LIMIT LINEAR SERIES: COMBINATORIAL THEORY

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ABSTRACT. The aim of this paper is to develop a purely combinatorial theory of limit linear series on metric graphs. This will be based on the formalism of hypercube rank functions and slope structures. Some applications and connections to other concepts in combinatorial algebraic geometry, as well as several open questions, are discussed in the paper.

CONTENTS

1. INTRODUCTION

One of the main open questions regarding the asymptotic geometry of curves is the problem of degeneration of linear series on smooth curves of given genus when they approach the boundary of their corresponding moduli space. That is, to fully describe all the possible limits of linear series of given rank and degree when smooth curves degenerate to singular ones. This question was studied in a series of works by Eisenbud and Harris [EH89, EH86, EH87b, EH87a, EH87c] for which they managed to provide a satisfactory answer in the case the limit curve is of compact type, and used this to make major progress in the study of curves. For curves of pseudo-compact type, these results were generalized by Esteves–Medeiros [EM02] (for curves with two components) and by Osserman [Oss19b, Oss19a, He19]. The generalization of this picture to more general curves is the subject of ongoing works [AE20a, AE20b, AE21, AE22]. The case of rank zero linear series in the pluricanonical systems is studied in recent works [BCG+18, BCG+19, MUW20, TT22].

Tropical geometry provides a modern perspective on degeneration methods in algebraic geometry and a new approach to classical questions in algebraic geometry. Developing the mathematics behind the tropical approach usually requires the introduction of new combinatorial structures, and it has become apparent now that from the viewpoint of applications, it is enough in many cases to understand the geometry behind these combinatorics. Two such examples are given in the development of a combinatorial theory of divisors on graphs [BN07, BJ16], and, more recently, in the development of tropical and combinatorial Hodge theories [AHK18, AP20].

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In a previous work [AB15], Baker and the first named author introduced linear series on hybrid objects called metrized complexes and used them to recover and partially generalize Eisenbud-Harris theory of limit linear series. In a subsequent work [Ami14], the formalism of slope structures on metric graphs was introduced as a way to describe the limiting behavior of Weierstrass points on degenerating families of curves. Slope structures were used in a recent work of Farkas–Jensen–Payne [FJP20] in the study of the geometry of moduli space of curves.

The aim of this paper is to take the tropical approach one step further by introducing a purely combinatorial theory of linear series of arbitrary rank and degree on metric graphs. This can be regarded as a combinatorial theory of limit linear series.

More precisely, we aim to draw relevant combinatorial properties of tropicalizations of linear series, regarding:

- the slopes taken by the tropicalizations of functions;
- the vectors of slopes taken by these functions around points;
- the tropical dependence between these functions, and
- the topological properties of tropicalizations,

in order to set a formalism of linear series on metric graphs. This will be based on two ingredients: hypercube rank functions and slope structures. In the rest of this introduction, we give an overview of the set-up and the results.

1.1. **Degeneration problem for linear series.** Recall that a linear series $\mathfrak{g}^r_d$ on a projective curve Y is by definition a vector subspace of dimension $r+1$ of the space of global sections $H^0(Y,L)$ of a line bundle L of degree d on Y.

Let $\mathscr{M}_g$ be the moduli space of smooth projective curves of genus g, and $\bar{\mathscr{M}}_g$ its Deligne–Mumford compactification. Let X be a stable curve of genus g over an algebraically closed field κ and x the corresponding point in $\bar{\mathscr{M}}_g$. The degeneration problem for linear series can be informally stated as follows.

Question 1.1. *Describe all the possible limits of linear series $\mathfrak{g}^r_d$ over any sequence of smooth projective curves of genus g when their corresponding points in $\mathscr{M}_g$ converge to x.*

1.2. **Metric graphs and their divisor theory.** Metric graphs arise as tropical limits of analytic one parameter families of smooth projective curves.

Recall that a metric graph Γ is a compact metric space isomorphic to the metric realization of a pair (G,ℓ) consisting of a finite graph $G=(V,E)$ and a length function $\ell\colon E\to\mathbb{R}_+$: this is obtained by associating to each edge e a copy of the interval $\mathcal{I}_e=[0,\ell_e]$, with the two extremities identified with those of e, and then further identifying the ends of different intervals corresponding to a same vertex v. The quotient topology on Γ is metrizable by the path metric. The pair (G,ℓ) is called a model of Γ.

The set of rational functions on Γ is denoted by $\mathrm{Rat}(\Gamma)$, and, by definition, consists of all continuous piecewise affine functions $f:\Gamma\to\mathbb{R}$ with integral slopes. Tropicalization of rational functions on curves gives rise to rational functions on metric graphs.

As in the algebraic setting, rational functions on metric graphs are linked to divisors. A divisor D on a metric graph Γ is a finite formal sum with integer coefficients of points of Γ. For any rational function $f\in\mathrm{Rat}(\Gamma)$, the corresponding divisor is denoted by

$$\mathrm{div}(f)\coloneqq\sum_x \mathrm{ord}_x(f)\,x,\qquad \text{where } \mathrm{ord}_x(f)\coloneqq-\sum_{\nu\in\mathrm{T}_x(\Gamma)}\mathrm{slope}_\nu(f)$$

where $\mathrm{T}_x(\Gamma)$ is the set of outgoing unit tangent vectors to Γ at x, and $\mathrm{slope}_\nu(f)$ is the slope of f along ν at x. A divisor obtained in this way is called principal. Two divisors D_1 and D_2 on a metric graphs are called linearly equivalent if their difference D_1-D_2 is principal.

Each divisor D defines a line bundle on Γ, and the space of its global section is denoted by $\mathrm{Rat}(D)$. Contrary to the algebraic setting, $\mathrm{Rat}(D)$ is not a vector space. Nevertheless, Baker and Norine discovered a way to associate a combinatorial notion of rank to $\mathrm{Rat}(D)$ [BN07, MZ08, GK08]. This is defined as the maximum integer among -1 and integers $r \geq 0$ such that for all points $x_1, \dots, x_r$ in Γ, the divisor $D - x_1 - \cdots - x_r$ is linearly equivalent to an effective divisor (that is, a divisor in which all points have non-negative coefficients). We refer to the survey paper by Baker and Jensen [BJ16] for details, extensions, and several applications.

The results of this paper are motivated by the question of describing the tropical limits of linear series $\mathfrak{g}^r_d$'s, when the metric graph arises as the tropical limit of a one-parameter family of smooth proper curves.

1.3. **Rank functions on hypercubes and permutation arrays.** As matroids provide a combinatorial axiomatization of linear algebra, hypercube rank functions give a combinatorial encoding for the intersection patterns of a finite collection of complete flags in a vector space. On a smooth projective curve, taking the orders of vanishing at a point leads to a complete flag in any finite dimensional vector space of rational functions. Such flags appear naturally in the degeneration of linear series.

Let r be a non-negative integer. We set $[r] \coloneqq \{0, \dots, r\}$. For a positive integer δ, the hypercube $\square^\delta_r$ of dimension δ and width r is the product $[r]^\delta$.

There is a lattice structure on $\square^\delta_r$ induced by the two operations $\vee$ and $\wedge$ defined by taking the maximum and the minimum coordinate-wise, respectively: for $\underline{a} = (a_1, \dots, a_\delta)$ and $\underline{b} = (b_1, \dots, b_\delta)$ in $\square^\delta_r$, we set

$$\underline{a} \vee \underline{b} \coloneqq (\max(a_1, b_1), \dots, \max(a_\delta, b_\delta)), \qquad \underline{a} \wedge \underline{b} \coloneqq (\min(a_1, b_1), \dots, \min(a_\delta, b_\delta)).$$

A function $f : \square^\delta_r \to \mathbb{Z}$ is called supermodular if for any two elements $\underline{a}$ and $\underline{b}$, we have

$$f(\underline{a}) + f(\underline{b}) \leq f(\underline{a} \vee \underline{b}) + f(\underline{a} \wedge \underline{b}).$$

A function $\rho : \square^\delta_r \to \mathbb{Z}$ is called a rank function if it is supermodular and, in addition, satisfies the following conditions:

(1) The values of ρ are in the set $[r] \cup \{-1\}$.
(2) ρ is non-increasing with respect to the partial order of $\square^\delta_r$, i.e., if $\underline{a} \leq \underline{b}$, then $\rho(\underline{b}) \leq \rho(\underline{a})$.
(3) for any $1 \leq i \leq \delta$, and all $0 \leq t \leq r$, we have $\rho\,(t\,\underline{e}_i) = r - t$.

