpha} \text{Rat}(\Sigma_\alpha) = \text{Rat}(|\Sigma|).$$

We define Σ'_α as the union of all faces of $\widehat{\Sigma}_\alpha$ that have nonempty intersection with $|\Sigma|$. It follows from the assumption made above that $|\Sigma'_\alpha| \subseteq U$.

Proposition 5.3. *Each facewise affine function F on $\widehat{\Sigma}_\alpha$ can be expressed as the difference of two strictly convex facewise affine functions H_1 and H_2 on $\widehat{\Sigma}_\alpha$. Moreover, if $F|_{\Sigma'_\alpha} \in \text{Rat}(\Sigma'_\alpha)$, we can choose H_1 and H_2 such that both $H_1|_{\Sigma'_\alpha}$ and $H_2|_{\Sigma'_\alpha}$ belong to $\text{Rat}(\Sigma'_\alpha)$.*

Proof. For sufficiently large $\lambda \in \mathbb{Z}_{>0}$, the functions $H_1 = \lambda\widehat{H}_\alpha + F$ and $H_2 = \lambda\widehat{H}_\alpha$ are both strictly convex. This proves the first part of the proposition.

To prove the second part, we observe that since $\widehat{H}_\alpha \in \text{Rat}(\widehat{\Sigma}_\alpha)$, the restriction $\widehat{H}_\alpha|_{\Sigma'_\alpha}$ belongs to $\text{Rat}(\Sigma'_\alpha)$. Therefore, we conclude that both H_1 and H_2 are elements of $\text{Rat}(\Sigma'_\alpha)$, as required. \square

Let $H \in \text{Rat}(\widehat{\Sigma}_\alpha, \mathbb{R})$ be any strictly convex facewise affine function on $\widehat{\Sigma}_\alpha$ such that the restriction $H|_{\Sigma'_\alpha}$ belongs to $\text{Rat}(\Sigma'_\alpha)$. For each $\sigma \in \widehat{\Sigma}_\alpha$, we choose an affine function L_σ such that $(H - L_\sigma)|_\sigma = 0$, and $(H - L_\sigma)|_{\eta \setminus \sigma} > 0$ for all $\eta \supseteq \sigma$. Moreover, since $H|_{\Sigma'_\alpha} \in \text{Rat}(\Sigma'_\alpha)$, we can assume that L_σ is integral \mathbb{Q} -affine for all $\sigma \in \Sigma_\alpha$.

Proposition 5.4. *Notation as above, we have*

$$H = \max_{\sigma \in \widehat{\Sigma}_\alpha} L_\sigma \quad \text{and} \quad H|_{\Sigma_\alpha} = \max_{\sigma \in \Sigma_\alpha} L_\sigma|_{\Sigma_\alpha}.$$

Proof. The inequality $H \leq \max_{\sigma \in \widehat{\Sigma}_\alpha} L_\sigma$ is straightforward. Indeed, let $x \in \mathbb{R}^{V_f'' \cup V_\infty''}$, and suppose $x \in \tau$ for $\tau \in \widehat{\Sigma}_\alpha$. Then,

$$H(x) = L_\tau(x) \leq \max_{\sigma \in \widehat{\Sigma}_\alpha} L_\sigma(x).$$

Similarly, we have $H|_{\Sigma_\alpha} \leq \max_{\sigma \in \Sigma_\alpha} L_\sigma|_{\Sigma_\alpha}$.

For the reverse inequality, we use the lemma below which asserts that a facewise affine function H on $\widehat{\Sigma}_\alpha$ that is convex according to Definition 2.3 is also convex as a real-valued function on $\mathbb{R}^{V_f'' \cup V_\infty''}$. We claim that for each $\sigma \in \widehat{\Sigma}_\alpha$, we have $H \geq L_\sigma$ globally on $\mathbb{R}^{V_f'' \cup V_\infty''}$.

