RESIDUE POLYTOPES

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Abstract. We associate to each finite graph G on n vertices a polytope in $\mathbb{R}^n$, defined using the first homology of G, which we call the *residue polytope*. We give a complete description of its face structure in terms of the so-called level structures on G, which are ordered partitions of its set of vertices. Remarkably, we show that the linear equations defining the faces of the residue polytope precisely encode the residue conditions recently discovered to be satisfied by limits of Abelian differentials along families of Riemann surfaces degenerating to a stable Riemann surface with dual graph G. In particular, the technically involved global residue conditions, previously obtained by application of Stokes theorem, are given a natural combinatorial interpretation via explicit deformations, associated with level structures, of flow conditions in graphs. In a subsequent work, we use this connection to describe and parameterize the collection of all the limits of the complete spaces of Abelian differentials on Riemann surfaces, for any given limit stable Riemann surface that is general for its topology.

1. Introduction

In this paper, we introduce residue polytopes, a class of polymatroid base polytopes associated with finite graphs. Their introduction is motivated by the study of Abelian differentials on compact Riemann surfaces and their limits, as we expand later in this section.

Let $G = (V, E)$ be a finite graph with vertex set V and edge set E. Let $\mathbf{k}$ be a field, and denote by $H_1(G, \mathbf{k})$ the first homology group of G with coefficients in $\mathbf{k}$. Equivalently, $H_1(G, \mathbf{k})$ can be interpreted as the space of $\mathbf{k}$-valued flows on G.

Let $\mathbb{E}$ denote the set of directed arrows assigned to the edges of G, where each edge $e = \{v, w\}$ gives rise to two arrows, vw and wv, in $\mathbb{E}$. If $v = w$, we still have two arrows $v \leftrightarrow w$, corresponding to the two half edges issued from e.

For each vertex v, let $\mathbb{E}_v \subseteq \mathbb{E}$ denote the set of arrows emanating from v. By decomposing each element of $H_1(G, \mathbf{k})$ according to its values on the arrows in $\mathbb{E}_v$ for $v \in V$, we obtain an embedding

$$H_1(G, \mathbf{k}) \hookrightarrow \bigoplus_{v \in V} \mathbf{k}^{\mathbb{E}_v}.$$

This embedding allows us to study the image of $H_1(G, \mathbf{k})$ via its coordinate projecions. To each subset $I \subseteq V$, we associate the dimension of the projection of $H_1(G, \mathbf{k})$ onto $\bigoplus_{v \in I} \mathbf{k}^{\mathbb{E}_v}$. This yields a submodular function on the power set of V, thereby giving rises to a polymatroid. The base polytope of this polymatroid is what we call the *residue polytope of the graph*, denoted $\mathbf{P}_{\rm res} = \mathbf{P}_{\rm res}(G)$; see the discussion leading to (1.1).

The goal of this paper is to describe the face structure of $\mathbf{P}_{\rm res}$. This is done in a way that relates to the asymptotic geometry of degenerating families of Riemann surfaces–though our approach is entirely combinatorial and does not rely on results from complex geometry.

In order to state our theorems, we introduce some terminology.

Date: May 2025.

For a subset of vertices $I \subseteq V$, we denote by $G[I]$ the induced subgraph on I. This subgraph has vertex set I and edge set consisting of those edges in G whose both endpoints lie in I.

For each positive integer r, let $[r] := \{1, \dots, r\}$. A *partition* of V is a collection of nonempty, pairwise disjoint subsets of V whose union is V. A *level structure* on G is a datum $\pi = (\pi_1, \dots, \pi_r)$, consisting of an ordered partition on the vertex set V; that is, a partition of V equipped with a total order on its parts. This data induces a surjection $h = h_\pi \colon V \to [r]$, which assigns to each vertex the index of the part to which it belongs. We refer to either (G, π) or (G, h) as a *level graph*, and call h the associated *level function*. For $v \in V$, $h(v)$ is called the *level* of v. Given two vertices $u, v \in V$, we write $u \prec_\pi v$ if $h(u) < h(v)$. We also denote by $\Pi = \Pi(G)$ the set of all ordered partitions of V. The terminology is adopted from [BCG⁺18]; see Section 1.2 for further discussion.

For an arrow $a = v \to w$ in $\mathbb{E}$, we refer to v as the *tail* and to w as the *head* of a, and write $\mathfrak{t}_a = v$ and $\mathfrak{h}_a = w$. The arrow in $\mathbb{E}$ with the reverse orientation is denoted $\bar{a}$; it has tail w and head v.

A *vertical edge* in a level graph (G, π) is an edge $e = \{u, v\}$ such that $h_\pi(u) \neq h_\pi(v)$. We denote the set of vertical edges by E_π. Edges for which $h_\pi(u) = h_\pi(v)$ are called *horizontal*, and form the complement $E_\pi^c = E \smallsetminus E_\pi$. We denote by $\mathbb{E}_\pi \subseteq \mathbb{E}$ the set of arrows lying on vertical edges; its elements are called *vertical arrows*. The complement $\mathbb{E}_\pi^c = \mathbb{E} \smallsetminus \mathbb{E}_\pi$ is the set of *horizontal arrows*.

An arrow $u \to v$ in $\mathbb{E}_\pi$ is said to be *compatible with the level structure* if $u \succ_\pi v$. (Visually, both $\to$ and $\succ_\pi$ point to v.) We denote by A_π the set of such arrows and refer to them as *upward arrows*. Its complement in $\mathbb{E}_\pi$, denoted $\bar{A}_\pi$, consists of *downward arrows*. This gives a bipartition

$$\mathbb{E}_\pi = A_\pi \sqcup \bar{A}_\pi.$$

Figure 1 illustrates a level graph G with two levels. In such figures, horizontal edges are drawn horizontally, and vertical edges vertically. Since the directions of upward and downward arrows are visually apparent, we omit the arrowheads.

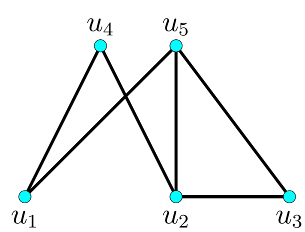

FIGURE 1. A level graph (G, π) with two levels: $\pi = (\pi_1, \pi_2)$ with $\pi_1 = \{u_4, u_5\}$ and $\pi_2 = \{u_1, u_2, u_3\}$. The arrows $u_1u_4, u_1u_5, u_2u_4, u_2u_5, u_3u_5$ are upward. The edge $\{u_2, u_3\}$ is horizontal.

Let $\mathbf{k}$ be a field and consider the vector space $\mathbf{k}^{\mathbb{E}}$. For each $\psi \in \mathbf{k}^{\mathbb{E}}$ and $a \in \mathbb{E}$, let ψ_a denote the a-coordinate of ψ. We decompose

$$\mathbf{k}^{\mathbb{E}} = \bigoplus_{v \in V} \mathbf{k}^{\mathbb{E}_v},$$

where $\mathbb{E}_v$ is the set of arrows vw in $\mathbb{E}$ with tail v.

Given a level graph $(G, \pi = (\pi_1, \dots, \pi_r))$ with level function h, define the *residue space of* (G, π), or simply π-*residue space* if G is fixed, as the subspace of $\mathbf{k}^{\mathbb{E}}$ denoted $\boldsymbol{\mathcal{G}}_\pi$, consisting of those elements $\psi \in \mathbf{k}^{\mathbb{E}}$ which satisfy *the residue conditions* (R1) through (R4) below, introduced in [BCG+18] (see Section 1.2 for a discussion of their geometric origin).

(R1) *Vanishing along downward arrows*

$$\psi_a = 0 \qquad \text{for every arrow } a \in \bar{A}_\pi.$$

(R2) *Local residue conditions*

$$\sum_{a \in \mathbb{E}_v} \psi_a = 0 \qquad \text{for every } v \in V.$$

(R3) *Rosenlicht conditions*

$$\psi_a + \psi_{\bar{a}} = 0 \qquad \text{for each horizontal arrow } a \text{ for } \pi.$$

(R4) *Global residue conditions*

$$\sum_{a \in A^{\Xi}_{\pi,n}} \psi_a = 0$$

for each level $n \in [r]$ and each connected component Ξ of the subgraph $G[V_{h<n}]$, induced on the set of vertices $V_{h<n} \subseteq V$ of level smaller than n, where $A^{\Xi}_{\pi,n} \subseteq A_\pi$ denotes the set of upward arrows with tail of level n and head in Ξ.

In Figure 1, for $n = 2$, the set $V_{h<2}$ consists of two vertices, u_4 and u_5, with the induced graph $G[V_{h<2}]$ containing no edges. Taking Ξ to be the connected component of $G[V_{h<2}]$ with vertex set $\{u_5\}$, the set $A^{\Xi}_{\pi,2}$ contains three arrows: u_1u_5, u_2u_5, and u_3u_5. The corresponding global residue condition is the equation

$$\psi_{u_1u_5} + \psi_{u_2u_5} + \psi_{u_3u_5} = 0.$$

Note that if $\pi_0 = \{V\}$ is the trivial partition of V, Conditions (R1) and (R4) become vacuous. The π_0-residue space $\boldsymbol{\mathcal{G}}_{\pi_0}$ can be identified with the first homology group $H_1(G, \mathbf{k})$, the two remaining conditions (R2) and (R3) being identified with the flow conditions in the graph. This space has dimension equal to the *genus* $g = g(G)$ of G given by

$$g = |E| - |V| + \mathfrak{c},$$

where $\mathfrak{c}$ denotes the number of connected components of G. Our first result establishes this for all residue spaces.

Theorem 1.1. *Let π be an ordered partition of V. We have*

$$\dim_{\mathbf{k}} \boldsymbol{\mathcal{G}}_\pi = |E| - |V| + \mathfrak{c}.$$

In the absence of horizontal edges, this theorem can be obtained from [BCG+19b, Prop. 6.3] by using Picard–Lefschetz Theory on a degenerating family of Riemann surfaces, which yields that the dimension of the space of vanishing cycles is the same as the genus of the dual graph of the limit stable Riemann surface. We provide a direct combinatorial proof of the general statement; an outline of the proof can be found in Section 1.1.

To each residue space $\boldsymbol{\mathcal{G}}_\pi$ for $\pi \in \Pi$, we associate a polytope $\mathbf{P}_\pi \subseteq \mathbb{R}^V$ as follows. For each subset $I \subseteq V$, consider the projection map

$$\theta_I : \bigoplus_{v \in V} \mathbf{k}^{\mathbb{E}_v} \to \bigoplus_{v \in I} \mathbf{k}^{\mathbb{E}_v}.$$

The π-residue space $\boldsymbol{\mathcal{G}}_\pi$ lives in $\mathbf{k}^{\mathbb{E}} = \bigoplus_{v\in V}\mathbf{k}^{\mathbb{E}_v}$. Define the function

$$\gamma_\pi \colon 2^V \to \mathbb{Z}, \qquad \gamma_\pi(I) = \dim_{\mathbf{k}} \theta_I(\boldsymbol{\mathcal{G}}_\pi) \quad \text{for each } I \subseteq V.$$

This function is *submodular*, meaning it satisfies $\gamma_\pi(\varnothing) = 0$ and the following inequality for all $I, J \subseteq V$:

$$\gamma_\pi(I) + \gamma_\pi(J) \geqslant \gamma_\pi(I \cup J) + \gamma_\pi(I \cap J).$$

We then define $\mathbf{P}_\pi$ as the base polytope of the polymatroid defined by γ_π, namely,

$$\mathbf{P}_\pi := \Big\{ q \in \mathbb{R}^V \;\Big|\; q(V) = g \text{ and } q(I) \leqslant \gamma_\pi(I) \text{ for all } I \subseteq V \Big\};$$

see [Sch03, Chap. 44]. Here, for $q \in \mathbb{R}^V$ and $I \subseteq V$, we set $q(I) = \sum_{v\in I} q(v)$.

The polytope $\mathbf{P}_\pi$ lives in the standard simplex Δ_g of width g in $\mathbb{R}^V_{\geqslant 0}$, which consists of points $q \in \mathbb{R}^V$ satisfying $q(v) \geqslant 0$ for all $v \in V$ and $q(V) = g$.