The geometric situation to have in mind is a vector space H of dimension $r+1$ over some field κ and a collection of δ complete flags $F^\bullet_1, \dots, F^\bullet_\delta$. That is, for $j = 1, \dots, \delta$, $F^\bullet_j$ consists of a chain of vector subspaces

$$H = F^0_j \supsetneq F^1_j \supsetneq \cdots \supsetneq F^{r-1}_j \supsetneq F^r_j \supsetneq (0)$$

with $\mathrm{codim}(F^i_j) = i$.

In this case, the dimensions of intersection patterns of these flags define a rank function. That is, the function $\rho : \square^\delta_r \to \mathbb{Z}$ defined by

$$\rho(a_1, \dots, a_\delta) \coloneqq \dim_\kappa \left(F^{a_1}_1 \cap \cdots \cap F^{a_\delta}_\delta \right) - 1$$

is a rank function. Hypercube rank functions appearing in this way are called realizable.

An alternative combinatorial approach to the study of intersection patterns of complete flags was introduced by Eriksson-Linusson in the setting of permutation arrays [EL00a, EL00b].

We prove in Theorem 3.9 that the two approaches are equivalent by providing a bijection between hypercube rank functions and permutation arrays. To this end, we show that rank functions are, in a precise sense, discrete analogues of concave functions, and that permutation arrays are discrete analogues of local concave functions. The theorem is then a consequence of a local-to-global principle for concavity proved in Theorem 3.6.

A hypercube rank function locally gives rise to a collection of matroids, see Section 3.4. Local obstructions for the realizability of permutation arrays and rank functions can then be formulated in terms of matroid realizability. This covers the examples found by Billey and Vakil [BV08] of non-realizable permutation arrays.

1.4. **Slope structures on graphs and metric graphs.** Slope structures encode the information regarding possible slopes of functions arising from tropicalizations of linear series.

For a simple graph $G = (V, E)$, we denote by $\mathbb{E}$ the set of all the orientations of edges of G. For an edge $\{u, v\}$ in E, we have two orientations $uv, vu \in \mathbb{E}$. The subset $\mathbb{E}_v \subset \mathbb{E}$ is the set of all the orientations vu of edges $\{v, u\}$ in G. A slope structure $\mathfrak{S}$ of order r on G, or simply an r-slope structure, is the data of

- For any oriented edge $e = uv \in \mathbb{E}$ of G, a collection S^e of $r + 1$ integers $s^e_0 < s^e_1 < \cdots < s^e_r$, subject to the requirement that $s^{uv}_i + s^{vu}_{r-i} = 0$ for any edge $\{u, v\} \in E$.
- For any vertex v of G, a rank function ρ_v on the hypercube $\square^{d_v}_r$.

Here, d_v is the valence of v in the graph. We denote by $S^v \subseteq \prod_{e\in\mathbb{E}_v} S^e$ the set of all points $s_{\underline{a}} = (s_{a_1}, \dots, s_{a_\delta})$ for $\underline{a} = (a_1, \dots, a_\delta)$ a point of the hypercube which is a jump of the rank function: that is, a point which verifies $\rho_v(\underline{a} + e_j) < \rho_v(\underline{a})$ whenever $\underline{a} + e_j$ is in the hypercube. Here e_j is the point of the hypercube with j-th coordinate equal to one and all the other coordinates zero. In the above notations, we then write $\mathfrak{S} = \{\, S^v; S^e \,\}_{v\in V, e\in\mathbb{E}}$.

Let Γ the a metric graph. By an r-slope structure on Γ we mean an r-slope structure $\mathfrak{S}$ on a simple graph model $G = (V, E)$ of Γ, that we enrich naturally by extending to any point of Γ (see Section 4.2).

Taking into account the slope structure $\mathfrak{S}$, we can define a relevant notion of rational functions. A function f in $\mathrm{Rat}(\Gamma)$ is said to be compatible with $\mathfrak{S}$ if the two conditions (i) and (ii) below are verified:

(i) for any point $x \in \Gamma$ and any tangent direction $\nu \in \mathrm{T}_x(\Gamma)$, the outgoing slope of f along ν lies in S^ν.

Denote by $\delta_x(f)$ the vector in $\prod_{\nu\in\mathrm{T}_x(\Gamma)} S^\nu$ which consists of outgoing slopes of f along $\nu \in \mathrm{T}_x(\Gamma)$. Then the second condition is:

(ii) for any point $x \in \Gamma$, the vector $\delta_x(f)$ belongs to S^x.

We denote by $\mathrm{Rat}(\Gamma, \mathfrak{S})$, or simply $\mathrm{Rat}(\mathfrak{S})$ if there is no risk of confusion, the space of rational functions on Γ compatible with $\mathfrak{S}$.

1.5. **Combinatorial limit linear series.** A linear series $\mathfrak{g}^r_d$ of rank r and degree d associated to $(D, \mathfrak{S})$ is a sub-semimodule M of $\mathrm{Rat}(D, \mathfrak{S})$ which is closed in $\mathrm{Rat}(D)$ (endowed with the infinity norm $\|\cdot\|_\infty$), and which satisfies the following property:

$(**)$ For any effective divisor E on Γ of degree r, there exists a rational function $f \in M$ such that

(1) For any point $x \in \Gamma$, $\rho_x(\delta_x(f)) \geq E(x)$; and in addition,
(2) $D + \mathrm{div}(f) - E \geq 0$.

For a $\mathfrak{g}^r_d$ M, the linear system $|M|$ is the space of all effective divisors E on Γ of the form $D + \mathrm{div}(f)$ for $f \in M$.

In the degeneration picture for linear series in a one-parameter family of smooth projective curves, the first condition is the combinatorial data underlying the reduction of rational functions when the point x is viewed inside the Berkovich analytification of the generic fiber of the family. The second one is the analogue of the Baker-Norine rank condition in this setting.

In order to define a refined notion of linear series, we use the concept of tropical rank introduced by Jensen and Payne in their series of works on applications of tropical divisor theory to the study of the geometry of generic curves [JP14, JP16] (for the definition, see Subsection 5.1). We define refined linear series $\mathfrak{g}^r_d$'s associated to $(D, \mathfrak{S})$ as those linear series which are of tropical rank r .

We also define strongly refined $\mathfrak{g}^r_d$'s associated to $(D, \mathfrak{S})$ to be linear series of tropical rank r which in addition verify the following stronger version of $(**)$:

$(***)$ For any effective divisor E on Γ of degree $s \leq r$, there exists a refined linear series M_E of rank $r-s$ associated to $(D, \mathfrak{S}_E)$ with $\mathfrak{S}_E$ a slope substructure of $\mathfrak{S}$ of rank $r-s$ such that for any function $f \in M_E$, we have

1. For any point $x \in \Gamma$, $\rho_x(\delta_x(f)) \geq E(x)$; and in addition,
2. $D + \mathrm{div}(f) - E \geq 0$.

1.6. **Basic properties.** Here is a list of interesting properties satisfied by slope structures and linear series.

(1) The vector of allowed slopes defined by a (crude) linear series is non-increasing along each edge (Proposition 4.16).

That is, as we move from one extremity of an edge to the other, the coordinates of the vector $(s^\nu_0, \dots, s^\nu_r)$ mutually decrease. This property turns out to be crucial in proving finiteness theorems about slopes structures underlying linear series $\mathfrak{g}^r_d$'s on metric graphs.

(2) The space of rational functions $\mathrm{Rat}(D, \mathfrak{S})$ is a semimodule over $\mathbb{R}$ (Proposition 5.1).

This is a consequence of the supermodularity of rank functions.

(3) A linear series M is closed in $\mathrm{Rat}(D)$ if, and only if, the corresponding linear system $|M|$ is closed in $\mathrm{Sym}^d(\Gamma)$ (Proposition 5.21).
(4) If $M \subset \mathrm{Rat}(D, \mathfrak{S})$ is a closed sub-semimodule, then it is generated by its extremal points (Proposition 5.22). Besides, fixing a function $f \in M$ and taking another $g \in M$, testing on a finite number of points is sufficient to determine whether $g = f$ (Lemma 5.24).

1.7. **Realization property for slope vectors.** An important feature of the linear series is the following realization property for jumps, proved in Theorem 5.27.

Theorem 1.2 (Realization of slope vectors)**.** *Take M a linear series associated to a pair $(D, \mathfrak{S})$. Then every vector of slopes around a point prescribed by the slope structure $\mathfrak{S}$ is realized by some function in M.*

One immediate consequence of this theorem is the fact that the slope structure underlying a linear series can be entirely retrieved from M.