Fix a point p in the relative interior of σ . Let x be any point of $\mathbb{R}^{V_f'' \cup V_\infty''}$, and consider the segment $[p, x]$. The restriction $H|_{[p, x]}$ is convex, and the restriction of L_σ to $[p, x]$ is affine. There exists a small segment $I \subseteq [p, x]$ containing p such that $H|_I \geq L_\sigma|_I$, with $H(p) = L_\sigma(p)$. The

function $(H - L_\sigma)|_{[p,x]}$ is convex, it takes value zero at p , and is non-negative on I . We conclude that $(H - L_\sigma)|_{[p,x]}$ is non-negative everywhere, i.e., $H|_{[p,x]} \geq L_\sigma|_{[p,x]}$. This gives $H(x) \geq L_\sigma(x)$.

Thus, we obtain both inequalities $H \geq \max_{\sigma \in \widehat{\Sigma}_\alpha} L_\sigma$ and $H|_{\Sigma_\alpha} \geq \max_{\sigma \in \Sigma_\alpha} L_\sigma|_{\Sigma_\alpha}$. \square

Lemma 5.5. *Let Σ be a polyhedral complex with support \mathbb{R}^n . Let $H: \mathbb{R}^n \rightarrow \mathbb{R}$ be a facewise affine function on Σ . The following properties are equivalent:*

- (1) H is convex according to Definition 2.3.
- (2) H is convex as a real-valued function on \mathbb{R}^n .

Proof. The result is well-known, and we provide only a sketch of the proof.

To show (1) implies (2), take a segment $[p, q]$ in \mathbb{R}^n . It will be enough to show that $F = H|_{[p,q]}$ is convex. Since F is a piecewise affine function, Definition 2.3 ensures that at points of $[p, q]$ where there is a change of slopes, the sum of outgoing slopes is positive, which proves the convexity.

To prove the other direction, consider the upper-graph Γ_H^+ of H , defined as the set of all points $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ with $t \geq H(x)$. Since H is convex, Γ_H^+ is a convex subset of \mathbb{R}^{n+1} .

Let p be a point in the relative interior of a face σ of Σ . There exists an affine hyperplane A passing through $(p, H(p))$ such that the upper-graph of H is entirely above A . Since $H|_\sigma$ is affine, we can assume that A contains the affine space generated by σ . Thus, A is of the form $(x, L_\sigma(x))$ for an affine function L_σ on \mathbb{R}^n . We have $H|_\sigma = L_\sigma|_\sigma$, and the inequalities $(H - L_\sigma)|_{\eta \setminus \sigma} \geq 0$ hold for all $\eta \supset \sigma$, as required. \square

5.2. Proof of Theorem 1.2. Taking the barycentric subdivision Δ' of Δ as above, we embed Δ' as an integral \mathbb{Q} -affine polyhedral complex Σ in some linear space $\mathbb{R}^n = \mathbb{R}^{V_f'' \cup V_\infty''}$, with the notation introduced above. We also keep the notation for $\widehat{\Sigma}$, $(\widehat{\Sigma}_\alpha)_\alpha$, Σ'_α . Let $\pi: U \rightarrow |\Sigma|$ be the fixed retraction map induced from the polyhedral tubular neighborhood U of $|\Sigma|$ to $|\Sigma|$.

Note that we have $\text{Rat}(|\Delta|) = \text{Rat}(|\Sigma|) = \text{Rat}(|\Sigma_\alpha|)$ for all α . We prove $\text{Rat}(|\Sigma|)$ is generated over \mathbb{TQ} by the restriction of coordinate functions x_1, \dots, x_n of \mathbb{R}^n to $|\Sigma|$.

Let $F \in \text{Rat}(|\Sigma|)$ be a piecewise integral \mathbb{Q} -affine function. There exists some α such that F belongs to $\text{Rat}(\Sigma_\alpha)$. Using the retraction map $\pi|_{\Sigma'_\alpha}: |\Sigma'_\alpha| \rightarrow |\Sigma|$, we extend F to an integral facewise \mathbb{Q} -affine function G on Σ'_α , i.e., we set $G = F \circ \pi|_{\Sigma'_\alpha} \in \text{Rat}(\Sigma'_\alpha)$.

Let $\widehat{\Upsilon}_\alpha$ denote the recession fan of $\widehat{\Sigma}_\alpha$.

Next, we extend G to a facewise (not necessarily integral \mathbb{Q} -) affine function \widehat{G} on $\widehat{\Sigma}_\alpha$. The extension process works as follows.