For the trivial ordered partition $\pi_0 = \{V\}$, we have

$$\gamma_{\pi_0}(I) = g - g(I^c) \qquad \text{for all } I \subseteq V,$$

where $I^c = V \smallsetminus I$, and $g(I^c)$ denotes the genus of the induced subgraph $G[I^c]$. Equivalently, $\gamma_{\pi_0}(I)$ is the genus of the graph obtained from G by contracting each connected component of $G[I^c]$ into a vertex. The residue polytope $\mathbf{P}_{\mathrm{res}}(G)$ is defined as $\mathbf{P}_{\pi_0}$,

$$\mathbf{P}_{\mathrm{res}}(G) := \mathbf{P}_{\pi_0}. \tag{1.1}$$

The residue polytope of the complete graph K_4 on 4 vertices in depicted in Figure 2.

Given two ordered partition $\pi = (\pi_1, \dots, \pi_r)$ and $\pi' = (\pi'_1, \dots, \pi'_s)$ of V, we say π' is a *coarsening* of π, and write $\pi' \geq \pi$, if each element of π is contained within an element of π', and the total orders are compatible. More precisely, $\pi' \geq \pi$ means that the surjection $h' \colon V \to [s]$ associated with π' factors through the surjection $h\colon V \to [r]$ associated with π, i.e., there exists a map $c\colon [r] \to [s]$ such that $h' = c \circ h$. Additionally, c satisfies the condition that $c(n_1) \leqslant c(n_2)$ for each $n_1, n_2 \in [r]$ with $n_1 \leqslant n_2$. In particular, note that π_0 is a coarsening of any $\pi \in \Pi$.

Our main theorem is stated as follows.

Theorem 1.2. *For each $\pi \in \Pi$, the polytope $\mathbf{P}_\pi$ is a face of $\mathbf{P}_{\mathrm{res}}$. Furthermore, each face of $\mathbf{P}_{\mathrm{res}}$ is of this form. In addition, if $\pi' \geq \pi$, then $\mathbf{P}_{\pi'} \supseteq \mathbf{P}_\pi$, and so $\mathbf{P}_\pi$ is a face of $\mathbf{P}_{\pi'}$.*

In the course of proving this result, we establish Theorem 6.1, which offers a new perspective on the technically involved global residue conditions. Specifically, Conditions (R1) and (R4) arise as explicit degenerations of the flow conditions in the graph, and the residue space $\boldsymbol{\mathcal{G}}_\pi$ is obtained by a simple combinatorial procedure called *splitting* (with respect to π) of $\boldsymbol{\mathcal{G}}_{\pi_0} = H_1(G, \mathbf{k})$. This gives a simple encoding of the residue conditions (R1)-(R4).

1.1. Outline of the proofs.

Given $\pi \in \Pi$, we define a flag of subspaces

$$\boldsymbol{\mathcal{G}}_\pi \subseteq \boldsymbol{\mathcal{R}}_\pi \subseteq \boldsymbol{\Upsilon}^0_\pi \subseteq \boldsymbol{\Upsilon}_\pi \subseteq \boldsymbol{\Upsilon} := \mathbf{k}^{\mathbb{E}}$$

by successively imposing Conditions (R1) through (R4). The subspace $\boldsymbol{\Upsilon}_\pi \subseteq \mathbf{k}^{\mathbb{E}}$ is obtained by imposing Condition (R1):

$$\boldsymbol{\Upsilon}_\pi := \big\{ \psi \in \mathbf{k}^{\mathbb{E}} \;\big|\; \psi_a = 0 \text{ for every } a \in \bar{A}_\pi \big\} \subseteq \mathbf{k}^{\mathbb{E}}. \tag{1.2}$$

Next, we define $\boldsymbol{\Upsilon}^0_\pi \subseteq \boldsymbol{\Upsilon}_\pi$ by imposing Condition (R2):

$$\boldsymbol{\Upsilon}^0_\pi := \Big\{ \psi \in \boldsymbol{\Upsilon}_\pi \;\Big|\; \sum_{a\in\mathbb{E}_v} \psi_a = 0 \text{ for each } v \in V \Big\}. \tag{1.3}$$

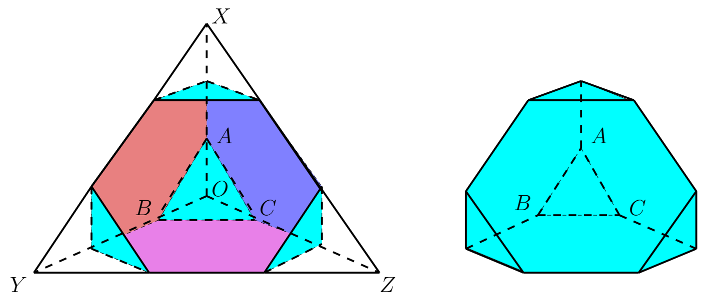

FIGURE 2. The residue polytope of the complete graph K_4 on four vertices on the right. The figure on the left shows the position of $\mathbf{P}_{\mathrm{res}}(K_4)$ within the simplex $OXYZ$ of width 3 in $\mathbb{R}^4$, with vertices $O = (3,0,0,0)$, $X = (0,3,0,0)$, $Y = (0,0,3,0)$ and $Z = (0,0,0,3)$, using the projection to $\mathbb{R}^3$ given by the last three coordinates.

Then, imposing Condition (R3) yields the subspace $\boldsymbol{\mathcal{R}}_\pi \subseteq \boldsymbol{\Upsilon}_\pi^0$:

$$\boldsymbol{\mathcal{R}}_\pi := \left\{\psi \in \boldsymbol{\Upsilon}_\pi^0 \;\middle|\; \psi_a + \psi_{\bar{a}} = 0 \quad \text{for all } a \in \mathbb{E} \smallsetminus \mathbb{E}_\pi\right\}. \tag{1.4}$$

Finally, the global residue space $\boldsymbol{\mathcal{G}}_\pi \subseteq \boldsymbol{\mathcal{R}}_\pi$ is obtained by imposing Condition (R4). The dimensions of all the spaces in this flag are computed in Sections 3 and 4, culminating in the proof of Theorem 1.1.

In order to prove Theorem 1.2, we show that for each pair of ordered partitions $\pi, \pi' \in \Pi$ with $\pi' \succeq \pi$, the π-residue space $\boldsymbol{\mathcal{G}}_\pi$ arises as a limit of a family of subspaces of $\mathbf{k}^{\mathbb{E}}$ sharing the same associated submodular function as $\gamma_{\pi'}$. More precisely, let $\mathbf{R} = \mathbf{k}[t, t^{-1}]$ be the ring of Laurent polynomials with $\mathbf{k}$-coefficients and denote by $\mathbf{G}_{\mathbf{m}}^V(\mathbf{R})$ the set of functions $x \colon V \to \mathbf{R}$ taking values in the set of invertible elements in $\mathbf{R}$. An integer valued function $d \colon V \to \mathbb{Z}$, $v \mapsto d_v$, determines an ordered partition of V induced by its level sets. Given such a function with π as the associated ordered partition, define $x \in \mathbf{G}_{\mathbf{m}}^V(\mathbf{R})$ by setting $x_v := t^{-d_v}$ for each $v \in V$, and put

$$(x \cdot \psi)_a := x_v \psi_a = t^{-d_v} \psi_a \qquad \text{for each } \psi \in \mathbf{k}^{\mathbb{E}}, \text{ and each } v \in V \text{ and } a \in \mathbb{E}_v.$$

This defines an action of $\mathbf{G}_{\mathbf{m}}^V(\mathbf{R})$ on $\boldsymbol{\Upsilon} \otimes_{\mathbf{k}} \mathbf{R}$. Analyzing this action and invoking Theorem 1.1, we prove in Theorem 6.1 that whenever $\pi' \succeq \pi$, we have

$$\lim_{t \to 0} x \cdot \boldsymbol{\mathcal{G}}_{\pi'} = \boldsymbol{\mathcal{G}}_\pi.$$

We deduce that $\boldsymbol{\mathcal{G}}_\pi$ is the splitting of $\boldsymbol{\mathcal{G}}_{\pi'}$ with respect to π, in the sense of Section 5.4. Thus, the submodular function γ_π is a splitting of the submodular function $\gamma_{\pi'}$; see Section 5.3 for details. Using the characterization of faces of polymatroids (see e.g. [Sch03, Chap. 44] or [AE24, Prop. 2.7]), we finish the proof of Theorem 1.2.

1.2. **Abelian differentials on Riemann surfaces and their limits.** The results above have their source of motivation in our companion work [AEG24], where we study the limits of complete spaces of Abelian differentials on degenerating families of Riemann surfaces. We briefly discuss the setup.

An Abelian differential on a compact Riemann surface is a globally defined holomorphic one-form, given in local coordinates by a differential $f(z)dz$ for a holomorphic function f.

For a smooth compact Riemann surface S of genus g, the space of Abelian differentials on S, denoted by H_S, is a complex vector space of dimension g.

For every family of smooth Riemann surfaces degenerating to any given stable Riemann surface X on the Deligne–Mumford boundary of the moduli space of curves, the g-dimensional spaces of Abelian differentials degenerate to a collection of g-dimensional linear series on X. As the degenerating family varies, so does the resulting collection. In this way, each stable Riemann surface X in the moduli space is associated with infinitely many such collections of linear series – one for each degenerating family. We are interested in describing all these collections. This has been thoroughly studied previously only for very special X, compact-type [EH87] and two-component [EM02], in the case the branches on the components of X over the nodes are in general position for each component.

Recently, Bainbridge, Chen, Gendron, Grushevsky and Möller [BCG+18] studied variations of one-dimensional subspaces of H_S along families of Riemann surfaces S, and constructed a compactification of their moduli. This was achieved by considering tuples of meromorphic differentials on components of each stable Riemann surface X that satisfy pole, zero, and residue conditions. Most importantly, they discovered the global residue conditions (R4) by an application of Stokes Formula, and put them along with the previously known residue conditions (R1)-(R2)-(R3). Their work led to a significant series of developments in the study of the geometry of moduli spaces; see [BCG+19b, CMSZ20, MUW20, GT21], and particularly [BCG+19a, §1] for furher references.

A stable Riemann surface X has a dual graph $G = (V, E)$ with each vertex $v \in V$ corresponding to a component $\mathbf{C}_v$ in the desingularization of X, and each edge $e \in E$ corresponding to a node p^e on X. If e connects u and v, then p^e lies on $\mathbf{C}_u$ and $\mathbf{C}_v$. For each arrow $a = u \to v$ in $\mathbb{E}$ over an edge $e = \{u, v\}$ of E, denote by p^a the point on $\mathbf{C}_u$ over the node p^e of X. The tropicalization of the family S_t yields an edge length function $\ell\colon E \to \mathbb{R}_{>0}$. Moreover, as we show in [AE24], tropicalization also associates a limit space $\mathrm{W}_h \subset \Omega := \bigoplus_{v\in V} \Omega_v$ for each function $h\colon V \to \mathbb{R}$, where Ω_v is the space of meromorphic differentials on $\mathbf{C}_v$. In particular, each element $\alpha \in \mathrm{W}_h$ can be viewed as a collection of meromorphic differentials $\alpha_v \in \Omega_v$ for $v \in V$. This yields a residue map

$$\mathrm{Res}\colon \mathrm{W}_h \to \mathbb{C}^{\mathbb{E}}, \qquad \alpha = (\alpha_v)_{v\in V} \mapsto \Big[a = vu \mapsto \mathrm{res}_{p^a}(\alpha_v)\Big].$$

Viewing each function h as a level function gives rise to an ordered partition π_h of V. Let $\boldsymbol{\mathcal{G}}_{\pi_h}$ be the corresponding π_h-residue space. Using results from [BCG+18, TT22] and [AE24], we prove in [AEG24, Thm. 5.1] that the image of the residue map lies in $\boldsymbol{\mathcal{G}}_{\pi_h}$, that is, any element $\psi = \mathrm{Res}(\alpha)$ for $\alpha \in \mathrm{W}_h$ satisfies Conditions (R1) through (R4).