Corollary 1.3. *The data of M determines the slope structure uniquely.*

1.8. **Finiteness of slope structures.** Let Γ be a metric graph, let D be a divisor on Γ. Let $G = (V, E)$ be a combinatorial graph underlying Γ supporting D. Using the decreasing property of the slopes along edges, we prove the following finiteness theorem.

Theorem 1.4 (Finiteness of slopes structures)**.** *For each integer r, there are finitely many subdivisions $H_1, \dots, H_k$ of G, and finitely many slope structures $\mathfrak{S}_1, \dots, \mathfrak{S}_k$ of rank r defined on them, respectively, such that any linear series $\mathfrak{g}^r_d$ with underlying pair $(D, \mathfrak{S})$ has a combinatorial model H_j among $H_1, \dots, H_k$ such that $\mathfrak{S}$ coincides with $\mathfrak{S}_j$.*

This theorem can be regarded as a first result in the direction of defining the moduli space of $\mathfrak{g}^r_d$'s over the moduli space of tropical curves of given genus.

1.9. **Classification of $\mathfrak{g}^1_d$'s.** In the case $r = 1$, we prove that, roughly speaking, the data of a $\mathfrak{g}^1_d$ on Γ is equivalent to the data of a finite harmonic map to a tree. This allows us to formulate a smoothing theorem for combinatorial $\mathfrak{g}^1_d$'s.

Theorem 1.5 (Classification of $\mathfrak{g}^1_d$'s on metric graphs)**.** *Let M be a refined $\mathfrak{g}^1_d$ on Γ associated to a pair $\mathrm{Rat}(D, \mathfrak{S})$ with D of degree d. Suppose that the constant functions are in M. Then, there exist a tropical modification $\alpha\colon \widetilde{\Gamma} \longrightarrow \Gamma$ of Γ and a finite harmonic morphism $\varphi\colon \widetilde{\Gamma} \to T$ of degree d to a metric tree such that M is the pre-image of the unique $\mathfrak{g}^1_1$ on the tree.*

Using the smoothing theorems proved in [ABBR15a, ABBR15b] for finite harmonic morphisms to trees, we deduce the following smoothing theorem.

Theorem 1.6 (Smoothing theorem for $\mathfrak{g}^1_d$'s)**.** *A refined $\mathfrak{g}^1_d$ $M \subset \mathrm{Rat}(D, \mathfrak{S})$ on Γ is smoothable, that is, it is the tropicalization of a $\mathfrak{g}^1_d$ from a smooth curve (see Subsection 8.1).*

The question of the existence of harmonic morphisms to metric trees is thoroughly studied by Draisma and Vargas [DV21a, DV21b], and by Cool and Draisma [CD18]. It would be interesting to make a link between combinatorial linear series introduced in this paper and these works.

1.10. **Specialization theorem.** Let now $\mathbb{K}$ be an algebraically closed field with a non-trivial non-Archimedean valuation val and C be a smooth proper curve over $\mathbb{K}$. We assume that $\mathbb{K}$ is complete with respect to val and we denote by κ the residue field of $\mathbb{K}$, which is also algebraically closed.

Let $\mathcal{D}$ be a divisor of degree d and rank r on C, and $(\mathcal{O}(\mathcal{D}), H)$ be a $\mathfrak{g}^r_d$ on C. We identify H with a subspace of $\mathbb{K}(C)$ of dimension $r+1$. Let Γ be a skeleton of the Berkovich analytification C^{an}. We define

$$M \coloneqq \mathrm{trop}(H) = \{\mathrm{trop}(f), f \in H \setminus \{0\}\}$$

with $\mathrm{trop}(f) = -\log(|f|)$ for any function $f \in H$, which is a piecewise affine function on Γ with integeral slopes.

Theorem 1.7 (Specialization of linear series)**.** *Notations as above, let $(\mathcal{O}(\mathcal{D}), H)$, $H \subseteq H^0(C, \mathcal{O}(\mathcal{D})) \subset \mathbb{K}(C)$, be a g^r_d on C. Let Γ be a skeleton of C^{an}. The slopes of rational functions F in M along edges in Γ define a well-defined slope structure $\mathfrak{S}$ on Γ. Let D be the specialization of $\mathcal{D}$ on Γ. Then, $M \subset \mathrm{Rat}(D, \mathfrak{S})$ is a strongly refined $\mathfrak{g}^r_d$ on Γ.*

A pluri-canonical linear series of rank r and order n is a vector subspace $H \subseteq H^0\left(C, \omega_C^{\otimes n}\right)$ of rank r, i.e., of dimension $r+1$, with ω_C the canonical sheaf of C.

Using Temkin metrization [Tem16], one can define tropicalizations of subspaces of global sections of pluri-canonical sheaves. We explain how they fit to our theory of combinatorial limit linear series.

Let Γ be the metric graph skeleton of the Berkovich analytification C^{an}, and K the canonical divisor of Γ.

Theorem 1.8 (Specialization of pluri-canonical linear series)**.** *Let $H \subseteq H^0\left(C, \omega_C^{\otimes n}\right)$ be a pluri-canonical linear series of rank r and order n. Let $M \coloneqq \mathrm{trop}(H) = \{\mathrm{trop}(\alpha) \bigm| \alpha \in H\}$. Then $M \subseteq \mathrm{Rat}(nK, \mathfrak{S})$, for the pluricanonical slope structure $\mathfrak{S}$ of S_n. Moreover, M is a refined $\mathfrak{g}^r_d$ on Γ, for $d = n(2g-2)$.*

Using the finiteness theorem, we prove that there are only finitely many combinatorial types for pluri-canonical slope structures of order n on augmented metric graphs Γ of a given combinatorial type, and leave as an interesting open question to classify all the pluri-canonical slope structures on a given graph G.

1.11. **Discussion of applications.** The formalism of this paper already has applications to the geometry of curves. In particular, the equidistribution theorem proved in [Ami14] is a consequence of the formalism of slope structures and the behavior of reduced divisors in a given combinatorial linear series. The results of [FJP20] also use slope structures and the notion of tropical independence, the underlying concepts of the materials presented in this paper. In a joint forthcoming work of the authors with Harry Richman, we apply the formalism of this paper to associate a Weierstrass multiplicity to any connected component of the naive Weierstrass locus of a given divisor on a metric graph. This solves a problem posed by Matt Baker from his original work on specialization of linear series from curves to graphs [Bak08]. Using these ideas, we explain the discrepancy between the naive counting of Weierstrass points on metric graphs in the work of Richman [Ric18], and the correct count of multiplicities.

Remark 1.9. When this paper was in the final stages of preparation, we learned of a notion of tropical linear series on metric graphs developed simultaneously and independently by Dave Jensen and Sam Payne [JP22], with many similarities to the framework introduced and studied here. There are also meaningful differences between the two approaches. One of the interesting features of the work by Jensen and Payne which does not appear in this paper is the valuated matroid property that allows to put global restrictions on the space of relations between elements of the tropical module, and to define maps to tropical linear spaces, as in the classical setting. The recursive condition on the existence of sublinear series is imposed on tangent vectors, which allows to add flexibility to the proofs. Clearly, understanding the precise connection between the two approaches, and making a combination of the properties discovered in both works, should lead to further progress in the development of tropical methods and their applications to the study of the geometry of curves and their moduli spaces. $\diamond$

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Further notations. For any non-negative integer r, we set $[r] \coloneqq \{0, \dots, r\}$. For a subset $S \subset \mathbb{R}^d$, $d \in \mathbb{N}$, and an element $a \in \mathbb{R}^d$, we define $S + a$ as the set of all elements $b + a$ for $b \in S$. For the rest of the article, Λ will denote a divisible subgroup of $\mathbb{R}$. Examples of such Λ are $\mathbb{Q}$ or $\mathbb{R}$ itself (a group is called divisible when multiplication by every positive integer n is surjective).

A Λ-metric graph, or Λ-rational metric graph, is a metric graph Γ whose edge lengths are Λ-rational, that is, that lie in Λ. A point $x \in \Gamma$ is said to be Λ-rational if its distances to the extremities of its incident edges are in Λ. In the case Γ is Λ-rational, a divisor whose support is made up of Λ-rational points is said to be a Λ-rational divisor. We also denote by $\mathrm{Rat}_\Lambda(\Gamma)$ the set of functions of $\mathrm{Rat}(\Gamma)$ which only change slopes at Λ-rational points of Γ.