First, for each ray ϱ in $\widehat{\Upsilon}_\alpha$, if ϱ is parallel to a ray ρ in Υ_α , we denote by s_ϱ the slope of F along any ray of Σ parallel to ρ . This is well-defined because $F \in \text{Rat}(\Sigma_\alpha)$.

Then, for rays ϱ of $\widehat{\Upsilon}_\alpha$ which are not parallel to any ray in Υ_α , we set $s_\varrho = 0$.

We define $\widehat{G}(v) = 0$ for every vertex v of $\widehat{\Sigma}_\alpha$ not contained in Σ'_α .

Once the values of \widehat{G} at the vertices of each polyhedron τ in $\widehat{\Sigma}_\alpha$ and the slopes along the rays of τ are fixed, since τ is simplicial, we can extend \widehat{G} by linear interpolation to the entire polyhedron τ . This way, we obtain an element $\widehat{G} \in \text{Rat}(\widehat{\Sigma}_\alpha, \mathbb{R})$. Note that the restriction $\widehat{G}|_{\Sigma'_\alpha}$ is integral \mathbb{Q} -affine.

By Proposition 5.3, we can write $\widehat{G} = H_1 - H_2$ for strictly convex functions H_1 and H_2 on $\widehat{\Sigma}_\alpha$ such that the restrictions $H_1|_{\Sigma'_\alpha}$ and $H_2|_{\Sigma'_\alpha}$ both belong to $\text{Rat}(\Sigma'_\alpha)$. By Proposition 5.4, we can express $H_1|_{\Sigma_\alpha}$ and $H_2|_{\Sigma_\alpha}$ as

$$H_1|_{\Sigma_\alpha} = \max_{\sigma \in \Sigma_\alpha} L_{1,\sigma}|_{\Sigma_\alpha} \quad \text{and} \quad H_2|_{\Sigma_\alpha} = \max_{\sigma \in \Sigma_\alpha} L_{2,\sigma}|_{\Sigma_\alpha}$$

where $L_{1,\sigma}, L_{2,\sigma}$ are integral \mathbb{Q} -affine functions on \mathbb{R}^n with

$$H_1|_\sigma = L_{1,\sigma}|_\sigma \quad \text{and} \quad H_2|_\sigma = L_{2,\sigma}|_\sigma$$

for all $\sigma \in \Sigma_\alpha$.

From the above, we conclude that F belongs to the semifield generated by \mathbb{TQ} and the restriction of the coordinate functions x_1, \dots, x_n of \mathbb{R}^n to $|\Sigma|$. Thus, the theorem is proven. \square

5.3. Proof of Theorem 1.4. To prove the first statement, using the work by Ewald and Ishida [EI06], we extend Σ to an integral \mathbb{Q} -affine polyhedral complex $\widehat{\Sigma}$ with full support \mathbb{R}^n which verifies the properties stated in Proposition 5.2. The rest of the arguments leading to the proof of Theorem 1.2 remains unchanged and gives the first part of Theorem 1.4.

We prove the second part. For $W \subseteq \mathbb{R}^n$ a subset of \mathbb{R}^n , let

$$E(W) = \{(F, G) \in \text{Rat}(\mathbb{R}^n)^2 \mid F(w) = G(w) \text{ for all } w \in W\}.$$

Then, $E(W)$ is a congruence on $\text{Rat}(\mathbb{R}^n)$. It is proved in [Son24] that the following are equivalent:

- (i) $E(W)$ is a finitely generated congruence.
- (ii) W is the support of an integral \mathbb{Q} -affine polyhedral complex.

(The result is stated over \mathbb{T} in *loc.cit.*, but the same proof works over \mathbb{TQ} .)