The results we prove here play a key role in our description in [AEG24] of all the limits of H_{S_t}, for all families of Riemann surfaces S_t approaching a stable Riemann surface X with an arbitrary number of components, meeting in whatever ways, as long as the branches on each component of its desingularization are in general position. Using this description, we prove that there exists a projective variety that parametrizes all these limits, and we describe this variety. This generalizes to all topologies the aforementioned work [EH87, EM02].

Our interest in the refinements of ordered partitions and their relation through residue polytopes stems from the theory of submodular functions and polymatroids, which play a crucial role in [AEG24]. There is however a more general notion of morphism between level graphs, introduced in [BCG+19a] for compactifying the moduli space of Abelian differentials.

It is likely that these morphisms induce maps between the corresponding residue polytopes, an interesting combinatorial question worth exploring.

Acknowledgment. We warmly thank Samuel Grushevsky for interesting exchanges about the content of this paper and its connection to [BCG+19b, BCG+19a].

We are grateful to École Polytechnique, IMPA, and the French-Brazilian network in mathematics for support and hospitality. This work was partially supported by the "CAPES-COFECUB" programme (project number: Ma 1017/24), funded by the French Ministry for Europe and Foreign Affairs, the French Ministry for Higher Education and CAPES.

2. Set-theoretically independent collections

We start by formulating two basic linear-algebra results, frequently used in the sequel.

We fix a field $\mathbf{k}$. For a finite set A, we denote by $\mathbf{k}^A$ the $\mathbf{k}$-vector space of functions $\psi\colon A \to \mathbf{k}$, $a \mapsto \psi_a$.

For each $\psi \in \mathbf{k}^A$, denote by $[\psi]$ the support of ψ defined by

$$[\psi] := \left\{ a \in A \;\middle|\; \psi_a \neq 0 \right\}.$$

For a finite collection of vectors $\Psi \subset \mathbf{k}^A$, let $[\Psi] \subseteq A$ be the union of $[\psi]$ for $\psi \in \Psi$. We say that Ψ is *set-theoretically independent* if the $[\psi]$ for $\psi \in \Psi$ form a partition of $[\Psi]$. Obviously, if this happens, then Ψ is linearly independent, and any subcollection $\Psi' \subseteq \Psi$ will be set-theoretically independent as well.

We say that two set-theoretically independent collections of vectors $\Phi, \Psi \subset \mathbf{k}^A$ are *related* when there are two nonempty subcollections $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$ that yield partitions of the same subset of A, that is, when $[\Phi'] = [\Psi']$. Otherwise, we call the two collections *unrelated*. Note that in the latter case, $\Phi \cap \Psi = \varnothing$.

Proposition 2.1. *Let $\Phi, \Psi \subset \mathbf{k}^A$ be two unrelated set-theoretically independent collections of vectors. Then, the union $\Phi \cup \Psi$ is linearly independent.*

Proof. For the sake of contradiction, assume there exists a nontrivial relation

$$\sum_{\varphi \in \Phi} c_\varphi \varphi + \sum_{\psi \in \Psi} d_\psi \psi = 0. \tag{2.1}$$

Since Φ and Ψ are unrelated, we have

$$\bigcup_{\varphi\colon c_\varphi \neq 0} [\varphi] \neq \bigcup_{\psi\colon d_\psi \neq 0} [\psi].$$

We may assume w.l.o.g. that there is an $a \in A$ included in the left-hand side but not in the right-hand side. Then, there exists $\varphi \in \Phi$ with $\varphi_a \neq 0$ and $c_\varphi \neq 0$, unique with this property because Φ is set-theoretically independent. Thus, evaluating (2.1) at a gives $c_\varphi \varphi_a = 0$, a contradiction. □

Two set-theoretically independent subsets $\Phi, \Psi \subseteq \mathbf{k}^A$ are called *properly unrelated* if for each nonempty subcollections $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$ with either $\Phi' \neq \Phi$ or $\Psi' \neq \Psi$, we have $[\Phi'] \neq [\Psi']$.

Proposition 2.2. *Let $\Phi, \Psi \subset \mathbf{k}^A$ be properly unrelated set-theoretically independent collections of vectors. Then, for each $\varphi \in \Phi$, the collection $(\Phi \cup \Psi) \smallsetminus \{\varphi\}$ is linearly independent.*

Proof. This follows from Proposition 2.1 applied to $\Phi \smallsetminus \{\varphi\}$ and Ψ. □

Remark 2.3. The above results generalize in an obvious way to the setup of a finite-dimensional vector space equipped with a basis. ◇

3. Imposing the first three conditions

Let $\mathbb{E}$ be as before the set of arrows on edges of G, each edge $e = \{u, v\}$ giving rise to two arrows uv and vu in $\mathbb{E}$. For a subset $A \subseteq \mathbb{E}$, we denote by $\bar{A}$ the set consisting of the arrows $\bar{a}$ for all $a \in A$. Given subsets $I, J \subseteq V$, we denote by A_I the set of arrows in A with tail in I, and denote by $A(I, J)$ the set of arrows in A with tail in I and head in J. If $I = J$, we simply write $A(I)$. Similarly, for $F \subseteq E$ and $I, J \subseteq V$, we denote by F_I the set of edges of F with one vertex in I, by $F(I, J)$ the set of edges of F with one vertex in I and the other in J, and by $F(I)$ the set of edges of F with both vertices in I.

We fix in this section and the next an ordered partition $\pi = (\pi_1, \ldots, \pi_r)$ on V, and consider the level graph (G, π). Let $h\colon V \to [r]$ be the corresponding level function. Let E_π, A_π, and $\bar{A}_\pi$ be, respectively, the set of vertical edges, upward arrows, and downward arrows in (G, π).

Given $n \in [r]$, a connected component of the graph $G[\pi_n]$, induced on vertices of level n, is called a *component of level* n or a *level* n *component* of (G, π). Denote by $\mathcal{LC}_n$ the collection of these level components.

A *summit* of (G, π) is a level component S containing no vertex which is the tail of an upward arrow. If S is a singleton (one vertex, no edges), then the summit is called *regular*; otherwise, it is called *singular*. We denote by $\mathrm{s}_{\mathrm{reg}}$, resp. $\mathrm{s}_{\mathrm{sing}}$, the number of regular, resp. singular, summits in G, and set $\mathrm{s} := \mathrm{s}_{\mathrm{reg}} + \mathrm{s}_{\mathrm{sing}}$, the total number of summits. For example, in Figure 3, the level components consisting of single vertices u_5, u_b and u_c are the regular summits, whereas, the level components $G[\{u_9, u_a\}]$ and $G[\{u_7, u_8\}]$ are the singular summits. We thus have $\mathrm{s}_{\mathrm{reg}} = 3$ and $\mathrm{s}_{\mathrm{sing}} = 2$; in total, there are $\mathrm{s} = 5$ summits.

Remark 3.1. When G is the dual graph of a stable Riemann surface X, each summit defines a subcurve of X. The summit is regular (respectively singular) if and only if the associated subcurve is regular (respectively singular). ◇

We consider in the sequel the $\mathbf{k}$-vector space $\boldsymbol{\Upsilon} = \mathbf{k}^{\mathbb{E}}$ of $\mathbf{k}$-valued functions on $\mathbb{E}$. Denote by $\langle \cdot\,, \cdot \rangle\colon \boldsymbol{\Upsilon} \times \boldsymbol{\Upsilon} \to \mathbf{k}$ the natural bilinear form. For $v \in V$ and $S \subseteq V$, define

$$\boldsymbol{\Upsilon}_v := \mathbf{k}^{\mathbb{E}_v} \qquad \text{and} \qquad \boldsymbol{\Upsilon}_S := \bigoplus_{v \in S} \boldsymbol{\Upsilon}_v.$$

For $a \in \mathbb{E}$, we denote by $\mathbf{1}_a$ the function on $\mathbb{E}$ that takes value 1 on a and 0 elsewhere.

3.1. Vanishing residue along downward arrows.

Let $\boldsymbol{\Upsilon}_\pi \subseteq \boldsymbol{\Upsilon}$ be the vector subspace given by the *vanishing conditions along downward arrows*

$$\langle \mathbf{1}_a, \cdot \rangle = 0 \qquad \text{for all } \ a \in \bar{A}_\pi. \tag{3.1}$$

Equivalently, $\boldsymbol{\Upsilon}_\pi$ is the set of $\psi \in \boldsymbol{\Upsilon}$ such that $\psi_a = 0$ for each downward arrow $a \in \bar{A}_\pi$. We have a decomposition $\boldsymbol{\Upsilon}_\pi = \bigoplus \boldsymbol{\Upsilon}_{\pi,v}$, where $\boldsymbol{\Upsilon}_{\pi,v} := \boldsymbol{\Upsilon}_\pi \cap \boldsymbol{\Upsilon}_v$. The following is straightforward.

Proposition 3.2. *We have* $\dim \boldsymbol{\Upsilon}_\pi = 2|E| - |E_\pi|$.

For each $S \subseteq V$, let $\boldsymbol{\Upsilon}_{\pi,S} := \boldsymbol{\Upsilon}_\pi \cap \boldsymbol{\Upsilon}_S$. Notice that $\boldsymbol{\Upsilon}_{\pi,S}$ has a natural basis consisting of the characteristic vectors $\mathbf{1}_a$ for all horizontal and upward arrows $a \in \mathbb{E}_S$. We will use this basis when applying the results in Section 2 to $\boldsymbol{\Upsilon}_{\pi,S}$.

3.2. **Local residue conditions.** For each $v \in V$, let $\mathbf{1}_v \in \boldsymbol{\Upsilon}_\pi$ be the characteristic vector of $\mathbb{E}_v \smallsetminus \bar{A}_\pi$, that is, $\mathbf{1}_v := \sum_{a \in \mathbb{E}_v \smallsetminus \bar{A}_\pi} \mathbf{1}_a$. By definition, $\boldsymbol{\Upsilon}_\pi^0 \subseteq \boldsymbol{\Upsilon}_\pi$ is the vector subspace defined by the *local residue conditions*

$$\langle \mathbf{1}_v, \cdot \rangle = 0 \qquad \text{for all } \ v \in V. \tag{3.2}$$

Let $\boldsymbol{\Upsilon}_{\pi,v}^0 \subseteq \boldsymbol{\Upsilon}_{\pi,v}$ be the subspace defined by

$$\boldsymbol{\Upsilon}_{\pi,v}^0 := \Big\{\psi \in \boldsymbol{\Upsilon}_{\pi,v} \;\Big|\; \langle \mathbf{1}_v, \psi \rangle = 0\Big\} = \Big\{\psi \in \boldsymbol{\Upsilon}_{\pi,v} \;\Big|\; \sum_{a \in \mathbb{E}_v \smallsetminus \bar{A}_\pi} \psi_a = \sum_{a \in \mathbb{E}_v} \psi_a = 0\Big\}.$$

We define $\boldsymbol{\Upsilon}_\pi^0 := \bigoplus_{v \in V} \boldsymbol{\Upsilon}_{\pi,v}^0$. Note that for each $v \in V$, $\boldsymbol{\Upsilon}_{\pi,v}^0 = \boldsymbol{\Upsilon}_\pi^0 \cap \boldsymbol{\Upsilon}_v$. For each $S \subseteq V$, we set $\boldsymbol{\Upsilon}_{\pi,S}^0 := \boldsymbol{\Upsilon}_\pi^0 \cap \boldsymbol{\Upsilon}_S$.

Denote by $\mathcal{T}_{\text{loc}} \subseteq \boldsymbol{\Upsilon}_\pi$ the collection of all nonzero vectors $\mathbf{1}_v$, $v \in V$,

$$\mathcal{T}_{\text{loc}} := \big\{\mathbf{1}_v \;\big|\; v \in V,\, \mathbf{1}_v \neq 0\big\}\,.$$

Note that $\mathbf{1}_v = 0$ if and only if $\mathbb{E}_v$ contains only downward arrows, that is, the singleton v is a regular summit of the level graph G. Also, clearly, $\mathcal{T}_{\text{loc}}$ is set-theoretically independent. Therefore, we get the following result, which by Proposition 3.2, gives the dimension of $\boldsymbol{\Upsilon}_\pi^0$.