2. Rank functions on hypercubes

Let r be a non-negative integer. For a positive integer δ, the *hypercube* $\square_r^\delta$ of *dimension* δ and *width* r is the product $[r]^\delta$. We denote the elements of $\square_r^\delta$ by vectors $\underline{a} = (a_1, \dots, a_\delta)$, for $0 \leq a_1, \dots, a_\delta \leq r$. There is a partial order $\leq$ on $\square_r^\delta$ where for two elements $\underline{a}, \underline{b} \in \square_r^\delta$, we say $\underline{a} \leq \underline{b}$ whenever for all $i = 1, \dots, \delta$, we have $a_i \leq b_i$.

The smallest (resp. largest) element of $\square_r^\delta$ with respect to this partial order is $\underline{0} \coloneqq (0, \dots, 0)$ (resp. $\underline{r} = (r, \dots, r)$). For every integer $1 \leq i \leq \delta$, we denote by $\underline{e}_i$ the vector whose coordinates are all zero expect the i-th coordinate which is equal to one. For $0 \leq t \leq r$, the vector $t\underline{e}_i$ lies in the hypercube $\square_r^\delta$. There is a *lattice* structure on $\square_r^\delta$ induced by the two operations $\vee$ and $\wedge$ defined by taking the maximum and the minimum coordinate-wise, respectively: for $\underline{a}$ and $\underline{b}$ in $\square_r^\delta$, we set

$$\underline{a} \vee \underline{b} := (\max(a_1, b_1), \dots, \max(a_\delta, b_\delta)), \qquad \underline{a} \wedge \underline{b} := (\min(a_1, b_1), \dots, \min(a_\delta, b_\delta)).$$

A function $f : \square_r^\delta \to \mathbb{Z}$ is called *supermodular* if for any two elements $\underline{a}$ and $\underline{b}$, we have

$$f(\underline{a}) + f(\underline{b}) \leq f(\underline{a} \vee \underline{b}) + f(\underline{a} \wedge \underline{b}).$$

We will be interested in a special kind of supermodular function on $\square_r^\delta$.

Definition 2.1 (Rank function)**.** A function $\rho : \square_r^\delta \to \mathbb{Z}$ is called a *rank function* if it is supermodular and, in addition, satisfies the following conditions:

(1) The values of ρ are in the set $[r] \cup \{-1\}$.
(2) ρ is non-increasing with respect to the partial order of $\square_r^\delta$, i.e., if $\underline{a} \leq \underline{b}$, then $\rho(\underline{b}) \leq \rho(\underline{a})$.
(3) for any $1 \leq i \leq \delta$, and all $0 \leq t \leq r$, we have $\rho\left(t\,\underline{e}_i\right) = r - t$.

$\diamond$

Remark 2.2. The above properties imply that if $\underline{a} \in \square_r^\delta$ has rank j (i.e., $\rho(\underline{a}) = j$), then $a_i \leq r - j$ for all $1 \leq i \leq \delta$. In particular, the point $\underline{0}$ is the only element of $\square_r^\delta$ whose image by ρ is r. $\diamond$

Definition 2.3 (Ranked hypercube)**.** A hypercube $\square_r^\delta$ endowed with a rank function will be called a *ranked hypercube*. $\diamond$

Convention 2.4 (Cases $\delta = 1, 2$)**.** In this article, a function ρ on $\square_r^1$ will be often described by a vector with $r + 1$ entries $(x_0, \dots, x_r)$, which means that $\rho(i) = x_i$. In the same way, a function ρ on $\square_r^2$ will be often described by a square matrix of size $r + 1$, $(x_{ij})_{0 \leq i,j \leq r}$, which means that ρ takes value x_{ij} on (i, j). $\diamond$

Proposition 2.5. *Let ρ be a rank function on $\square_r^\delta$. Let $1 \leq i \leq \delta$. For all $\underline{a} \in \square_r^\delta$ such that $\underline{a} + \underline{e}_i \in \square_r^\delta$, we have*

$$\rho(\underline{a}) - 1 \leq \rho(\underline{a} + \underline{e}_i) \leq \rho(\underline{a}).$$

Proof. Let $\underline{b} = (a_i + 1)\,\underline{e}_i$, and note that $\underline{a} \vee \underline{b} = \underline{a} + \underline{e}_i$, and $\underline{a} \wedge \underline{b} = a_i\,\underline{e}_i$. Applying the supermodularity of ρ to the vectors $\underline{a}$ and $\underline{b}$, and using (3) in Definition 2.1, we get $\rho(\underline{a}) - 1 \leq \rho(\underline{a} + \underline{e}_i)$. The other inequality follows from the decreasing property of ρ. $\square$

The proposition leads to the following definition.

Definition 2.6 (Jumps of a rank function)**.** Let ρ be a rank function on $\square_r^\delta$. A point $\underline{a}$ is called a *jump* for ρ if

(1) $\rho(\underline{a}) \geq 0$, and
(2) for any index $1 \leq i \leq \delta$ such that $\underline{a} + \underline{e}_i$ belongs to $\square_r^\delta$, we have $\rho(\underline{a} + \underline{e}_i) = \rho(\underline{a}) - 1$.

The set of jumps of ρ is denoted by J_ρ. $\diamond$

We immediately get the following fact.

Fact 2.7. Suppose that $\underline{a} \leq \underline{b}$ are two different jumps of ρ. Then $\rho(\underline{a}) > \rho(\underline{b})$. $\diamond$

Proposition 2.8 (Stability of jumps under meet)**.** *The set of jumps J_ρ of a rank function ρ on $\square_r^\delta$ is stable under $\wedge$.*

Proof. Let $\underline{a}$ and $\underline{b}$ be elements of $\square_r^\delta$ and let $\underline{c} = \underline{a} \wedge \underline{b}$. Let $1 \le i \le \delta$ be such that $\underline{c} + \underline{e}_i$ belongs to $\square_r^\delta$. We want to show that $\rho(\underline{c} + \underline{e}_i) = \rho(\underline{c}) - 1$. Thanks to Proposition 2.5, it will be enough to show that $\rho(\underline{c} + \underline{e}_i) \le \rho(\underline{c}) - 1$.

Without loss of generality, we can suppose that $a_i \le b_i$. We now consider the elements $\underline{a}$ and $\underline{c} + \underline{e}_i$ in $\square_r^\delta$. We have $\underline{a} \vee (\underline{c} + \underline{e}_i) = \underline{a} + \underline{e}_i$ and $\underline{a} \wedge (\underline{c} + \underline{e}_i) = \underline{c}$. We then apply supermodularity to these two elements and get:

$$\rho(\underline{a}) + \rho(\underline{c} + \underline{e}_i) \le \rho(\underline{a} + \underline{e}_i) + \rho(\underline{c}).$$

Since $\underline{a} \in J_\rho$, the right-hand side is equal to $\rho(\underline{a}) - 1 + \rho(c)$, which proves the required inequality. We infer that $\underline{c} \in J_\rho$. $\square$

The following fact will be useful in the sequel.

Fact 2.9. Let ρ be a rank function on $\square_r^\delta$. Assume that $\underline{r} := (r, \dots, r) \in J_\rho$. Then $\underline{r}$ is the only element of J_ρ having some coordinate equal to r. $\diamond$

Proof. Suppose for the sake of a contradiction that $\underline{x} \ne \underline{r}$ is an element of J_ρ with some coordinate equal to r, say the first one. The inequality $\underline{x} \ge r\,\underline{e}_1$ implies that $\rho(\underline{x}) \le 0$. Let $1 \le i \le \delta$ such that $\underline{x} + \underline{e}_i \in \square_r^\delta$. Then, $\rho(\underline{r}) \le \rho(\underline{x} + \underline{e}_i) = \rho(\underline{x}) - 1 = -1$, which is a contradiction because by our assumption, $\underline{r}$ is a jump. $\square$

Some other elementary results about rank functions are given in Subsections 4.8 and 5.4, for application to crude linear series (see Definition 4.9) or linear series (see Definition 5.12).

2.1. **Standard rank functions.**

We now define a simple kind of rank functions which will be useful in the sequel.