Consider the map $\pi: \text{Rat}(\mathbb{R}^n) \rightarrow \text{Rat}(|\Sigma|)$ from the introduction. The kernel of π is by definition equal to $E(|\Sigma|)$. Since $|\Sigma|$ verifies (ii), $E(|\Sigma|) = \text{Ker}(\pi)$ is a finitely generated congruence. \square

5.4. Proof of Theorem 1.3. Let X be a smooth projective variety over K with a strictly semistable pair $(\mathcal{X}, \mathcal{H})$. As in the introduction, we assume that X and the components of \mathcal{H} are geometrically connected. Denote by $\Delta(\mathcal{X}, \mathcal{H})$ the dual complex of $(\mathcal{X}, \mathcal{H})$ with the set of vertices V_f and the set of rays V_∞ , and we identify $|\Delta(\mathcal{X}, \mathcal{H})|$ with $\text{Sk}(\mathcal{X}, \mathcal{H})$.

Let

$$\varphi: |\Delta(\mathcal{X}, \mathcal{H})'| \rightarrow |\Sigma| \subseteq \mathbb{R}^{V_f'' \cup V_\infty''}$$

be the map in Proposition 5.1, so that φ is bijective and that both φ and φ^{-1} preserve the integral \mathbb{Q} -affine structures. Here, $\Delta(\mathcal{X}, \mathcal{H})'$ is a subdivision of $\Delta(\mathcal{X}, \mathcal{H})$ with the same recession fan, and we have $|\Delta(\mathcal{X}, \mathcal{H})'| = |\Delta(\mathcal{X}, \mathcal{H})| = \text{Sk}(\mathcal{X}, \mathcal{H})$.

Let x_1, \dots, x_n be the coordinate functions on $\mathbb{R}^{V_f'' \cup V_\infty''}$, where $n = \dim \mathbb{R}^{V_f'' \cup V_\infty''}$. We define $F_i := \varphi^* x_i \in \text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})'|)$. Then we have $\varphi = (F_1, \dots, F_n)$. Since $\text{Rat}(|\Sigma|)$ is generated by x_1, \dots, x_n over \mathbb{TQ} by Theorem 1.2, $\text{Rat}(|\Delta(\mathcal{X}, \mathcal{H})'|)$ is generated by F_1, \dots, F_n over \mathbb{TQ} .

By Theorem 1.1, we take $f_i \in \text{Rat}(X_{\bar{K}})$ such that $\text{trop}(f_i) = F_i$. Then setting $U = X_{\bar{K}} \setminus \bigcup_{i=1}^n |\text{div}(f_i)|$, the map

$$\psi: U^{\text{an}} \rightarrow \mathbb{R}, \quad x \mapsto (-\log |f_1(x)|, \dots, -\log |f_n(x)|) = (\nu_x(f_1), \dots, \nu_x(f_n))$$

gives a faithful tropicalization of the skeleton $\text{Sk}(\mathcal{X}, \mathcal{H})$. This completes the proof of Theorem 1.3. \square

REFERENCES

- [AI22] Omid Amini and Hernan Iriarte. Geometry of higher rank valuations. *arXiv preprint arXiv:2208.06237*, 2022.
- [Ber90] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.
- [Ber99] Vladimir G. Berkovich. Smooth p -adic analytic spaces are locally contractible. *Invent. Math.*, 137(1):1–84, 1999.
- [Ber04] Vladimir G. Berkovich. Smooth p -adic analytic spaces are locally contractible. II. In *Geometric aspects of Dwork theory. Vol. I, II*, pages 293–370. Walter de Gruyter, Berlin, 2004.
- [BFJ16] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Singular semipositive metrics in non-Archimedean geometry. *J. Algebraic Geom.*, 25(1):77–139, 2016.
- [BGS11] José Ignacio Burgos Gil and Martín Sombra. When do the recession cones of a polyhedral complex form a fan? *Discrete & Computational Geometry*, 46(4):789–798, 2011.
- [BPR16] Matthew Baker, Sam Payne, and Joseph Rabinoff. Nonarchimedean geometry, tropicalization, and metrics on curves. *Algebr. Geom.*, 3(1):63–105, 2016.

