On the Path-width of Planar Graphs *

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Abstract

In this paper, we present a result concerning the relation between the path-with of a planar graph and the path-width of its dual. More precisely, we prove that for a 3-connected planar graph G, $pw(G) \leq 3pw(G^*) + 2$. For 4-connected planar graphs, and more generally for Hamiltonian planar graphs, we prove a stronger bound $\mathrm{pw}(G^*) \leq 2\ \mathrm{pw}(G) + c$. The best previously known bound was obtained by Fomin and Thilikos who proved that $\mathrm{pw}(G^*) \leq 6\ \mathrm{pw}(G) + cte$. The proof is based on an algorithm which, given a fixed spanning tree of G, transforms any given decomposition of G into one of G^*. The ratio of the corresponding parameters is bounded by the maximum degree of the spanning tree.

1 Introduction

A *planar graph* is a graph that can be embedded in the plane without crossing edges. It is said to be *outerplanar* if it can be embedded in the plane without crossing edges and such that all its vertices are incident to the unbounded face. For any graph G, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. The *dual* of a planar graph G, denoted by G^*, is the graph with one vertex for each face, and joining two vertices by one edge in G^* for each edge that the corresponding faces in G share. The weak dual $\mathcal{T}_G$ is the induced subgraph of G^* obtained by removing the vertex corresponding to the unbounded face. Note that the dual of a planar graph can be computed in linear time.

The notion of path-width was introduced by Robertson and Seymour [11]. A *path decomposition* of a graph G is a set system $(X_1, \ldots, X_r)$ of $V(G)$ (X_is are called *bags*) such that

1. $\bigcup_{i=1}^{r} X_i = V(G)$;
2. $\forall\ xy \in E, \exists\ i \in \{1, \ldots, r\} : \quad \{x, y\} \subseteq X_i$;
3. for all $1 \leq i_0 < i_1 < i_2 \leq r$, $X_{i_0} \cap X_{i_2} \subseteq X_{i_1}$.

The *width* of the path-decomposition $(X_1, \ldots, X_r)$ is $\max_{1 \leq i \leq r} |X_i| - 1$. The *path-width* of G, denoted by $\mathrm{pw}(G)$, is the minimum width over its path decompositions. For the definition of other width-parameters, *branch-width* and *tree-width*, we refer to the survey of Bodlaender [3] and Reed [10]. We denote the tree-width and branch-width of G by $tw(G)$ and $bw(G)$, respectively.

Comparing the width-parameters of G and G^* seems to be a very natural question. Indeed, a more interesting (algorithmic) problem should ask for a natural way of transforming a given decomposition of G to a decomposition of G^* without changing "too much" the width of the corresponding decompositions.

It is a consequence of Seymour and Thomas work [13] that such a comparison exists for branch-width:

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Theorem 1 (Seymour and Thomas [13]) *For every bridgeless planar graph G, $bw(G) = bw(G^*)$.*

Calculating branch-width can be also done in polynomial time for planar graphs.

For tree-width, Lapoire [9] proved the following theorem using algebraic methods:

Theorem 2 (Lapoire [9]) *For every planar graph G, $\mathrm{tw}(G) \leq \mathrm{tw}(G^*) + 1$.*

This was a conjecture of Robertson and Seymour [12] and a combinatorial shorter proof of this theorem can be found in Bouchitté et al [4]. Remark that it is an open question to see if tree-width can be calculated in polynomial time in planar graphs.

But how about the path-width? Is there any relation? Note that computing the path-width of graphs is an NP-complete problem even for planar graphs with maximum degree 3. For biconnected outerplanar graphs, Bodlaender and Fomin [2] provided a linear time algorithm which approximates the path-width of biconnected outerplanar graphs with a multiplicative factor of 2. To do so, they exhibit a relationship between the path-width of an outerplanar graph and the path-width of its dual. More precisely they prove that for any biconnected outerplanar graph G without loops and multiple edges, $\mathrm{pw}(G^*) \leq \mathrm{pw}(G) \leq 2\ \mathrm{pw}(G^*) + 2$.