Definition 2.10 (Standard rank function)**.** The *standard rank function* $\rho = \rho^{\mathrm{st}}$ of dimension δ and width r is given by

$$\rho(i_1, \dots, i_\delta) := \begin{cases} r - i_1 - \cdots i_\delta & \text{if } i_1 + \cdots + i_\delta \le r \\ -1 & \text{else} \end{cases},$$

for $(i_1, \dots, i_\delta) \in \square_r^\delta$. $\diamond$

For example, if $\delta = 1$, ρ is given by the vector

$$(r, r-1, r-2, \dots, 1, 0),$$

and, if $\delta = 2$, ρ is given by the matrix

$$\begin{pmatrix} r & r-1 & r-2 & \cdots & 1 & 0 \\ r-1 & r-2 & \cdots & 1 & 0 & -1 \\ r-2 & \vdots & 1 & \iddots & \iddots & \vdots \\ \vdots & 1 & \iddots & \iddots & & \vdots \\ 1 & 0 & \iddots & & & \vdots \\ 0 & -1 & \cdots & \cdots & \cdots & -1 \end{pmatrix}.$$

Proposition 2.11. *Any rank function ρ on the hypercube $\square_r^\delta$ dominates the standard rank function ρ^{st}. That is, for any $\underline{x} \in \square_r^\delta$, we have $\rho(\underline{x}) \ge \rho^{\mathrm{st}}(\underline{x})$.*

Proof. It will be enough to show that for any $\underline{i} = (i_1, \dots, i_\delta) \in \square_r^\delta$, we have $\rho(\underline{i}) \ge r - i_1 - \cdots i_\delta$. This can be proved by induction on δ, using the supermodularity of ρ. $\square$

2.2. **Rank functions induced by complete flags.** Let r be a non-negative integer, and let H be a vector space of dimension $r+1$ over some field κ. A complete flag of H consists of a chain of vector subspaces

$$H = F^0 \supsetneq F^1 \supsetneq \dots \supsetneq F^{r-1} \supsetneq F^r \supsetneq (0),$$

where for each $i \in [r]$, F^i is a vector subspace of codimension i in H. Complete flags of H are points of a smooth projective variety $\mathfrak{Fl}(H)$ of dimension $\binom{r+1}{2}$ called the *flag variety*.

Let δ be a positive integer, and let $F_1^\bullet, \dots, F_\delta^\bullet$ be a collection of δ complete flags of H. Define the function $\rho : \square_r^\delta \to \mathbb{Z}$ by

(1) $$\rho(a_1, \dots, a_\delta) := \dim_\kappa \left(F_1^{a_1} \cap \dots \cap F_\delta^{a_\delta} \right) - 1.$$

Proposition 2.12. *The function ρ defined in Equation 1 is a rank function on $\square_r^\delta$.*

Proof. Let $\underline{a}$ and $\underline{b}$ be two points $\square_r^\delta$, and let $\underline{m} = \underline{a} \wedge \underline{b}$ and $\underline{M} = \underline{a} \vee \underline{b}$. We have an injection

$$F_1^{a_1} \cap \dots \cap F_\delta^{a_\delta} / F_1^{M_1} \cap \dots \cap F_\delta^{M_\delta} \hookrightarrow F_1^{m_1} \cap \dots \cap F_\delta^{m_\delta} / F_1^{b_1} \cap \dots \cap F_\delta^{b_\delta},$$

from which, comparing the dimensions, we get the inequality

$$\rho(\underline{a}) - \rho(\underline{M}) \leq \rho(\underline{m}) - \rho(\underline{b}).$$

This proves the supermodularity of ρ. Properties (1), (2), (3) in the definition of a rank function 2.1 are trivially verified. $\square$

Remark 2.13. The standard rank function is induced by complete flags which are in *general relative position*. That is, by complete flags whose intersection patterns have the smallest possible dimension. ◇

Definition 2.14 (Realizable rank functions)**.** A rank function ρ on $\square_r^\delta$ is called *realizable* over a field κ if it is the rank function associated to a collection of δ complete flags $F_1^\bullet, \dots, F_\delta^\bullet$ on some vector space E of dimension $r+1$ over κ. ◇

We will discuss in the next section obstructions to the realizability of rank functions.

2.3. **Geometric rank functions.** We now describe a geometric situation which naturally leads to rank functions.

Let κ be an algebraically closed field, and let C be a smooth proper curve over κ. Let r be a non-negative integer, and let x be a κ-point on C. Let $\kappa(C)$ be the function field of C, and let $H \subset \kappa(C)$ be a vector subspace of rational functions of dimension $r+1$ over κ. The point x leads to a complete flag $F_x^\bullet$ of H by looking at the orders of vanishing of functions in H at the point x. Define the set $S_x := \{\mathrm{ord}_x(f) \,|\, f \in H \setminus \{0\}\}$. We have the following basic result.

Proposition 2.15. *The set S_x is finite of cardinality $r+1$.*

Denote by $s_0^x < \dots < s_r^x$ the elements of S_x, enumerated in increasing order. The flag $F_x^\bullet$ is defined by setting, for $j = 0, \dots, r$,

$$F_x^j := \left\{ f \in H - \{0\} \,|\, \mathrm{ord}_x(f) \geq s_j^x \right\} \cup \{0\}.$$

The fact that F_x^j has codimension j in H follows from the previous proposition.

Let now δ be a natural number, and let $A = \{x_1, \dots, x_\delta\}$ be a collection of δ distinct κ-points on C. By the construction above, each point x_i leads to a complete flag $F_i^\bullet$. Denoting $S_i := \{\mathrm{ord}_{x_i}(f) \,|\, f \in H - \{0\}\}$, and enumerating the elements of S_i in increasing order $s_0^i < \dots < s_r^i$, the flag $F_i^\bullet$ is defined by setting

$$F_i^j := \left\{ f \in H - \{0\} \,|\, \mathrm{ord}_{x_i}(f) \geq s_j^i \right\} \cup \{0\}.$$

This leads to a rank function ρ on the hypercube $\square_r^\delta$ using Equation 1.

We say a rank function ρ is geometric if it arises from the above construction for a curve C over over an algebraically closed field κ. Obviously, geometric rank functions are all realizable (over the same field). We end this section with the following interesting question.

Question 2.16 (Geometric origin of realizable rank functions). *Is it true that all realizable rank functions are geometric? What is the smallest possible genus of a curve realizing ρ?*

3. Permutation arrays, rank functions and realizability

The aim of this section is to provide a comparison of the definitions in the previous section with the Eriksson-Linusson setting of permutation arrays [EL00a, EL00b], introduced as a combinatorial setup in the study of the intersection patterns of complete flags. At the same time it provides interesting properties of hypercube rank functions.

We however note that it is somehow independent of the rest of the paper, and can be omitted at first reading.

3.1. Permutation arrays.

First we recall some terminology from [EL00a]. Note that the presentation below is slightly different as our indexing of complete flags is by codimension while in the Eriksson-Linusson setting the indexing is by dimension. (Concretely, this amounts to having lower blocks in [EL00a, EL00b] replaced here by upper blocks.)

Let $r_1, \dots, r_\delta$ be δ non-negative integers. A *δ-dimensional dot array* P is a δ-dimensional $[r_1] \times \cdots \times [r_\delta]$-type array where some of the entries are dotted. For $1 \le i \le \delta$ and $0 \le t \le r_i$, we denote by L^i_t the *t-th layer in the direction i*, namely $L^i_t \coloneqq \{\underline{x} \in P, x_i = t\}$.

For a dot array P, and $\underline{x} \in [r_1] \times \cdots \times [r_\delta]$, we denote by $P[\underline{x}]$ the upper principal subarray of P consisting of all $\underline{y}$ with $\underline{y} \ge \underline{x}$. To be precise, for $P[\underline{x}]$ to become a dot array, we must coordinate-wise substract the point $(x_1, \dots, x_\delta)$ to all its elements. In the following, we will use both parametrization conventions freely for the sake of convenience.

For a dot array P and $1 \le j \le \delta$, the *rank along the j-axis*, denoted by $\mathrm{rank}_j(P)$, is the total number of $0 \le t \le r_j$ such that there is at least one dot in some position whose j-th index is equal to t. A dot array P is called *rankable* if we have $\mathrm{rank}_j(P) = \mathrm{rank}_i(P)$ for all $1 \le i, j \le \delta$. If P is rankable, then we call $\mathrm{rank}_j(P)$ the *rank* of P for any $1 \le j \le \delta$.

A dot array P is called *totally rankable* if any upper principal subarray of P is rankable.

We recall that in the terminology of [EL00a] and [EL00b], a position $\underline{x}$ is *redundant* if there exist dot positions $\underline{y}_1, \dots, \underline{y}_m \neq \underline{x}$ for some $m \ge 2$, such that each $\underline{y}_i$ has at least one coordinate in common with $\underline{x}$, and such that $\underline{x} = \bigwedge_{i=1}^m \underline{y}_i$. The set of redundant positions of P is denoted by $R(P)$. A *redundant dot* is a redundant position that is dotted. The reason for the term "redundant" is that placing or removing a redundant dot does not change the rank of any principal subarray of P. If A is a subset of $[r]^\delta$, then $P \cup A$ (resp. $P \setminus A$) denotes the dot array based on P where, for every $\underline{x} \in A$, we dot (resp. undot) the position $\underline{x}$ in P, if necessary.