Proposition 3.3. *We have* $\operatorname{codim}(\boldsymbol{\Upsilon}_\pi^0, \boldsymbol{\Upsilon}_\pi) = |V| - \mathrm{s}_{\mathrm{reg}}$.

3.3. **Rosenlicht residue conditions.** For each horizontal edge $e \in E_\pi^c = E \smallsetminus E_\pi$, let $\mathbf{1}_e := \mathbf{1}_a + \mathbf{1}_{\bar{a}}$ be the "characteristic" vector of e, where a and $\bar{a}$ are the two arrows on e.

Define the vector subspace $\boldsymbol{\mathcal{R}}_\pi \subseteq \boldsymbol{\Upsilon}_\pi^0$ by *Rosenlicht residue conditions*

$$\langle \mathbf{1}_e, \cdot \rangle = 0 \qquad \text{for all } \ e \in E_\pi^c. \tag{3.3}$$

Let $\mathcal{T}_{\text{Ros}} \subseteq \boldsymbol{\Upsilon}_\pi$ be the collection of characteristic vectors $\mathbf{1}_e$ for e horizontal, that is,

$$\mathcal{T}_{\text{Ros}} := \big\{\mathbf{1}_e \;\big|\; e \in E_\pi^c\big\}.$$

This collection is set-theoretically independent in $\boldsymbol{\Upsilon}_\pi$. However, Equations 3.3 are defined in the subspace $\boldsymbol{\Upsilon}_\pi^0$, and we need to do some extra work to get the dimension of $\boldsymbol{\mathcal{R}}_\pi$.

For each singular summit with set of vertices $S \subseteq V$, each arrow in $\mathbb{E}_S$ is either horizontal or downward. Therefore, we have the equation

$$\sum_{v \in S} \mathbf{1}_v = \sum_{e \in E_\pi^c(S)} \mathbf{1}_e + \sum_{a \in \bar{A}_{\pi,S}} \mathbf{1}_a,$$

from which we deduce the following equation

$$\sum_{e \in E_\pi^c(S)} \langle \mathbf{1}_e, \cdot \rangle = 0 \quad \text{on } \boldsymbol{\Upsilon}_\pi^0.$$

We will show below that these are the only relations the Rosenlicht conditions satisfy; see Lemma 3.5. From this, we will deduce the following.

Proposition 3.4. *We have* $\operatorname{codim}(\boldsymbol{\mathcal{R}}_\pi, \boldsymbol{\Upsilon}_\pi^0) = |E_\pi^c| - \mathrm{s}_{\mathrm{sing}}$.

In preparation for the proof, for each $n \in [r]$, consider the set $\pi_n \subseteq V$ of vertices of level n, and the set $\mathcal{LC}_n$ of level n components. Since each element $\mathrm{C} \in \mathcal{LC}_n$ is entirely determined by its set of vertices, abusing notation, we simply use C for the set $V(\mathrm{C})$. We thus have

$$\pi_n = \bigsqcup_{\mathrm{C} \in \mathcal{LC}_n} \mathrm{C}.$$

We can decompose

$$\boldsymbol{\Upsilon}^0_\pi = \bigoplus_{n\in[r]} \boldsymbol{\Upsilon}^0_{\pi,\pi_n} \qquad \text{and} \qquad \boldsymbol{\Upsilon}^0_{\pi,\pi_n} = \bigoplus_{\mathrm{C}\in\mathcal{LC}_n} \boldsymbol{\Upsilon}^0_{\pi,\mathrm{C}}.$$

Note that each vertex $v \in V$ belongs to a unique level component $\mathrm{C} \in \mathcal{LC}_n$ for unique $n \in [r]$, and we have $\mathbf{1}_v \in \boldsymbol{\Upsilon}_{\pi,\mathrm{C}}$. Also, each horizontal edge $e \in E^c_\pi$ connects a pair of vertices in a unique level component $\mathrm{C} \in \mathcal{LC}_n$, $n \in [r]$, in particular $\mathbf{1}_e \in \boldsymbol{\Upsilon}_{\pi,\mathrm{C}}$. We thus get a decomposition

$$\boldsymbol{\mathcal{R}}_\pi = \bigoplus_{n\in[r],\, \mathrm{C}\in\mathcal{LC}_n} \boldsymbol{\mathcal{R}}_\pi \cap \boldsymbol{\Upsilon}_{\pi,\mathrm{C}}.$$

This implies that

$$\operatorname{codim}(\boldsymbol{\mathcal{R}}_\pi, \boldsymbol{\Upsilon}^0_\pi) = \sum_{n\in[r],\, \mathrm{C}\in\mathcal{LC}_n} \operatorname{codim}\Big(\boldsymbol{\mathcal{R}}_\pi \cap \boldsymbol{\Upsilon}_{\pi,\mathrm{C}}, \boldsymbol{\Upsilon}^0_{\pi,\mathrm{C}}\Big). \tag{3.4}$$

For each $n \in [r]$ and $\mathrm{C} \in \mathcal{LC}_n$, define the two collections of vectors $\mathcal{T}^{n,\mathrm{C}}_{\text{loc}}, \mathcal{T}^{n,\mathrm{C}}_{\text{Ros}} \subset \boldsymbol{\Upsilon}_{\pi,\mathrm{C}}$ by

$$\mathcal{T}^{n,\mathrm{C}}_{\text{loc}} := \Big\{\mathbf{1}_v \;\Big|\; v \in \mathrm{C},\, \mathbf{1}_v \neq 0\Big\} \qquad \text{and} \qquad \mathcal{T}^{n,\mathrm{C}}_{\text{Ros}} := \Big\{\mathbf{1}_e \;\Big|\; e \in E^c_\pi(\mathrm{C})\Big\}.$$

Lemma 3.5. *Notation as above, for each $n \in [r]$ and $\mathrm{C} \in \mathcal{LC}_n$, the two collections $\mathcal{T}^{n,\mathrm{C}}_{\text{loc}}$ and $\mathcal{T}^{n,\mathrm{C}}_{\text{Ros}}$ of vectors in $\boldsymbol{\Upsilon}_{\pi,\mathrm{C}}$ are properly unrelated. Moreover, they are related if and only if the level component C is a singular summit of the level graph G.*

Assuming this, we prove the proposition.

Proof of Proposition 3.4. We combine the above lemma with Propositions 2.1 and 2.2 applied to the two collections of vectors $\mathcal{T}^{n,\mathrm{C}}_{\text{loc}}$ and $\mathcal{T}^{n,\mathrm{C}}_{\text{Ros}}$ in $\boldsymbol{\Upsilon}_{\pi,\mathrm{C}}$, for $n \in [r]$ and $\mathrm{C} \in \mathcal{LC}_n$, to infer that

$$\operatorname{codim}\big(\boldsymbol{\mathcal{R}}_\pi \cap \boldsymbol{\Upsilon}_{\pi,\mathrm{C}}, \boldsymbol{\Upsilon}^0_{\pi,\mathrm{C}}\big) = \big|\mathcal{T}^{n,\mathrm{C}}_{\text{Ros}}\big| - \epsilon(\mathrm{C}),$$

where $\epsilon(\mathrm{C}) = 1$ if C is a singular summit and $\epsilon(\mathrm{C}) = 0$ otherwise. Combining Equation (3.4) with the observation that

$$\sum_{n\in[r],\, \mathrm{C}\in\mathcal{LC}_n} \big|\mathcal{T}^{n,\mathrm{C}}_{\text{Ros}}\big| = |E^c_\pi|$$

concludes the proof. □

Proof of Lemma 3.5. Let $\Phi \subseteq \mathcal{T}^{n,\mathrm{C}}_{\text{loc}}$ and $\Psi \subseteq \mathcal{T}^{n,\mathrm{C}}_{\text{Ros}}$ be two subcollections of vectors with $[\Phi] = [\Psi]$, that is,

$$\bigcup_{\varphi\in\Phi} [\varphi] = \bigcup_{\psi\in\Psi} [\psi], \tag{3.5}$$

so that they yield partitions of the same subset of $\mathbb{E} \smallsetminus \bar{A}_\pi$.

We claim that $\Phi = \mathcal{T}^{n,\mathrm{C}}_{\text{loc}}$. For the sake of contradiction, suppose this is not the case and consider the proper subset $Z \subset \mathrm{C}$ consisting of those vertices $u \in \mathrm{C}$ with $\mathbf{1}_u \in \Phi$. Since C is connected, there is an edge $e = \{u, v\}$ connecting a pair of vertices u, v of C such that $u \in Z$ and $v \in \mathrm{C} \smallsetminus Z$. This implies that $\mathbf{1}_u \in \Phi$ but $\mathbf{1}_v \notin \Phi$. Note that e is horizontal, i.e., $e \in E^c_\pi$. Denote by uv and vu the two arrows on e. Since $uv \in [\mathbf{1}_u]$, the equality in (3.5) implies that $uv \in [\psi]$ for some $\psi \in \Psi$. Necessarily, we have $\psi = \mathbf{1}_e = \mathbf{1}_{uv} + \mathbf{1}_{vu}$. But then, since $[\mathbf{1}_e] = \{uv, vu\}$, we also have $vu \in \bigcup_{\psi\in\Psi}[\psi]$, and hence $vu \in [\varphi]$ for some $\varphi \in \Phi$. This is possible only if $\varphi = \mathbf{1}_v$, a contradiction, proving the claim.

The union of the sets $[\varphi]$ for $\varphi \in \mathcal{T}_{\text{loc}}^{n,\mathrm{C}}$ is $\mathbb{E}_{\mathrm{C}} \smallsetminus \bar{A}_\pi$, which clearly contains $\mathbb{E}(\mathrm{C})$, the set of (horizontal) arrows connecting vertices of C. This set can be in the union of the $[\psi]$ for $\psi \in \Psi \subseteq \mathcal{T}_{\text{Ros}}^{n,\mathrm{C}}$ only if $\Psi = \mathcal{T}_{\text{Ros}}^{n,\mathrm{C}}$. We conclude that $\mathcal{T}_{\text{loc}}^{n,\mathrm{C}}$ and $\mathcal{T}_{\text{Ros}}^{n,\mathrm{C}}$ are properly unrelated.

To prove the second statement, note that the two collections of vectors $\mathcal{T}_{\text{loc}}^{n,\mathrm{C}}$ and $\mathcal{T}_{\text{Ros}}^{n,\mathrm{C}}$ are related if and only if they are both nonempty and we have

$$\bigcup_{\varphi \in \mathcal{T}_{\text{loc}}^{n,\mathrm{C}}} [\varphi] = \bigcup_{\psi \in \mathcal{T}_{\text{Ros}}^{n,\mathrm{C}}} [\psi].$$

The above equality implies that all the arrows appearing in $[\varphi]$, $\varphi \in \mathcal{T}_{\text{loc}}^{n,\mathrm{C}}$, are horizontal, that is, C is a summit. Moreover, if $\mathcal{T}_{\text{Ros}}^{n,\mathrm{C}}$ is nonempty, the summit must be singular. This proves one direction of the statement. To finish, note that if C is a singular summit, then $\mathcal{T}_{\text{loc}}^{n,\mathrm{C}}$ and $\mathcal{T}_{\text{Ros}}^{n,\mathrm{C}}$ are nonempty, and $[\mathcal{T}_{\text{loc}}^{n,\mathrm{C}}] = \mathbb{E}(\mathrm{C}) = [\mathcal{T}_{\text{Ros}}^{n,\mathrm{C}}]$. ☐

4. Global residue conditions and proof of Theorem 1.1

We keep the notation as in the previous section: $\pi = (\pi_1, \dots, \pi_r)$ is a level structure on G, and $h\colon V \to [r]$ is the corresponding level function.