By the results of Coudert, Huc and Sereni [5] it is impossible to have $pw(G) = pw(G^*)$ (Fomin and Thilikos provided similar constructions in [8]): they constructed an infinite family of outerplanar graphs such that each one has path-width twice the path-width of its dual. Indeed they proved the following theorem:

Theorem 3 (Coudert et al. [5]) *For every biconnected outerplanar graph G, $pw(G^*) \leq pw(G) \leq 2\ pw(G^*) - 1$. Furthermore, for every integer $p \geq 1$ and every integer $k \in \{1, 2, \ldots, p+1\}$, there exists a biconnected outerplanar graph of path-width $p + k$ whose dual has path-width $p + 1$.*

Fomin and Thilikos showed in [8] a linear inequality between the two parameters:

Theorem 4 (Fomin and Thilikos [8]) *There is a constant c such that for every 3-connected planar graph G we have $pw(G^*) \leq 6\ pw(G) + c$.*

In this paper, we propose an algorithm which given a spanning tree of G, transforms a given decomposition of G into one of G^*. The ratio of the corresponding parameters is bounded by the maximum degree of the spanning tree. Our transformation then reduces the question of comparing the different width-parameters of G and G^* to the problem of finding spanning trees of low maximum degree in a given planar graph.

Theorems 5 and 6 are the main theorems of this paper.

Theorem 5 *For every 3-connected planar graph G, we have $pw(G^*) \leq 3\ pw(G) + 2$.*

Remark that Theorem 5 improves Theorem 4.

Theorem 6 *If G is a planar graph with a Hamiltonian path, then $pw(G^*) \leq 2\ pw(G) + 1$.*

Theorem 6 in particular proves that for a 4-connected planar graph G we always have $\mathrm{pw}(G^*) \leq 2\ \mathrm{pw}(G) + 1$. Indeed, by a theorem of Tutte [14], every such graph has a Hamiltonian cycle.

2 Main Theorem

In this section we present the proofs of Theorems 5 and 6. We will use the following notations:

Given a planar graph G on vertex set $V(G)$ and edge set $E(G)$ of maximum degree $\Delta(G)$, by $F(G)$ we mean the set of faces of G, which is also the vertex set of its dual. The number of faces, edges and vertices of G are respectively denoted by f_G, e_G and n_G. Given a face $F \in F(G)$, we denote the set of vertices belonging to this face by $V(F)$. $E(F)$ is the set of edges appearing on the boundary of F. Given a set A, by $\mathcal{P}(A)$ we denote the family of all subsets of A.

Definition 1 Let G and H be two graphs and σ a map from $V(G)$ to $\mathcal{P}(V(H))$. We say that σ is a *connected map from G to H* if it satisfies the following two properties:

1. for every $v \in V(G)$, the subgraph of H induced by $\sigma(v)$ is connected.
2. for every edge $vw \in E(G)$, the subgraph of H induced by $\sigma(v) \cup \sigma(w)$ is also connected.

For every vertex $w \in V(H)$, we define $\sigma^{-1}(w) := \{v \in V(G) |\ w \in \sigma(v)\}$. The *degree* of σ is the integer $k = \max |\sigma^{-1}(w)|$.

Lemma 1 *Let G and H be two graphs. If there exists a connected map σ of degree at most k from G to H, we have:*

$$pw(G) \leq k \cdot pw(H) + k - 1$$

Proof

Let $(X_1, \ldots, X_r)$ be a path-decomposition of H. We first show that the sequence $(\sigma^{-1}(X_1), \ldots, \sigma^{-1}(X_r))$ provides a path-decomposition of G. For this, we should prove the three properties of a path-decomposition:

- Every vertex v of G appears in one $\sigma^{-1}(X_i)$. To show this, let $u \in \sigma(v)$. As $(X_1, \ldots, X_r)$ forms a path-decomposition of H, there exists an i such that $u \in X_i$. It is clear that for this bag, v appears in $\sigma^{-1}(X_i)$.
- For every edge $xy \in E(G)$, there is one $\sigma^{-1}(X_i)$ which contains both x and y. To prove this, let $A = \sigma(x)$ and $B = \sigma(y)$. The graph induced by H on $A \cup B$ is connected, so at least one of the two following two cases appears:
  - $A \cap B \neq \emptyset$: let $u \in A \cap B$ and X_i be the bag which contains u. Then $\sigma^{-1}(X_i)$ contains both x and y.
  - There exist $a \in A$ and $b \in B$ such that $ab \in E(H)$. Let X_i be the bag which contains both a and b. It is clear that $\sigma^{-1}(X_i)$ contains both x and y.
- For all $1 \leq i_0 < i_1 < i_2 \leq r$, we should prove $\sigma^{-1}(X_{i_0}) \cap \sigma^{-1}(X_{i_2}) \subseteq \sigma^{-1}(X_{i_1})$. Let $v \in \sigma^{-1}(X_{i_0}) \cap \sigma^{-1}(X_{i_2})$. The graph induced by $\sigma(v)$ in H, i.e. $H[\sigma(v)]$, is connected and intersects both X_{i_0} and X_{i_2}. The graph $H[\sigma(v)] \setminus X_{i_1}$ is not connected. We infer that $\sigma(v) \cap X_{i_1} \neq \emptyset$, which implies $v \in \sigma^{-1}(X_{i_1})$.

As the degree of σ is at most k and $|X_i| \leq pw(H) + 1$, we have $|\sigma^{-1}(X_i)| \leq k(pw(H) + 1)$, which proves that the width of the path-decomposition $(\sigma^{-1}(X_1), \ldots, \sigma^{-1}(X_r))$ is at most $k.pw(H) + k - 1$. This finishes the proof of the lemma. □

Remark that the same proof applies for other types of decompositions.

From now on, our aim will be to find a way to produce low degree connected maps from G^* to G. The key role will be played by spanning trees of G: every spanning tree of maximum degree k produces a connected map from G^* to G of degree at most k. Before we proceed, we need some new definitions:

An *edge-assignment* to faces of G is a one-to-one map from the faces of G to the edges of G which to each face F of G, associates one edge of $E(F)$. More formally:

Definition 2 An edge-assignment is a function: $\tau : F(G) \rightarrow E(G)$ such that

1. for every face $F \in F(G)$, $\tau(F)$ is an edge on the boundary of F, and
2. $\tau(F) \neq \tau(F')$ for all distinct faces $F, F' \in F(G)$.

Given an edge-assignment, we define the map $\sigma_\tau : F(G) \rightarrow \mathcal{P}(V(G))$ as follows: σ_τ associates to every face F in $F(G)$ the subset $V(F) \setminus V(\tau(F))$.

Proposition 1 *For every edge-assignment τ, the map σ_τ is connected.*

Proof It is clear that the graph induced by $\sigma_\tau(F)$ is connected, since it forms a path. Two faces F_1, F_2 sharing an edge e can not be both associated to e (since they are associated to different edges). Consequently $\sigma_\tau(F_1) \cup \sigma_\tau(F_2)$ induces also a connected subgraph of G. This proves that σ_τ is connected. □

Given an edge-assignment τ, let H be the subgraph of G consisting of non selected edges; i.e. $H = G \setminus \{\tau(F) | F \in V(G^*)\}$. Using Euler's Formula ($f_G + n_G = e_G + 2$), we infer that H contains exactly $n_G - 2$ edges. We have

Proposition 2 *For all $v \in V(G)$, $|\sigma_\tau^{-1}(v)| = deg_H(v)$.*

Proof A selected edge (an edge of $G \setminus H$) is associated to one of the two faces containing it. Given a vertex v of G of degree d, it appears exactly in d faces. Suppose r edges incident to v are selected; they are associated to exactly r faces incident with v. The image of this faces by σ_τ does not contain v, and v appears in $\sigma_\tau(F)$ for all other faces F incident to v. So $|\sigma_\tau^{-1}(v)| = d - r = deg_H(v)$. □

Corollary 1 *σ_τ is of degree $\Delta(H)$.*

Remark that the average degree in H is always < 2.

Definition 3 A subgraph $H \subseteq G$ is *nice* if it has $n_G - 2$ edges and if there exists an edge-assignment τ with $\tau(F) \in E(G) \setminus E(H)$.