A *permutation array* of width r and dimension δ is a totally rankable dot array of shape $[r]^\delta$, with rank $r+1$, and with no redundant dots.

3.2. Weak rank functions and supermodularity.

In the sequel, we will prove a theorem stating an equivalence between permutation arrays and rank functions. To this end, we introduce here a weaker version of rank functions, following the axioms used in course of proof of [EL00a, Proposition 4.3] up to some changes in the conventions, to our setting.

Definition 3.1 (Weak supermodularity). A function ρ on $\square^\delta_r$ is said to be a *weak rank function in dimension one and distance one* if ρ satisfies properties (1), (2) and (3) in Definition 2.1, and moreover, it satisfies the following property that we call *weak supermodularity in dimension one and distance one*:

For all $1 \leq i \leq \delta$ and for all points $\underline{x} \leq \underline{y} \in \square_r^\delta$ such that $\underline{x} + \underline{e}_i \in \square_r^\delta$ and $x_i = y_i$, we have

$$\rho(\underline{x}) - \rho(\underline{x} + \underline{e}_i) \geq \rho\left(\underline{y}\right) - \rho\left(\underline{y} + \underline{e}_i\right). \tag{2}$$

◇

There is a useful generalization of this in *higher dimension* and at *higher distance*.

Definition 3.2 (Generalized weak supermodularity)**.** For positive integers k and n, we define the property $(*)_k^n$ called *weak supermodularity in dimension k and distance n* as follows.

$(*)_k^n$: Pick any integer $1 \leq s \leq k$, any integers $1 \leq i_1 < \cdots < i_s \leq \delta$ and $0 \leq n_{i_1}, \ldots, n_{i_s} \leq n$. Then, for any pair of elements $\underline{x} \leq \underline{y} \in \square_r^\delta$ such that $x_{i_j} = y_{i_j}$ for all $1 \leq j \leq s$, and $\underline{x} + \sum_{1 \leq j \leq s} n_j \underline{e}_{i_j} \in \square_r^\delta$, we have

$$\rho(\underline{x}) - \rho\left(\underline{x} + \sum_{1 \leq k \leq s} n_{i_j} \underline{e}_{i_j}\right) \geq \rho\left(\underline{y}\right) - \rho\left(\underline{y} + \sum_{1 \leq j \leq s} n_{i_j} \underline{e}_{i_j}\right).$$

◇

Remark 3.3. Notice that the property stated in Inequality (2) in Definition 3.1 is exactly $(*)_1^1$ as defined above. So our notations are consistent.

In addition, notice that any $(*)_k^n$ with $k, n \geq 1$ implies $(*)_1^1$. Although it might appear surprising at first sight, we will show in Theorem 3.6 that the converse is also true. ◇

Remark 3.4 (Simplified version of weak supermodularity)**.** Property $(*)_1^1$ is implied by the following *simplified weak supermodularity property*: for all point $\underline{x} \in \square_r^\delta$ and integers $1 \leq i, j \leq \delta$ such that $\underline{x} + \underline{e}_i, \underline{x} + \underline{e}_j \in \square_r^\delta$ and $i \neq j$, we have

$$\rho(\underline{x}) - \rho(\underline{x} + \underline{e}_i) \geq \rho\left(\underline{x} + \underline{e}_j\right) - \rho\left(\underline{x} + \underline{e}_i + \underline{e}_j\right). \tag{3}$$

The fact that $x_i = y_i$ like in Definition 3.1 implies that $\underline{y}$ can be written as $\underline{y} = \underline{x} + \sum_{j \neq i} n_j \underline{e}_j$ with $n_j \geq 0$, and we can sum inequalities of the form (3) to get Inequality (2). ◇

Remark 3.5 (Alternative description of $(*)_k^n$)**.** Using the notations of Definition 3.2, after the change of variables

$$\underline{a} := \underline{x}, \quad \underline{b} := \underline{y} - \sum_{1 \leq j \leq s} n_{i_j} \underline{e}_{i_j},$$

property $(*)_k^n$ can be rewritten as follows.

For all elements $\underline{a}$ and $\underline{b} \in \square_r^\delta$, we have the supermodularity relation

$$\rho(\underline{a}) + \rho(\underline{b}) \geq \rho(\underline{a} \vee \underline{b}) + \rho(\underline{a} \wedge \underline{b})$$

as long as there exists an integer $1 \leq s \leq k$ and integers $1 \leq i_1 < \cdots < i_s \leq \delta$ such that $\underline{b} + \sum_{1 \leq j \leq s} (a_{i_j} - b_{i_j}) \underline{e}_{i_j}$ is an element of $\square_r^\delta$ greater than or equal to $\underline{a}$ and, that, for all $1 \leq j \leq s$, we have $0 \leq a_{i_j} - b_{i_j} \leq n$.

This parametrization using $\underline{a}$ and $\underline{b}$ enables us to see instantaneously that the supermodularity property in the hypercube implies all the properties $(*)_k^n$. The other parametrization, using $\underline{x}$ and $\underline{y}$, will be useful to prove Theorem 3.6 below, in that it behaves linearly (contrary to formulas involving the symbols $\wedge$ and $\vee$). ◇

Theorem 3.6 (Equivalence of supermodularity and weak supermodularity)**.** *Property $(*)_1^1$ implies $(*)_k^n$ for all $k, n \geq 1$.*

Proof. We first show that $(*)_1^1$ implies $(*)_1^n$. Let $1 \leq i \leq \delta$ and $0 \leq n_i \leq n$, and let $\underline{x} \leq \underline{y}$ be elements of $\square_r^\delta$ such that $\underline{x} + n_i\, \underline{e}_i \in \square_r^\delta$ and $x_i = y_i$.

For all $0 \le t < n_i$, the pair $(\underline{x} + t\,\underline{e}_i, \underline{y} + t\,\underline{e}_i)$ satisfies the hypotheses needed to apply $(*)_1^1$ in direction i, so we know that

$$\rho(\underline{x} + t\,\underline{e}_i) - \rho(\underline{x} + (t+1)\,\underline{e}_i) \ge \rho\left(\underline{y} + t\,\underline{e}_i\right) - \rho\left(\underline{y} + (t+1)\,\underline{e}_i\right).$$

Summing all these inequalities for $0 \le t < n_i$ cancels out almost all terms and yields

$$\rho(\underline{x}) - \rho(\underline{x} + n_i\,\underline{e}_i) \ge \rho\left(\underline{y}\right) - \rho\left(\underline{y} + n_i\,\underline{e}_i\right),$$

which gives $(*)_1^n$.

We now show that properties $(*)_1^n$ for $n \ge 1$ imply properties $(*)_2^n$. Let $1 \le i, j \le \delta$ and $0 \le n_i, n_j \le n$, and let $\underline{x} \le \underline{y}$ be elements of $\square_r^\delta$ such that $\underline{x} + n_i\,\underline{e}_i + n_j\,\underline{e}_j \in \square_r^\delta$, $x_i = y_i$ and $x_j = y_j$.