- [BR15] Matthew Baker and Joseph Rabinoff. The skeleton of the Jacobian, the Jacobian of the skeleton, and lifting meromorphic functions from tropical to algebraic curves. *International Mathematics Research Notices*, 2015(16):7436–7472, 2015.
- [Car21] Dustin Cartwright. A specialization inequality for tropical complexes. *Compos. Math.*, 157(5):1051–1078, 2021.
- [CHW14] Maria Angelica Cueto, Mathias Hübich, and Annette Werner. Faithful tropicalization of the Grassmannian of planes. *Mathematische Annalen*, 360:391–437, 2014.
- [CM19] Maria Angelica Cueto and Hannah Markwig. Tropical geometry of genus two curves. *Journal of Algebra*, 517:457–512, 2019.
- [DHLY24] Antoine Ducros, Ehud Hrushovski, François Loeser, and Jinhe Ye. Tropical functions on a skeleton. *Journal de l'École Polytechnique*, 11:613–654, 2024.
- [dJ96] A. J. de Jong. Smoothness, semi-stability and alterations. *Inst. Hautes Études Sci. Publ. Math.*, (83):51–93, 1996.
- [DP16] Jan Draisma and Elisa Postingshel. Faithful tropicalisation and torus actions. *Manuscripta Mathematica*, 149:315–338, 2016.
- [Duc12] Antoine Ducros. Espaces de Berkovich, polytopes, squelettes et théorie des modèles. *Confluentes Mathematici*, 4(4), 2012.
- [EI06] Günter Ewald and Masa-Nori Ishida. Completion of real fans and Zariski–Riemann spaces. *Tohoku Mathematical Journal, Second Series*, 58(2):189–218, 2006.
- [GRW16] Walter Gubler, Joseph Rabinoff, and Annette Werner. Skeletons and tropicalizations. *Advances in Mathematics*, 294:150–215, 2016.
- [Gub07] Walter Gubler. Tropical varieties for non-Archimedean analytic spaces. *Invent. Math.*, 169(2):321–376, 2007.
- [Har01] Urs T Hartl. Semi-stability and base change. *Archiv der Mathematik*, 77(3):215–221, 2001.
- [Hel19] Paul Alexander Helminck. Faithful tropicalizations of elliptic curves using minimal models and inflection points. *Arnold Mathematical Journal*, 5(4):401–434, 2019.
- [HW93] Udo Hebisch and Hanns Joachim Weinert. *Semirings: algebraic theory and applications in computer science*, volume 5. World Scientific, 1993.
- [JM12] Mattias Jonsson and Mircea Mustață. Valuations and asymptotic invariants for sequences of ideals. 62(6):2145–2209, 2012.
- [KKMSD73] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal embeddings. I*, volume Vol. 339 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1973.
- [KMM08] Eric Katz, Hannah Markwig, and Thomas Markwig. The j -invariant of a plane tropical cubic. *Journal of Algebra*, 320(10):3832–3848, 2008.
- [Kut16] Max B Kutler. Faithful tropicalization of hypertoric varieties. *arXiv preprint arXiv:1604.08271*, 2016.
- [KY19] Shu Kawaguchi and Kazuhiko Yamaki. Effective faithful tropicalizations associated to adjoint linear systems. *Int. Math. Res. Not. IMRN*, (19):6089–6112, 2019.
- [KY21] Shu Kawaguchi and Kazuhiko Yamaki. *Effective faithful tropicalizations associated to linear systems on curves*, volume 270 of *Memoirs of the AMS*. American Mathematical Society, 2021.
- [Mat80] Hideyuki Matsumura. *Commutative algebra*, volume 56 of *Mathematics Lecture Note Series*. Benjamin/Cummings Publishing Co., Inc., Reading, MA, second edition, 1980.
- [MN15] Mircea Mustață and Johannes Nicaise. Weight functions on non-Archimedean analytic spaces and the Kontsevich-Soibelman skeleton. *Algebr. Geom.*, 2(3):365–404, 2015.
- [MRS23] Hannah Markwig, Lukas Ristau, and Victoria Schleis. Faithful tropicalization of hyperelliptic curves. *arXiv preprint arXiv:2310.02947*, 2023.
- [Ran17] Dhruv Ranganathan. Skeletons of stable maps I: rational curves in toric varieties. *Journal of the London Mathematical Society*, 95(3):804–832, 2017.
- [Son24] JuAe Song. Finitely generated congruences on tropical rational function semifields. *arXiv preprint arXiv:2405.14087*, 2024.

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