Hence to prove Theorem 5, by using Lemma 1, we need to find a nice subgraph with maximum degree at most 3. To proceed we need the following lemma:

Lemma 2 *Let G be a planar graph and T a spanning tree of maximun degree k in G. Let e be an edge of T. The subgraph $H = T \setminus e$ is a nice subgraph of maximum degree at most k.*

Proof To prove that H is nice we will apply Hall's matching theorem to the adjacency graph A between faces of G and edges of $G \setminus H$. More precisely, A is the bipartite graph on vertex set $F(G) \sqcup (E(G) \setminus E(H))$. An edge of A connects $F \in F(G)$ (i.e. F a face of G) to $e \in E(G) \setminus E(H)$, if e belongs to $E(F)$. We want to prove that there exists a matching in A covering all the vertices of $F(G)$. Given a set of faces $\{F_1, \ldots, F_i\}$, we need to prove that the corresponding set has at least i neighbours in A. Let us consider the planar graph S obtained by taking the union of F_i's, we have:

- $f_S \geq i + 1$ (because G^* is connected), and
- $f_S + n_S = e_S + 2$ (Euler's formula).

We conclude that $e_S - (n_S - 1) \geq i$. As H is a forest, the number of edges of H incident to vertices of S is at most $n_S - 1$. So the hypothesis of Hall's theorem are satisfied. This proves that H is a nice subgraph. □

Barnette proved in [1] the following theorem:

Theorem 7 (Barnette [1]) *Every 3-connected planar graph has a spanning tree of maximum degree 3.*

Later, Czumaj and Strothmann [6] proved that such a spanning tree in G can be found in linear time. We can now present the proof of Theorems 5 and 6:

Proof of Theorem 5 Barnette's theorem insures the existence of a spanning tree of maximun degree three, which by Lemma 2 provides a nice subgraph of maximun degree 3. The associated edge-assignment gives a connected map of degree three (Proposition 1 and Corollary 1), hence Theorem 5 follows from Lemma 1. The algorithm of Czumaj and Strothmann [6] combined with the algorithm of finding a maximum matching in A (see the proof of Lemma 2) results in a polynomial time algorithm to find the corresponding decompositions. □

Proof of Theorem 6 Using Lemma 2, deleting an edge of the Hamiltonian path gives a nice subgraph of G of maximum degree 2. □

3 Discussion

A 3-connected planar graph does not contain in general a nice subgraph of maximum degree two. Indeed, it can happen that the graph G does not contain a subgraph of maximum degree 2 with $n-2$ edges. Even more, for every constant k, there exist planar graphs such that for every subgraph H, containing $n-2$ edges and covering all vertices, we have $\sum_{deg(v)\geq 2} deg(v) \geq k$. Examples of such graphs are planar graphs which can not be covered by less than k disjoint paths. In [7], an inductive construction of an infinite family of non Hamiltonian Delaunay triangulations is presented. Two examples of graphs of this family are given in Figure 3. By looking at the central separating triangle and using induction, one can prove that for any k, a large graph of this family does not contain k disjoint paths covering all its vertices. This shows that we can not expect to improve Theorem 5 via nice subgraphs.

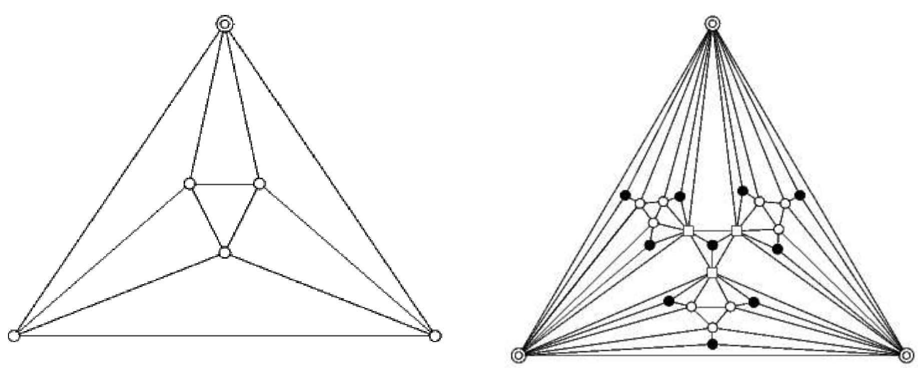

Figure 1: Two graphs of the family constructed in [7]

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