We can apply $(*)_1^n$ to the pair $(\underline{x}, \underline{y})$ in direction i and get

$$\rho(\underline{x}) - \rho(\underline{x} + n_i\,\underline{e}_i) \ge \rho\left(\underline{y}\right) - \rho\left(\underline{y} + n_i\,\underline{e}_i\right).$$

Now the pair $(\underline{x} + n_i\,\underline{e}_i, \underline{y} + n_i\,\underline{e}_i)$ satisfies the hypotheses needed to apply $(*)_1^n$ again, but this time in direction j, which yields

$$\rho(\underline{x} + n_i\,\underline{e}_i) - \rho(\underline{x} + n_i\,\underline{e}_i + n_j\,\underline{e}_j) \ge \rho\left(\underline{y} + n_i\,\underline{e}_i\right) - \rho\left(\underline{y} + n_i\,\underline{e}_i + n_j\,\underline{e}_j\right).$$

Summing up these two inequalities shows that ρ satisfies $(*)_2^n$, and the same procedure inductively proves that ρ satisfies all $(*)_k^n$, i.e., ρ is supermodular in the strong sense. ☐

Remark 3.7 (Discrete partial derivatives and transverse local concavity)**.** For $1 \le i \le \delta$, we can define the *discrete partial derivative of ρ in the direction i* as the function $\partial_i \rho$ defined by

$$\partial_i \rho(\underline{x}) \coloneqq \rho(\underline{x}) - \rho(\underline{x} + \underline{e}_i), \qquad x \in \square_r^\delta.$$

We notice that property $(*)_1^1$ is equivalent to the fact that for all $1 \le i \le \delta$ and for all $0 \le t \le r$, $\partial_i \rho|_{L_t^i}$ is non-increasing. This is why $(*)_1^1$ may be alternatively called *transverse local concavity.* Supermodularity is thus equivalent to transverse local concavity. ⋄

Remark 3.8. In other contexts, supermodularity is sometimes referred to as the discrete analogue of convexity (or concavity according to the conventions of this paper): see, for example, [Sch03, Theorem 44.1]. While this is fully relevant for a function ρ defined on the collection $\mathcal{P}(S)$ of all subsets of a given set S, that is on the hypercube $\square_1^\delta$, it is not exactly true for supermodular functions on $\square_r^\delta$ for larger values of r. This is because the functions $\partial_i \rho$ are non-increasing only on directions different from i. In the following example of a rank function defined on $\square_2^2$, we have $\rho((2,0)) - \rho((2,1)) \not\ge \rho((2,1)) - \rho((2,2))$, i.e., $\partial_2 \rho$ is not non-increasing in the direction 2.

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

⋄

3.3. **Equivalence of rank functions with permutation arrays.** We now state the theorem which shows an equivalence between permutation arrays of width r and dimension δ, and rank functions on the hypercube $\square_r^\delta$.

Theorem 3.9. *Let P be a permutation array. Then the function ρ_P defined by $\rho_P(\underline{a}) \coloneqq \operatorname{rank}(P[\underline{a}]) - 1$ for any $\underline{a} \in [r]^\delta$ is a rank function according to Definition 2.1, and the set of jumps of ρ_P is precisely the set of dot positions of $P \cup R(P)$.*

Conversely, any rank function ρ on the hypercube $\square_r^\delta$ defines a dot array P' with dots positioned on the set of jumps of ρ. Then $P_\rho \coloneqq P' \setminus R(P')$ is a permutation array.

Proof. The theorem follows from the equivalence stated in Proposition 4.3 in [EL00a] between weak rank functions and permutation arrays, and our theorem above establishing an equivalence between weak rank functions and rank functions. Note however that some changes in the conventions in [EL00a] are needed. For the sake of completeness, and since, to the best of our understanding, some non-trivial arguments are lacking in the proof of the cited proposition, we provide a proof of this theorem in Appendix A. □

3.4. **Coherent complexes of matroids and realizability of rank functions.** Let κ be a field.

By Theorem 3.9, the realizability of a rank function $\square_r^\delta$ is equivalent to the realizability of the corresponding permutation array in the terminology of [EL00b]. Billey and Vakil [BV08] provided several examples which show the existence of non-realizable permutation arrays. In this section, we formulate a local obstruction for the realizabiliy of rank functions, which cover all the examples in [BV08].

In the following, for all $\underline{a} \in \square_r^\delta$, we define $I_{\underline{a}}$ as the set of all $1 \leq i \leq \delta$ such that $\underline{a}+\underline{e}_i \in \square_r^\delta$. Consider first a realizable rank function, given by δ complete flags $F_1^\bullet, \dots, F_\delta^\bullet$. Let

$$F_{\underline{a}} \coloneqq \bigcap_{i \in I_{\underline{a}}} F_i^{a_i}.$$

Let $J_{\underline{a}}$ be the set of $i \in I_{\underline{a}}$ such that $F_{\underline{a}+\underline{e}_i}$ is proper inside $F_{\underline{a}}$, and thus, is a hyperplane of $F_{\underline{a}}$. The arrangement of hyperplanes in $F_{\underline{a}}$ given by $J_{\underline{a}}$ defines a matroid $M_{\underline{a}}$ on the ground set $I_{\underline{a}}$ whose rank one elements are precisely given by $J_{\underline{a}}$ and the elements in $I_{\underline{a}} \setminus J_{\underline{a}}$ are all loops. For the terminology regarding matroids we refer to [Oxl06].

We now explain how to generalize the above picture to any rank function and define local matroids associated to elements of the hypercube. Let ρ be a rank function on $\square_r^\delta$. Let $\underline{a} \in \square_r^\delta$, and define a function $\rho_{\underline{a}} \colon 2^{I_{\underline{a}}} \to \mathbb{N}$ as follows. For any subset $X \subset I_{\underline{a}}$, set

$$\rho_{\underline{a}}(X) \coloneqq \rho(\underline{a}) - \rho\left(\underline{a} + \sum_{i \in X} \underline{e}_i\right)$$

Proposition 3.10. *With notations as above, the pair $(I_{\underline{a}}, \rho_{\underline{a}})$ defines a matroid $M_{\underline{a}}$ on the set of elements $I_{\underline{a}}$.*

Proof. Obviously, $\rho_{\underline{a}}$ takes values in the set $\left\{0, \dots, \left|I_{\underline{a}}\right|\right\}$. So it will be enough to show that ρ_a is submodular, in the sense that

$$\forall\, X, Y \subseteq I_{\underline{a}} \quad \rho_{\underline{a}}(X \cup Y) + \rho_{\underline{a}}(X \cap Y) \leq \rho_{\underline{a}}(X) + \rho_{\underline{a}}(Y).$$

This follows from the supermodularity of ρ applied to $\underline{b} = \underline{a} + \sum_{i \in X} \underline{e}_i$ and $\underline{c} = \underline{a} + \sum_{i \in Y} \underline{e}_i$. □

For the sake of convenience, if $i \in I_{\underline{a}}$, we sometimes write $\rho_{\underline{a}}(i)$ or $\rho_{\underline{a}}(\underline{e}_i)$ instead of $\rho_{\underline{a}}(\{i\})$. We will now see that the data of a rank function of $\square_r^\delta$ is equivalent to the data of a set of matroids indexed by $\square_r^\delta$ satisfying certain properties. We start with the definitions below.

Definition 3.11 (Increasing path)**.** Let $\underline{a}$ and $\underline{b}$ be points of $\square_r^\delta$ such that $\underline{a} \leq \underline{b}$. We define an *increasing path* from $\underline{a}$ to $\underline{b}$ to be any finite sequence

$$\underline{a} = \underline{c}_0, \underline{c}_1, \dots, \underline{c}_k = \underline{b}$$

such that for all $0 \leq j < k$, we have $\underline{c}_{j+1} = \underline{c}_j + e_\ell$ for some $1 \leq \ell \leq \delta$. k is entirely determined by $\underline{a}$ and $\underline{b}$ and is equal to $\sum_{i=1}^{\delta}(b_i - a_i)$. ⋄

Definition 3.12 (Coherent complex of matroids)**.** Let $\left(M_{\underline{a}}\right)_{\underline{a}}$ be a set of matroids indexed by $\square_r^\delta$, with $M_{\underline{a}}$ a matroid on the set $I_{\underline{a}}$ and with rank function $\rho_{\underline{a}}$. We call $\left(M_{\underline{a}}\right)$ a *coherent complex of matroids* if the following three conditions are met:

(i) For all $1 \leq i \leq \delta$ and $0 \leq t < r$, we have $\rho_{t\underline{e}_i}(\underline{e}_i) = 1$.
(ii) The matroids satisfy the following relation.

$$M_{\underline{a}+\underline{e}_i} = \begin{cases} M_{\underline{a}}/i & \text{if } a_i = r-1 \\ M_{\underline{a}}/i \sqcup \{i\} & \text{else} \end{cases}.$$

(iii) For every increasing path $\underline{0} = \underline{c}_0, \underline{c}_2, \dots, \underline{c}_k = \underline{r} = (r, \dots, r)$, from $\underline{0}$ to $\underline{r}$ (note that $k = r\delta$), we have

$$\sum_{j=0}^{k-1} \rho_{\underline{c}_j}\left(\underline{c}_{j+1} - \underline{c}_j\right) \leq r+1.$$

Here, M/e denotes the contraction of a matroid M by its element e, and i is the element of the matroid set corresponding to the direction i. Moreover, $M_{\underline{a}}/i \sqcup \{i\}$ denotes an extension of $M_{\underline{a}}/i$ by a single element denoted i. ◇

In the following, we denote by $\rho_{\underline{a}/i}$ the rank function on $I_{\underline{a}} \setminus \{i\}$ defining the matroid $M_{\underline{a}/i}$. We recall this is given by the following equation in terms of the rank function $\rho_{\underline{a}}$:

$$\rho_{\underline{a}/i}(X) = \rho_{\underline{a}}(X \cup \{i\}) - \rho_{\underline{a}}(\{i\}) \qquad \text{for all } X \subset I_{\underline{a}} \setminus \{i\}.$$

Remark 3.13. Property (ii) above implies the following property: let $\underline{x} \leq \underline{y}$ be two points of $\square_r^\delta$ and $1 \leq i \leq \delta$ such that $x_i = y_i$ and $i \in I_{\underline{x}}$. Then the fact that i is a loop in $M_{\underline{x}}$ implies that it is a loop in $M_{\underline{y}}$. Indeed, $M_{\underline{y}}$ is obtained from $M_{\underline{x}}$ through contractions of elements different from i and extensions. Those operations do not change the fact that i is a loop. ◇

We now discuss how one can associate a coherent complex of matroids to a rank function on the hypercube, and vice-versa; and we show that these constructions yield a one-to-one correspondence between both objects.