For each $n \in [r]$, let

$$V_{h<n} := \bigcup_{i<n} \pi_i \qquad \text{and} \qquad V_{h\leqslant n} := \bigcup_{i \leqslant n} \pi_i = V_{h<n} \sqcup \pi_n.$$

Let $\mathcal{CC}_{h<n}$ and $\mathcal{CC}_{h\leqslant n}$ be the set of connected components of $G[V_{h<n}]$ and $G[V_{h\leqslant n}]$, respectively.

A connected component Ξ of $G[V_{h<n}]$ is called *special* if there is an arrow in $\mathbb{E}$ with tail in π_n and head in Ξ. Such an arrow is necessarily upward. Let $\mathcal{CC}^*_{h<n} \subseteq \mathcal{CC}_{h<n}$ be the set of special connected components of $G[V_{h<n}]$. In Figure 3, the connected component of the two upper levels formed by the vertices u_9 and u_a is special. For each $\Xi \in \mathcal{CC}^*_{h<n}$, let $A^{\Xi}_{\pi,n} \subseteq A_\pi$ be the set of upward arrows with tail in π_n and head in Ξ, namely,

$$A^{\Xi}_{\pi,n} := \left\{ a \in A_\pi \;\middle|\; \mathfrak{t}_a \in \pi_n \,, \mathfrak{h}_a \in \Xi \right\}.$$

Consider the characteristic vector $\mathbf{1}^{\Xi}_n \in \boldsymbol{\Upsilon}_\pi$ defined by

$$\mathbf{1}^{\Xi}_n := \mathbf{1}_{A^{\Xi}_{\pi,n}} = \sum_{a \in A^{\Xi}_{\pi,n}} \mathbf{1}_a.$$

Consider the vector subspace $\boldsymbol{\mathcal{G}}_\pi \subseteq \boldsymbol{\mathcal{R}}_\pi$ defined by the *global residue conditions*

$$\langle \mathbf{1}^{\Xi}_n, \cdot \rangle = 0 \qquad \text{for all } \; n \in [r] \text{ and } \Xi \in \mathcal{CC}^*_{h<n}. \tag{4.1}$$

Denote by $\mathcal{T}_{\text{glob}} \subseteq \boldsymbol{\Upsilon}_\pi$ the collection of all the vectors $\mathbf{1}^{\Xi}_n$, that is,

$$\mathcal{T}_{\text{glob}} := \left\{ \mathbf{1}^{\Xi}_n \;\middle|\; n \in [r],\, \Xi \in \mathcal{CC}^*_{h<n} \right\}.$$

This collection is obviously set-theoretically independent in $\boldsymbol{\Upsilon}_\pi$. We show the following result.

Proposition 4.1. *We have* $\operatorname{codim}(\boldsymbol{\mathcal{G}}_\pi, \boldsymbol{\mathcal{R}}_\pi) = \mathrm{s} - \mathfrak{c}$.

For each connected component $\mathrm{C} \in \mathcal{CC}_{h\leqslant n}$, denote by $\mathrm{C}_n = \pi_n \cap \mathrm{C}$ the set of vertices of level n in C. Note that for each $n \in [r]$, each special component $\Xi \in \mathcal{CC}^*_{h<n}$ is contained in a unique component $\mathrm{C} \in \mathcal{CC}_{h\leqslant n}$.

We use a reasoning similar to the one that led to Equation (3.4) to get

$$\operatorname{codim}(\boldsymbol{\mathcal{G}}_\pi, \boldsymbol{\Upsilon}^0_\pi) = \sum_{n\in[r]} \sum_{\mathrm{C}\in\mathcal{CC}_{h\leqslant n}} \operatorname{codim}\Big(\boldsymbol{\mathcal{G}}_\pi \cap \boldsymbol{\Upsilon}_{\pi,\mathrm{C}_n}, \boldsymbol{\Upsilon}^0_{\pi,\mathrm{C}_n}\Big). \tag{4.2}$$

For each $n \in [r]$ and $\mathrm{C} \in \mathcal{CC}_{h\leqslant n}$, define the three collections of characteristic vectors in $\boldsymbol{\Upsilon}_{\pi,\mathrm{C}_n}$:

$$\mathcal{T}^{n,\mathrm{C}}_{\text{loc}} := \big\{\, \mathbf{1}_v \;\big|\; v \in \mathrm{C}_n \text{ and } \mathbf{1}_v \neq 0 \,\big\}, \qquad \mathcal{T}^{n,\mathrm{C}}_{\text{Ros}} := \big\{\, \mathbf{1}_e \;\big|\; e \in E^c_\pi(\mathrm{C}_n) \,\big\}, \quad \text{and}$$
$$\mathcal{T}^{n,\mathrm{C}}_{\text{glob}} := \big\{\, \mathbf{1}^{\Xi}_n \;\big|\; \Xi \in \mathcal{CC}^*_{h<n} \text{ and } \Xi \subseteq \mathrm{C} \big\}.$$

Clearly, each of $\mathcal{T}^{n,\mathrm{C}}_{\text{glob}} \cup \mathcal{T}^{n,\mathrm{C}}_{\text{Ros}}$ and $\mathcal{T}^{n,\mathrm{C}}_{\text{loc}}$ are set-theoretically independent collections of vectors in $\boldsymbol{\Upsilon}_{\pi,\mathrm{C}_n}$. However, the two collections are related since

$$\bigcup_{\varphi\in\mathcal{T}^{n,\mathrm{C}}_{\text{glob}}\cup\mathcal{T}^{n,\mathrm{C}}_{\text{Ros}}} [\varphi] = \mathbb{E}_{\mathrm{C}_n} \smallsetminus \bar{A}_\pi = \bigcup_{\psi\in\mathcal{T}^{n,\mathrm{C}}_{\text{loc}}} [\psi].$$

Lemma 4.2. *Notation as above, for each $n \in [r]$ and $\mathrm{C} \in \mathcal{CC}_{h\leqslant n}$, the two collections of vectors $\mathcal{T}^{n,\mathrm{C}}_{\text{glob}} \cup \mathcal{T}^{n,\mathrm{C}}_{\text{Ros}}$ and $\mathcal{T}^{n,\mathrm{C}}_{\text{loc}}$ in $\boldsymbol{\Upsilon}_{\pi,\mathrm{C}_n}$ are related but properly unrelated.*

Proof. We need only show that the two collections are properly unrelated. Let $\Phi \subseteq \mathcal{T}^{n,\mathrm{C}}_{\text{glob}} \cup \mathcal{T}^{n,\mathrm{C}}_{\text{Ros}}$ and $\Psi \subseteq \mathcal{T}^{n,\mathrm{C}}_{\text{loc}}$ be two nonempty subsets such that $[\Phi] = [\Psi]$, that is

$$\bigcup_{\varphi\in\Phi} [\varphi] = \bigcup_{\psi\in\Psi} [\psi]. \tag{4.3}$$

If $\Psi = \mathcal{T}^{n,\mathrm{C}}_{\text{loc}}$, then the right-hand side of (4.3) is $\mathbb{E}_{\mathrm{C}_n} \smallsetminus \bar{A}_\pi$, which yields $\Phi = \mathcal{T}^{n,\mathrm{C}}_{\text{glob}} \cup \mathcal{T}^{n,\mathrm{C}}_{\text{Ros}}$.

Assume for the sake of contradiction that $\Psi \neq \mathcal{T}^{n,\mathrm{C}}_{\text{loc}}$. Let Z be the proper subset of C_n consisting of vertices v in C_n such that $\mathbf{1}_v \in \Psi$.

If there were a horizontal edge e connecting a vertex $v \in Z$ to a vertex $w \in \mathrm{C}_n \smallsetminus Z$, then the corresponding horizontal arrow $a = vw$ would be in $[\mathbf{1}_v]$, whence, in the right-hand side of (4.3). This would be possible only if $\mathbf{1}_e \in \Phi$, and so, wv would belong to the left- and therefore, right-hand side of (4.3), yielding $w \in Z$, a contradiction.

Thus, we can assume there is no horizontal edge connecting Z to $\mathrm{C}_n \smallsetminus Z$. Since C is connected, there has to be an upward arrow $a = vu$ connecting a vertex v in Z to a vertex u in a component $\Xi \in \mathcal{CC}^*_{h<n}$. Moreover, since C is connected, Ξ is contained in C, and there is a vertical arrow $b = wz$ connecting a vertex $w \in \mathrm{C}_n \smallsetminus Z$ to a vertex z in Ξ. Since $v \in Z$, the arrow a appears in the right-hand side of (4.3), and thus, in the left-hand side as well. This means that $\mathbf{1}^{\Xi}_n \in \Phi$. We infer that b appears in the left-hand side of (4.3), and so in the right-hand side as well. This yields $w \in Z$, a contradiction. ☐

Lemma 4.3. *For each $\mathrm{C} \in \mathcal{CC}_{h\leqslant n}$, the following holds:*

- $|\mathcal{T}^{n,\mathrm{C}}_{\text{loc}}| = |\mathrm{C}_n|$ *unless* C *is a singleton of level n, in which case $|\mathrm{C}_n| = 1$ but $|\mathcal{T}^{n,\mathrm{C}}_{\text{loc}}| = 0$.*
- $|\mathcal{T}^{n,\mathrm{C}}_{\text{Ros}}|$ *is the number of horizontal edges in C_n.*
- $|\mathcal{T}^{n,\mathrm{C}}_{\text{glob}}|$ *is the number of connected components in $\mathcal{CC}^*_{h<n}$ that are included in* C.

Proof. The proof can be obtained by direct verification. We omit the details. ☐

We can now finish the proof of the proposition.

Proof of Proposition 4.1. We will combine the previous lemmas with Proposition 2.2. Notice first that, for each $n \in [r]$,

$$\mathcal{CC}_{h<n} \smallsetminus \mathcal{CC}^*_{h<n} = \mathcal{CC}_{h\leqslant n} \cap \mathcal{CC}_{h<n}. \tag{4.4}$$

Then, for each $\mathrm{C} \in \mathcal{CC}_{h\leqslant n}$, we have that $\mathrm{C}_n = \varnothing$ if and only if $\mathrm{C} \in \mathcal{CC}_{h<n} \smallsetminus \mathcal{CC}^*_{h<n}$, in which case, $\mathcal{T}^{n,\mathrm{C}}_{\text{glob}}$, $\mathcal{T}^{n,\mathrm{C}}_{\text{Ros}}$ and $\mathcal{T}^{n,\mathrm{C}}_{\text{loc}}$ are all empty. In fact, these collections are all empty if and only if either, $\mathrm{C} \in \mathcal{CC}_{h<n} \smallsetminus \mathcal{CC}^*_{h<n}$ or, C is a singleton whose only vertex has level n, in which case C is a regular summit of the level graph G. We apply Proposition 2.2 and deduce from (4.2) and Lemma 4.2 that

$$\operatorname{codim}\big(\boldsymbol{\mathcal{G}}_\pi, \boldsymbol{\Upsilon}_\pi\big) = \sum_{n\in[r]} \Big(\mathrm{s}_{\text{reg}}(n) + \sum_{\mathrm{C}\in\mathcal{CC}^\dagger_{h\leqslant n}} \big(|\mathcal{T}^{n,\mathrm{C}}_{\text{glob}}| + |\mathcal{T}^{n,\mathrm{C}}_{\text{Ros}}| + |\mathcal{T}^{n,\mathrm{C}}_{\text{loc}}| - 1\big)\Big) \tag{4.5}$$

where, for each $n \in [r]$, the quantity $\mathrm{s}_{\text{reg}}(n)$ is the number of regular summits of the level graph G whose unique vertex has level n, and

$$\mathcal{CC}^\dagger_{h\leqslant n} = \mathcal{CC}_{h\leqslant n} \smallsetminus \big(\mathcal{CC}_{h\leqslant n} \cap \mathcal{CC}_{h<n}\big). \tag{4.6}$$

Now, applying Lemma 4.3 to each $\mathrm{C} \in \mathcal{CC}^\dagger_{h\leqslant n}$, we obtain the cardinalities in (4.5), leading, for each $n \in [r]$, to

$$\sum_{\mathrm{C}\in\mathcal{CC}^\dagger_{h\leqslant n}} \Big(|\mathcal{T}^{n,\mathrm{C}}_{\text{glob}}| + |\mathcal{T}^{n,\mathrm{C}}_{\text{Ros}}| + |\mathcal{T}^{n,\mathrm{C}}_{\text{loc}}| - 1\Big) = |\mathcal{CC}^*_{h<n}| + |E(\pi_n)| + |\pi_n| - \mathrm{s}_{\text{reg}}(n) - |\mathcal{CC}^\dagger_{h\leqslant n}|. \tag{4.7}$$

For each $n \in [r]$, by (4.4) and (4.6), we have $|\mathcal{CC}^*_{h<n}| - |\mathcal{CC}^\dagger_{h\leqslant n}| = |\mathcal{CC}_{h<n}| - |\mathcal{CC}_{h\leqslant n}|$. Therefore, adding the equations (4.7) for all n, and using (4.5), we deduce that

$$\operatorname{codim}\big(\boldsymbol{\mathcal{G}}_\pi, \boldsymbol{\Upsilon}_\pi\big) = -\mathfrak{c} + |E^c_\pi| + |V|.$$

Finally, using Proposition 3.3 and Proposition 3.4, we get

$$\operatorname{codim}\big(\boldsymbol{\mathcal{G}}_\pi, \boldsymbol{\mathcal{R}}_\pi\big) = -\mathfrak{c} + |E^c_\pi| + |V| - (|V| - \mathrm{s}_{\text{reg}}) - (|E^c_\pi| - \mathrm{s}_{\text{sing}}) = \mathrm{s} - \mathfrak{c}.$$

□

Remark 4.4 (The case of level graphs with a unique summit)**.** Notice that $\operatorname{codim}\big(\boldsymbol{\mathcal{G}}_\pi, \boldsymbol{\mathcal{R}}_\pi\big) = 0$ if the level graph (G, π) has a unique summit (and is therefore connected). This is always the case if the underlying graph G is (multi)complete, that is, each two vertices are connected by at least one edge. In this case, the global residue conditions are a consequence of the downward vanishing, local and Rosenlicht residue conditions. In the applications to the study of limits of spaces of Abelian differentials, we note that this is the setting of the works [EM02] and [ES07], in which global residue conditions do not play any role. ⋄

4.1. **Proof of Theorem 1.1.** We are now in position to prove the dimension count. Propositions 3.2 and 3.3 yield

$$\dim_{\mathbf{k}} \boldsymbol{\Upsilon}^0_\pi = 2|E^c_\pi| + |E_\pi| - |V| + \mathrm{s}_{\text{reg}}.$$

Also, Proposition 3.4 yields $\operatorname{codim}(\boldsymbol{\mathcal{R}}_\pi, \boldsymbol{\Upsilon}^0_\pi) = |E^c_\pi| - \mathrm{s}_{\text{sing}}$, whereas Proposition 4.1 yields $\operatorname{codim}(\boldsymbol{\mathcal{G}}_\pi, \boldsymbol{\mathcal{R}}_\pi) = \mathrm{s} - \mathfrak{c}$. Finally, $|E| = |E_\pi| + |E^c_\pi|$. Combining these all, we get the desired formula $\dim_{\mathbf{k}} \boldsymbol{\mathcal{G}}_\pi = |E| - |V| + \mathfrak{c} = g(G)$, as required. □

4.2. **An example.** We discuss the example of the level graph $(G, \pi = (\pi_1, \pi_2, \pi_3))$ with three levels depicted in Figure 3. For simplicity, we let ij denote the arrow from u_i to u_j for distinct i, j. Let $h \colon V \to [3]$ be the level function. For each $n \in [3]$, let

$$\mathcal{T}^n_{\text{loc}} := \big\{\mathbf{1}_v \;\big|\; v \in \pi_n \text{ and } \mathbf{1}_v \neq 0\big\}, \quad \mathcal{T}^n_{\text{Ros}} := \big\{\mathbf{1}_e \,|\, e \in E(\pi_n)\big\}, \quad \mathcal{T}^n_{\text{glob}} := \big\{\mathbf{1}^\Xi_n \;\big|\; \Xi \in \mathcal{CC}^*_{h<n}\big\}$$

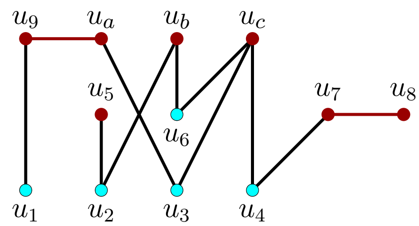

FIGURE 3. A level graph with three levels. Summits are drawn in red. There are three regular and two singular summits. The subgraph $G[\{u_9, u_a\}]$ is a special connected component in $\mathcal{CC}_{h<3}$, but it is nonspecial in $\mathcal{CC}_{h<2}$.

be subsets of $\boldsymbol{\Upsilon}_\pi$. Then, we obtain

$$\begin{aligned}
&\mathcal{T}^3_{\text{loc}} = \{\mathbf{1}_{19},\ \mathbf{1}_{25} + \mathbf{1}_{2b},\ \mathbf{1}_{3a} + \mathbf{1}_{3c},\ \mathbf{1}_{4c} + \mathbf{1}_{47}\}, \qquad \mathcal{T}^3_{\text{Ros}} = \varnothing,\\
&\mathcal{T}^3_{\text{glob}} = \{\mathbf{1}_{19} + \mathbf{1}_{3a},\ \mathbf{1}_{25},\ \mathbf{1}_{2b} + \mathbf{1}_{3c} + \mathbf{1}_{4c},\ \mathbf{1}_{47}\},\\
&\mathcal{T}^2_{\text{loc}} = \{\mathbf{1}_{6b} + \mathbf{1}_{6c},\ \mathbf{1}_{78},\ \mathbf{1}_{87}\}, \qquad \mathcal{T}^2_{\text{Ros}} = \{\mathbf{1}_{78} + \mathbf{1}_{87}\}, \qquad \mathcal{T}^2_{\text{glob}} = \{\mathbf{1}_{6b},\ \mathbf{1}_{6c}\},\\
&\mathcal{T}^1_{\text{loc}} = \{\mathbf{1}_{9a},\ \mathbf{1}_{a9}\}, \qquad \mathcal{T}^1_{\text{Ros}} = \{\mathbf{1}_{9a} + \mathbf{1}_{a9}\}, \qquad \mathcal{T}^1_{\text{glob}} = \varnothing.
\end{aligned}$$

Notice that $G[\pi_3]$ has four vertices (nonisolated in G) and no edges, and $G[V_{h<3}]$ has four connected components. Hence, $\mathcal{T}^3_{\text{loc}}$ and $\mathcal{T}^3_{\text{glob}}$ have four elements each, whereas $\mathcal{T}^3_{\text{Ros}} = \varnothing$. Also, $\mathcal{T}^3_{\text{loc}}$ and $\mathcal{T}^3_{\text{glob}}$ are related but properly unrelated, hence $\operatorname{codim}(\boldsymbol{\mathcal{G}}_\pi \cap \boldsymbol{\Upsilon}_{\pi,\pi_3}, \boldsymbol{\Upsilon}_{\pi,\pi_3}) = 7$; in other words, $\boldsymbol{\mathcal{G}}_\pi \cap \boldsymbol{\Upsilon}_{\pi,\pi_3} = 0$. Also, $\boldsymbol{\mathcal{R}}_\pi \cap \boldsymbol{\Upsilon}_{\pi,\pi_3} = \boldsymbol{\Upsilon}^0_{\pi,\pi_3}$, and $\operatorname{codim}(\boldsymbol{\mathcal{R}}_\pi \cap \boldsymbol{\Upsilon}_{\pi,\pi_3}, \boldsymbol{\Upsilon}_{\pi,\pi_3}) = 4$.

Further up, $G[\pi_2]$ has four vertices, one isolated in $G[V_{h\leqslant 2}]$, and one horizontal edge, whereas $G[V_{h<2}]$ has three connected components, only two connected to a vertex in π_2. Then, $\mathcal{T}^2_{\text{loc}}$, $\mathcal{T}^2_{\text{Ros}}$ and $\mathcal{T}^2_{\text{glob}}$ have respectively three, one and two elements. We have $\operatorname{codim}(\boldsymbol{\mathcal{R}}_\pi \cap \boldsymbol{\Upsilon}_{\pi,\pi_2}, \boldsymbol{\Upsilon}_{\pi,\pi_2}) = 3$, and $\operatorname{codim}(\boldsymbol{\mathcal{G}}_\pi \cap \boldsymbol{\Upsilon}_{\pi,\pi_2}, \boldsymbol{\Upsilon}_{\pi,\pi_2}) = 4$, the maximum possible.

Finally, $G[\pi_1]$ has four vertices and one horizontal edge, whereas $G[V_{h<1}]$ is empty. Then, $\mathcal{T}^1_{\text{loc}}$, $\mathcal{T}^1_{\text{Ros}}$ and $\mathcal{T}^1_{\text{glob}}$ have respectively two, one and zero elements. The codimension of $\boldsymbol{\mathcal{G}}_\pi \cap \boldsymbol{\Upsilon}_{\pi,\pi_1} = \boldsymbol{\mathcal{R}}_\pi \cap \boldsymbol{\Upsilon}_{\pi,\pi_1}$ in $\boldsymbol{\Upsilon}_{\pi,\pi_1}$ is 2, the maximum possible.

It follows that $\dim_{\mathbf{k}} \boldsymbol{\mathcal{G}}_\pi = 0$ and $\dim_{\mathbf{k}} \boldsymbol{\mathcal{R}}_\pi = 4$. Clearly, $\dim_{\mathbf{k}} \boldsymbol{\Upsilon}^0_\pi = 4$. Notice that G has 2 horizontal arrows and five summits, two of which are singular and three of them are regular, so $\mathrm{s}_{\text{sing}} = 2$, $\mathrm{s}_{\text{reg}} = 3$, and $\mathrm{s} = 5$.

5. Splitting, realization and degeneration

Let V be a finite nonempty set, and denote by 2^V the family of subsets of V. We say a function $\eta\colon 2^V \to \mathbb{R}$ is *nonnegative* if $\eta \geqslant 0$. We say η is *nonincreasing* if $\eta(J) \geqslant \eta(I)$ for $J \subseteq I \subseteq V$, and *nondecreasing* if the reverse inclusions hold.

A function $\eta\colon 2^V \to \mathbb{R}$ with $\eta(\varnothing) = 0$ is called *submodular* if we have the inequalities

$$\eta(I_1) + \eta(I_2) \geqslant \eta(I_1 \cup I_2) + \eta(I_1 \cap I_2) \quad \text{for each } I_1, I_2 \subseteq V,$$

and *supermodular* if all the inequalities are reversed. In any case, the quantity $\eta(V)$ is called the *range* of η. A function which is both submodular and supermodular is called *modular*. Modular functions are in bijection with elements of $\mathbb{R}^V$: each $q \in \mathbb{R}^V$ can be viewed as a modular function $q\colon 2^V \to \mathbb{R}$ by setting $q(I) := \sum_{v\in I} q(v)$ for each $I \subseteq V$, with the convention that $q(\varnothing) := 0$.

For $\eta\colon 2^V \to \mathbb{R}$ with $\eta(\varnothing) = 0$, we define the *adjoint to* η, denoted $\eta^*\colon 2^V \to \mathbb{R}$, by

$$\eta^*(I) := \eta(V) - \eta(I^c) \quad \text{for each } I \subseteq V,$$

where $I^c := V \smallsetminus I$. It is easy to see that η is submodular, resp. supermodular, if and only if η^* is supermodular, resp. submodular. Furthermore, η and η^* have the same range, and $(\eta^*)^* = \eta$. If η is supermodular, then $\eta \leqslant \eta^*$, and we refer to the ordered pair (η, η^*) as a *modular pair.* Note that for a modular function q, we have $q^* = q$, so (q, q) is a modular pair.

To each modular pair (η, η^*) we associate a polytope:

$$\mathbf{Q} := \left\{q \in \mathbb{R}^V \;\middle|\; \eta(I) \leqslant q(I) \leqslant \eta^*(I) \quad \forall I \subseteq V\right\}.$$

Clearly, $q \in \mathbf{Q}$ if and only if $q(V) = \eta(V) = \eta^*(V)$ and either $q(I) \geqslant \eta(I)$ for every I or $q(I) \leqslant \eta^*(I)$ for every I. (In other words, the lower bounds imply the upper bounds, and vice-versa, if $\eta(V) = \eta^*(V)$.) The polytope is called the base polytope of the polymatroid, or simply the (*base*) *polytope* associated to (η, η^*), or to η, or to η^*, and is denoted $\mathbf{Q}_\eta$ or $\mathbf{Q}_{\eta^*}$, depending on context.

5.1. **Modular pairs associated to subspaces.** Let $\mathbf{k}$ be a field. For each $v \in V$, let $\mathbf{U}_v$ be a vector space over $\mathbf{k}$. Let $\mathbf{U} := \bigoplus_{v\in V} \mathbf{U}_v$. For each subset $I \subseteq V$, let $\mathbf{U}_I := \bigoplus_{v\in I} \mathbf{U}_v$ and denote by $\iota_I\colon \mathbf{U}_I \to \mathbf{U}$ and $\theta_I\colon \mathbf{U} \to \mathbf{U}_I$ the corresponding natural inclusion and projection maps.

A finite-dimensional vector subspace $\mathrm{W} \subseteq \mathbf{U}$ gives rise to two functions $\nu_{\mathrm{W}}, \nu^*_{\mathrm{W}}\colon 2^V \to \mathbb{Z}$ defined by setting $\nu_{\mathrm{W}}(I) := \dim_{\mathbf{k}}(\mathrm{W}^I)$ and $\nu^*_{\mathrm{W}}(I) := \dim_{\mathbf{k}}(\mathrm{W}_I)$ for each $I \subseteq V$, with $\mathrm{W}^I := \iota_I^{-1}(\mathrm{W})$ and $\mathrm{W}_I := \theta_I(\mathrm{W})$. Clearly, we have a short exact sequence for each $I \subseteq V$:

$$0 \to \mathrm{W}^{I^c} \to \mathrm{W} \to \mathrm{W}_I \to 0. \tag{5.1}$$

Proposition 5.1. *The pair $(\nu_{\mathrm{W}}, \nu^*_{\mathrm{W}})$ is modular, and both ν_{W} and ν^*_{W} are nonnegative and nondecreasing.*

Proof. From (5.1), we get $\mathrm{W}_I \simeq \mathrm{W}/\mathrm{W}^{I^c}$, and so the adjoint to ν_W is ν^*_{W}. Also, ν_{W} is supermodular, that is, for $I, J \subset V$, we have

$$\dim_{\mathbf{k}} \mathrm{W}^I + \dim_{\mathbf{k}} \mathrm{W}^J \leqslant \dim_{\mathbf{k}} \mathrm{W}^{I\cup J} + \dim_{\mathbf{k}} \mathrm{W}^{I\cap J}.$$

This follows from the observation that W^I and W^J are subspaces of $\mathrm{W}^{I\cup J}$ with intersection $\mathrm{W}^{I\cap J}$. The rest is clear. ☐

5.2. **Filtration associated to an ordered partition.** Let $\pi = (\pi_1, \dots, \pi_r)$ be an ordered partition of V, with level function $h = h_\pi\colon V \to [r]$.

Let $\mathrm{F}_0 = \varnothing$, and for each $n \in [r]$, let $\mathrm{F}_n := \pi_1 \cup \cdots \cup \pi_n = V_{h\leqslant n}$ be the set of all $v \in V$ with $h(v) \leqslant n$. We obtain an ascending filtration $\mathrm{F}_\bullet$ indexed by $0, 1, \dots, r$, and the data of π is equivalent to that of $\mathrm{F}_\bullet$.

5.3. **Splitting.** Let $\pi = (\pi_1, \dots, \pi_r)$ be an ordered partition of V, and $\mathrm{F}_\bullet$ the corresponding ascending filtration associated to π.

To each supermodular function $\mu\colon 2^V \to \mathbb{R}$ we associate another supermodular function μ_π called the *splitting* of μ with respect to π, or the π-splitting of μ, as follows.

First, for each $n \in [r]$, we define the function

$$\mu_{\mathrm{F}_n/\mathrm{F}_{n-1}}\colon 2^V \to \mathbb{Z}$$

by setting

$$\mu_{\mathrm{F}_n/\mathrm{F}_{n-1}}(I) := \mu((I \cap \mathrm{F}_n) \cup \mathrm{F}_{n-1}) - \mu(\mathrm{F}_{n-1}) \quad \text{for each } I \subseteq V.$$

This is easily seen to be a supermodular function; see [AE24, §2.4]. Then, we define μ_π by

$$\mu_\pi := \sum_{n=1}^{r} \mu_{\mathrm{F}_n/\mathrm{F}_{n-1}}.$$

Taking adjoints in the equality above, we have $\mu^*_\pi = \sum_i \mu^*_{\mathrm{F}_n/\mathrm{F}_{n-1}}$. Furthermore,

$$\mu^*_{\mathrm{F}_n/\mathrm{F}_{n-1}}(I) = \mu(\mathrm{F}_n) - \mu((I^c \cap \mathrm{F}_n) \cup \mathrm{F}_{n-1}) \quad \text{for each } I \subseteq V.$$

We say as well that μ^*_π (resp. (μ_π, μ^*_π)) is the π-splitting of μ^* (resp. (μ, μ^*)).

5.4. **Realization of splittings.** We show now that if the modular pair (μ, μ^*) is associated to a subspace of $\mathbf{U}$, then each of its splittings is also associated to a subspace. More precisely, let $\pi = (\pi_1, \dots, \pi_r)$ be an ordered partition of V and denote by $\mathrm{F}_\bullet$ the corresponding filtration of V. Consider a subspace $\mathrm{W} \subseteq \mathbf{U}$. We associate to π a subspace $\mathrm{W}(\pi) \subseteq \mathbf{U}$, called the π-*splitting* of W, defined as follows.

Let

$$0 \subseteq \mathrm{W}^{\mathrm{F}_1} \subseteq \mathrm{W}^{\mathrm{F}_2} \subseteq \cdots \subseteq \mathrm{W}^{\mathrm{F}_n} = \mathrm{W}.$$

For each $n \in [r]$, define $\mathrm{W}_n := \theta_{\pi_n}(\mathrm{W}^{\mathrm{F}_n})$. Clearly, we have the following short exact sequence

$$0 \to \mathrm{W}^{\mathrm{F}_{n-1}} \to \mathrm{W}^{\mathrm{F}_n} \to \mathrm{W}_n \to 0. \tag{5.2}$$

Define

$$\mathrm{W}(\pi) := \bigoplus_{n=1}^{r} \mathrm{W}_n.$$

Since each W_n lives naturally in $\mathbf{U}_{\pi_n} = \bigoplus_{v \in \pi_n} \mathbf{U}_v$, by using $\mathbf{U} \simeq \bigoplus_{n \in [r]} \mathbf{U}_{\pi_n}$, we view

$$\mathrm{W}(\pi) \subseteq \mathbf{U} = \bigoplus_{v \in V} \mathbf{U}_v.$$

Proposition 5.2. *For each ordered partition π of V, the π-splitting of the modular pair $(\nu_{\mathrm{W}}, \nu^*_{\mathrm{W}})$ associated to a subspace $\mathrm{W} \subseteq \mathbf{U}$ is the modular pair associated to $\mathrm{W}(\pi) \subseteq \mathbf{U}$.*

Proof. Let $\nu := \nu_{\mathrm{W}}$ for a subspace $\mathrm{W} \subseteq \mathbf{U}$. For each $I \subseteq V$, we have the natural commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{W}^{\mathrm{F}_{n-1}} & \longrightarrow & \mathrm{W}^{\mathrm{F}_n} & \longrightarrow & (\mathrm{W}^{\mathrm{F}_n})_{\pi_n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{W}^{\mathrm{F}_n \cap (I \cup \pi_n^c)} & \longrightarrow & \mathrm{W}^{\mathrm{F}_n} & \longrightarrow & (\mathrm{W}^{\mathrm{F}_n})_{I^c \cap \pi_n} & \longrightarrow & 0,
\end{array}$$

with each of the two rows forming a short exact sequence. Using the above diagram, and the identity $\mathrm{W}_n = \big(\mathrm{W}^{\mathrm{F}_n}\big)_{\pi_n}$, we deduce the equality

$$\nu_{\mathrm{W}_n}(I) = \nu(\mathrm{F}_n) - \nu(\mathrm{F}_{n-1}) - \Big(\nu(\mathrm{F}_n) - \nu\big(\mathrm{F}_{n-1} \cup (\mathrm{F}_n \cap I)\big)\Big) = \nu_{\mathrm{F}_n/\mathrm{F}_{n-1}}(I) \qquad \text{for each } I \subseteq V.$$

Therefore, we have $\nu_{\mathrm{W}_n} = \nu_{\mathrm{F}_n/\mathrm{F}_{n-1}}$, and thus

$$\nu_{\mathrm{W}(\pi)} = \sum_{n=1}^{r} \nu_{\mathrm{W}_n} = \sum_{n=1}^{r} \nu_{\mathrm{F}_n/\mathrm{F}_{n-1}} = \nu_\pi,$$

as required. □

5.5. **Degeneration.** An alternate point of view on $\mathrm{W}(\pi)$ is through degenerations, as follows.

First, denote by $\mathbf{G_m}$ the multiplicative group scheme. The group $\mathbf{G_m}^V(\mathbf{k}) = \big(\mathbf{k}^\times\big)^V$ acts componentwise on $\mathbf{U}$. Also, letting $\mathbf{R} := \mathbf{k}[t, t^{-1}]$ be the ring of Laurent polynomials with coefficients in $\mathbf{k}$, the group $\mathbf{G_m}^V(\mathbf{R}) = \big(\mathbf{R}^\times\big)^V$ acts componentwise on $\mathbf{U_R} = \bigoplus_{v\in V}(\mathbf{U}_v \otimes_{\mathbf{k}} \mathbf{R})$.

For each integer-valued level function $d\colon V \to \mathbb{Z}$, $v \mapsto d_v$, with induced ordered partition given by π, let $x \in \mathbf{G_m}^V(\mathbf{R})$ be given by $x_v := t^{-d_v}$ for each $v \in V$. Consider the multiplication by x in $\mathbf{U_R}$, $\varphi \mapsto x\cdot\varphi$, given by

$$(x\cdot\varphi)_v := x_v\varphi_v = t^{-d_v}\varphi_v \qquad \text{for each } \varphi \in \mathbf{U_R} \text{ and } v \in V.$$

Given a subspace $\mathrm{W} \subseteq \mathbf{U}$, we obtain a submodule $x\cdot \mathrm{W_R} \subseteq \mathbf{U_R}$ for $\mathrm{W_R} := \mathrm{W}\otimes_{\mathbf{k}} \mathbf{R}$. Then, evaluation at $t = \lambda \in \mathbf{k} \smallsetminus \{0\}$ gives a subspace $x(\lambda)\cdot \mathrm{W} \subset \mathbf{U}$. We denote by $\lim_{t\to 0} x\cdot \mathrm{W}$ the limit of the subspaces $x(\lambda)\cdot\mathrm{W} \subset \mathbf{U}$ for $\lambda \neq 0$ in the Grassmannian $\mathrm{Grass}(m, \mathbf{U})$ of m-dimensional subspaces of $\mathbf{U}$ with $m = \dim_{\mathbf{k}} \mathrm{W}$.

Proposition 5.3. *We have*

$$\lim_{t\to 0} x\cdot \mathrm{W} = \mathrm{W}(\pi).$$

Proof. To simplify, for each $n \in [r]$, put $\mathrm{W}^n := \mathrm{W}^{\mathrm{F}_n}$. Recall that $\mathrm{W}_n = \big(\mathrm{W}^n\big)_{\pi_n} \subseteq \mathbf{U}_{\pi_n}$. Let $m_n := \dim \mathrm{W}_n$ and d_n be the value taken by d on π_n. We have $d_1 < d_2 < \cdots < d_r$.

For each $n \in [r]$, we choose a lifting $y_{n,1}, \dots, y_{n,m_n}$ to W^n of a basis $z_{n,1}, \dots, z_{n,m_n}$ of W_n. Let $z_{n,1}, \dots, z_{n,m_n}, \dots, z_{n,M_n}$ be an extension of the basis of W_n to one of $\mathbf{U}_{\pi_n}$. Order the basis $z_{n,j}$ of $\mathbf{U}$ lexographically, so that $z_{n,j} < z_{n',j'}$ if either $n < n'$, or $n = n'$ and $j < j'$. Extend this order to one for the basis obtained from the $z_{i,j}$ for the m-th exterior product $\bigwedge^m \mathbf{U}$, where $m = m_1 + \cdots + m_r = \dim_{\mathbf{k}} \mathrm{W}$.

The $y_{n,j}$ form a basis of W. Their exterior product $y_{1,1}\wedge y_{1,2}\wedge\cdots\wedge y_{r,m_r}$ can be written as the exterior product $z_{1,1}\wedge z_{1,2}\wedge\cdots\wedge z_{r,m_r}$ plus a linear combination of terms of higher order (with respect to the lexicographic order).

Likewise, the $x\cdot y_{n,j}$ form a basis of $x\cdot \mathrm{W_R}$ over the ring $\mathbf{R}$, and we have

$$(x\cdot y_{1,1})\wedge\cdots\wedge(x\cdot y_{r,m_r}) = t^{-\sum_n m_n d_n} z_{1,1}\wedge z_{1,2}\wedge\cdots\wedge z_{r,m_r} + t^l z'$$

for some z' in $\bigwedge^m \mathbf{U_R}$, where $l > -\sum_n m_n d_n$. Clearly, the limit of $(x\cdot y_{1,1})\wedge\cdots\wedge(x\cdot y_{r,m_r})$ as t approaches 0 is $z_{1,1}\wedge z_{1,2}\wedge\cdots\wedge z_{r,m_r}$. Since the $z_{n,j}$ for $j \leqslant m_n$ form a basis of the sum $\mathrm{W}(\pi) = \bigoplus_n \mathrm{W}_n$, the proof is complete. □

6. Proof of Theorem 1.2

Consider the trivial level structure π_0 on G given by the ordered partition of V into a unique set V. In this case, vanishing along downward arrows and global residue conditions are vacuum. The residue space $\boldsymbol{\mathcal{G}}_{\pi_0} \subset \boldsymbol{\Upsilon} = \bigoplus_{v\in V} \boldsymbol{\Upsilon}_v$ is the first homology of G with $\mathbf{k}$-coefficients, and the polytope associated to $\nu^*_{\boldsymbol{\mathcal{G}}_{\pi_0}}$ is the residue polytope of G, denoted by $\mathbf{P}_{\mathrm{res}} = \mathbf{P}_{\mathrm{res}}(G)$.

Since $\dim \boldsymbol{\mathcal{G}}_{\pi_0} = g(G)$ and both $\nu_{\boldsymbol{\mathcal{G}}_{\pi_0}}$ and $\nu^*_{\boldsymbol{\mathcal{G}}_{\pi_0}}$ are nonnegative, the residue polytope $\mathbf{P}_{\mathrm{res}}$ lives in the standard simplex $\Delta_g \subset \mathbb{R}^V_{\geqslant 0}$, consisting of points whose coordinates sum up to g.

For the ordered partition $\pi = (\pi_1, \dots, \pi_r)$, consider the subspace $\boldsymbol{\mathcal{G}}_\pi \subseteq \Upsilon$. Denote by $\gamma_\pi = \nu^*_{\boldsymbol{\mathcal{G}}_\pi}$ the corresponding submodular function and by $\mathbf{P}_\pi$ the corresponding polytope, called the π-residue polytope. Again, we have $\mathbf{P}_\pi \subseteq \Delta_g$.

Consider a coarsening $\pi' = (\pi'_1, \dots, \pi'_s)$ of π, that is, an ordered partition of V such that the natural surjection $h'\colon V \to [s]$ factors through $h\colon V \to [r]$, and the induced map $c\colon [r] \to [s]$

satisfies $c(n_1) \leqslant c(n_2)$ for each $n_1, n_2 \in [r]$ with $n_1 \leqslant n_2$. Notation as in the previous section, the group $\mathbf{G}_{\mathbf{m}}^V(\mathbf{R})$ acts componentwise on $\boldsymbol{\Upsilon}_{\mathbf{R}} = \bigoplus_v \big(\boldsymbol{\Upsilon}_v \otimes_{\mathbf{k}} \mathbf{R}\big)$, with $\boldsymbol{\Upsilon}_v = \mathbf{k}^{\mathbb{E}_v}$.

Consider a function $d\colon V \to \mathbb{Z}$, $v \mapsto d_v$, with induced ordered partition given by π, and let $x \in \mathbf{G}_{\mathbf{m}}^V(\mathbf{R})$ be given by $x_v := t^{-d_v}$ for each $v \in V$. Multiplication by x is given by

$$(x \cdot \varphi)_a := x_v \varphi_a = t^{-d_v} \varphi_a \qquad \text{for each } \varphi \in \boldsymbol{\Upsilon}, \text{ and each } v \in V \text{ and } a \in \mathbb{E}_v.$$

Applying Proposition 5.3 to the subspace $\boldsymbol{\mathcal{G}}_{\pi'} \subset \boldsymbol{\Upsilon}$, we get

$$\lim_{t \to 0} x \cdot \boldsymbol{\mathcal{G}}_{\pi'} = \boldsymbol{\mathcal{G}}_{\pi'}(\pi). \tag{6.1}$$

Theorem 6.1. *We have*

$$\lim_{t \to 0} x \cdot \boldsymbol{\mathcal{G}}_{\pi'} = \boldsymbol{\mathcal{G}}_{\pi}.$$

In particular, $\boldsymbol{\mathcal{G}}_{\pi}$ coincides with the splitting $\boldsymbol{\mathcal{G}}_{\pi'}(\pi)$ of $\boldsymbol{\mathcal{G}}_{\pi'}$ with respect to π, and so γ_π is the π-splitting of $\gamma_{\pi'}$. As a consequence, we have $\gamma_\pi \leqslant \gamma_{\pi'}$.

Proof. Multiplying by the appropriate power of t each of certain equations describing $x \cdot \boldsymbol{\mathcal{R}}_{\pi'}$, and then putting $t = 0$, we get already some of the equations that describe the limit of $x \cdot \boldsymbol{\mathcal{G}}_{\pi'}$, to wit, vanishing along downward arrows in $\bar{A}_\pi$, local residue conditions, and Rosenlicht conditions for π:

$$\begin{aligned} \varphi_a &= 0 && \text{for each downward } a \in \bar{A}_\pi, \\ \sum_{a \in \mathbb{E}_v} \varphi_a &= 0 && \text{for each } v \in V, \\ \varphi_a + \varphi_{\bar{a}} &= 0 && \text{for each } a \in \mathbb{E} \smallsetminus \mathbb{E}_\pi \text{ (horizontal for } \pi). \end{aligned}$$

For each $i \in [s]$, let $L_{i,1}, \dots, L_{i,q_i}$ be the parts of π which are contained in π'_i in ascending order, and $d_{i,1}, \dots, d_{i,q_i}$ the ascending sequence of levels attributed by d to each of them. The order given by π on the $L_{i,j}$, $i \in [s]$, $j \in [q_i]$, is thus the lexographic on the subindices. Let $V_{h' < i}$ denote the union of the π'_n for $n < i$ in $[s]$, and let $V_{h \prec_{\text{lex}} (i,j)}$ denote the union of the $L_{n,m}$ for $(n,m) \prec_{\text{lex}} (i,j)$. We need to show that, for each $i \in [s]$ and $j = 1, \dots, q_i$, the limit of $x \cdot \boldsymbol{\mathcal{G}}_{\pi'}$ satisfies the global residue conditions for π, namely, the equations

$$\sum_{a \in \mathbb{E}(L_{i,j}, \Xi)} \varphi_a = 0 \qquad \text{for each connected component } \Xi \text{ of } G[V_{\prec (i,j)}]. \tag{6.2}$$

Now, for a component Ξ of $G[V_{h \prec_{\text{lex}} (i,j)}]$, let S_l be the subset of vertices of $L_{i,l}$ it contains for each $l = 1, \dots, j-1$. Then, Ξ is the union of certain connected components $\Xi_1, \dots, \Xi_k$ of $G[V_{h' < i}]$ to which we add the vertices in the S_l for $l = 1, \dots, j-1$, and the edges connecting the vertices in these sets. The arrows that connect $L_{i,j}$ to Ξ are all upward and have head either in S_l for some $l = 1, \dots, j-1$ or in Ξ_l for some $l = 1, \dots, k$. We get Equation (6.2) summing up the following series of equations satisfied by $x \cdot \boldsymbol{\mathcal{G}}_{\pi'}$ for $t \neq 0$, by multiplying by

$t^{-d_{i,j}}$ and then putting $t=0$:

$$\begin{aligned} \sum_{l=1}^{k}\sum_{p=1}^{q_i} t^{d_{i,p}} \sum_{a\in\mathbb{E}(L_{i,p},\Xi_l)} \varphi_a &= 0 && \text{(Global residue conditions for } \pi'\text{)} \\ -\sum_{l=1}^{j-1} t^{d_{i,l}} \sum_{a\in\mathbb{E}_{S_l}} \varphi_a &= 0 && \text{(Local residue conditions)} \\ \sum_{l=1}^{j-1}\sum_{p=1}^{q_i}\sum_{a\in\mathbb{E}(S_l,L_{i,p})} \left(t^{d_{i,l}}\varphi_a + t^{d_{i,p}}\varphi_{\bar a}\right) &= 0 && \text{(Rosenlicht conditions for } \pi'\text{)} \\ \sum_{l=1}^{j-1}\sum_{a\in\mathbb{E}(S_l,V_{h'>i})} t^{d_{i,l}}\varphi_a &= 0. && \text{(Vanishing along downward arrows for } \pi'\text{)} \end{aligned}$$

Having established that the limit of $x\cdot\boldsymbol{\mathcal{G}}_{\pi'}$ as t approaches 0 satisfies all the equations that $\boldsymbol{\mathcal{G}}_\pi$ does finishes the proof of the first statement, as the two spaces $x\cdot\boldsymbol{\mathcal{G}}_{\pi'}$ and $\boldsymbol{\mathcal{G}}_\pi$ have the same dimension.

Combining this with (6.1), we get $\boldsymbol{\mathcal{G}}_\pi = \boldsymbol{\mathcal{G}}_{\pi'}(\pi)$, which implies that γ_π is the splitting of $\gamma_{\pi'}$ with respect to π by Proposition 5.2.

The last assertion follows from the splitting statement, using [AE24, Prop. 2.2], or directly from the limit, using lower semicontinuity of the dimensions of projections in the limit. □

We can now prove Theorem 1.2.

Proof of Theorem 1.2. We show that for each ordered partition π, the polytope $\mathbf{P}_\pi$ is a face of the residue polytope $\mathbf{P}_{\text{res}}$. Moreover, each face of $\mathbf{P}_{\text{res}}$ is of this form.

By [AE24, Prop. 2.7], the faces of $\mathbf{P}_{\text{res}}(G)$ are the base polytopes of the π-splittings of the submodular function γ_{π_0} for all ordered partitions π of V; see [AE24, §2.4]. But the π-splitting of γ_{π_0} is the submodular function of $\boldsymbol{\mathcal{G}}_{\pi_0}(\pi)$ by Proposition 5.2. By Theorem 6.1, we have $\boldsymbol{\mathcal{G}}_{\pi_0}(\pi) = \boldsymbol{\mathcal{G}}_\pi$. Therefore, the π-splitting of γ_{π_0} is γ_π. We conclude that the faces of $\mathbf{P}_{\text{res}}(G)$ are the base polytopes of γ_π, and the theorem follows. □

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