Proposition 3.14. *There is a one-to-one correspondence between coherent complexes of matroids indexed by the hypercube and rank functions on the hypercube.*

Proof. $\Longleftarrow$ If we start with a rank function ρ on $\square_r^\delta$, the collection of matroids $M_{\underline{a}}$ defined above forms a coherent complex of matroids. Indeed, condition (i) is of course satisfied because of (3) in Definition 2.1. We now check condition (ii). Let $\underline{a} \in \square_r^\delta$ and $i \in I_{\underline{a}}$. If $a_i = r-1$, then $I_{\underline{a}+\underline{e}_i} = I_{\underline{a}} \setminus \{i\}$, which is the ground set of the matroid $M_{\underline{a}}/i$. If $a_i < r-1$, $I_{\underline{a}+\underline{e}_i} = I_{\underline{a}}$, which is the ground set of the matroid $M_{\underline{a}}/i \sqcup \{i\}$. To finish, we have to check equality of the rank functions of the matroids on subsets of $I_{\underline{a}} \setminus \{i\}$. That is, it is sufficient to consider $X \subset I_{\underline{a}+\underline{e}_i}$ not containing the element i, and show that

$$\rho_{\underline{a}+\underline{e}_i}(X) = \rho_{\underline{a}}(X \cup \{i\}) - \rho_{\underline{a}}(\{i\}).$$

But the left-hand side is by definition $\rho(\underline{a}+\underline{e}_i) - \rho\left(\underline{a}+\underline{e}_i + \sum_{j\in X} \underline{e}_j\right)$, and the right-hand side is

$$\rho(\underline{a}) - \rho\left(\underline{a} + \underline{e}_i + \sum_{j\in X} \underline{e}_j\right) - \rho(\underline{a}) + \rho(\underline{a}+\underline{e}_i),$$

so both sides are equal.

To prove (iii), we notice that if $\underline{0} = \underline{c}_0, \underline{c}_2, \dots, \underline{c}_k = ((r, \dots, r))$ is an increasing path from $\underline{0}$ to $((r, \dots, r))$, then

$$\sum_{j=0}^{k-1} \rho_{\underline{c}_j}\left(\underline{c}_{j+1} - \underline{c}_j\right) = \rho(\underline{0}) - \rho((r, \dots, r)) \in \{r, r+1\}.$$

Remember that the supermodularity of ρ is necessary for the $M_{\underline{a}}$ to be matroids (see Proposition 3.10).

$\Longrightarrow$ The other way around, we consider a coherent complex of matroids $(M_{\underline{a}})_{\underline{a}}$ and associate a rank function ρ on $\square_r^\delta$. Let $\underline{a} \in \square_r^\delta$. We take any increasing path

$$\underline{0} = \underline{b}_0, \underline{b}_1, \dots, \underline{b}_k = \underline{a}$$

from $\underline{0}$ to $\underline{a}$ (see Definition 3.11), and define

$$\rho(\underline{a}) \coloneqq r - \sum_{j=0}^{k-1} \rho_{\underline{b}_j}\left(\underline{b}_{j+1} - \underline{b}_j\right).$$

We first prove that ρ is well-defined, which amounts to showing that $\rho(\underline{a})$ does not depend on the path $(\underline{b}_j)$ chosen. Two different such paths can be linked by a finite sequence of paths such that between two consecutive paths in this sequence, the only change is an inversion between two consecutive elementary moves $\underline{e}_i$ and $\underline{e}_j$, $i \neq j$. We thus have to check that, for any $\underline{a} \in \square_r^\delta$ and $i, j \in I_{\underline{a}}$ with $i \neq j$, we have

$$\rho_{\underline{a}}(\underline{e}_i) + \rho_{\underline{a}+\underline{e}_i}(\underline{e}_j) = \rho_{\underline{a}}(\underline{e}_j) + \rho_{\underline{a}+\underline{e}_j}(\underline{e}_i).$$

But by point (ii) in Definition 3.12, $\rho_{\underline{a}+\underline{e}_i}(\underline{e}_j) = \rho_{\underline{a}/i}(\underline{e}_j) = \rho_{\underline{a}}(\underline{e}_j + \underline{e}_i) - \rho_{\underline{a}}(\underline{e}_i)$ in the left-hand part and $\rho_{\underline{a}+\underline{e}_j}(\underline{e}_i) = \rho_{\underline{a}/j}(\underline{e}_i) = \rho_{\underline{a}}(\underline{e}_i + \underline{e}_j) - \rho_{\underline{a}}(\underline{e}_j)$ in the right-hand part, so the desired equality holds.

We now check the properties in Definition 2.1 and the supermodularity of ρ. It is obvious by construction that ρ takes values $\leq r$, is non-increasing and that $\rho(t\underline{e}_i) = r - t$ for all $1 \leq i \leq \delta$ and $0 \leq t \leq r$. The fact that $\rho((r, \dots, r)) \geq -1$ is implied by property (iii) in Definition 3.12.

To finish, we show that ρ is supermodular. By Theorem 3.6, it is sufficient to check property $(*)_1^1$. Let thus $1 \leq i \leq \delta$ and $\underline{x}, \underline{y} \in \square_r^\delta$ such that $\underline{x} \leq \underline{y}$, $i \in I_{\underline{x}}$ and $x_i = y_i$. We assume that $\rho_{\underline{x}}(\underline{e}_i) = 0$ and show that $\rho_{\underline{y}}(\underline{e}_i) = 0$. This has been shown to be true in Remark 3.13.

Now that we have defined two maps linking coherent complex of matroids and rank functions, it is straightforward to check that they are inverse of each other, for instance by induction on the sum of coordinates of the points of $\square_r^\delta$, and using recursively the formula

$$\rho_M(X) = \rho_M(\{e\}) + \rho_{M/e}(X \setminus \{e\})$$

for any matroid M on a set E of rank function ρ and any set $X \subset E$ containing e. □

Proposition 3.15. *A necessary condition for the realizability of a rank function ρ on $\square_r^\delta$ is the realizability of all the matroids $M_{\underline{a}}$ for all $\underline{a} \in \square_r^\delta$.*

Proof. This follows directly from the above discussions. □

The local obstructions given by Proposition 3.15 are probably not the only obstructions for the realizability of a rank function, so we formulate the following interesting open question.

Question 3.16 (Realizability of rank functions)**.** *Is it possible to formulate the realizability of a rank function ρ in terms of (joint) matroid realizability for a certain collection of matroids?*

4. Slope structures

In this section and the following one, we define combinatorial linear series on metric graphs with the help of an auxiliary data called a *slope structure*. Slope structures provide a parametrized version of rank functions of order r, verifying a finiteness condition.

4.1. **Slope structures on graphs.** Let first $G = (V, E)$ be a simple graph. We denote by $\mathbb{E}$ the set of all the orientations of edges of G, so that for an edge $\{u, v\}$ in E, we have two orientations $uv, vu \in \mathbb{E}$. For an oriented edge $e = uv \in \mathbb{E}$, we call u the tail and v the head of e. We denote by $\overline{e} = vu$ the oriented edge of $\mathbb{E}$ with reverse orientation. For a vertex $v \in V$, we denote by $\mathbb{E}_v \subset \mathbb{E}$ the set of all the oriented edges which have tail v, that is, all $vu \in \mathbb{E}$ for edges $\{v, u\} \in E$.

A *slope structure* $\mathfrak{S} = \{\, S^v; S^e \,\}_{v\in V, e\in\mathbb{E}}$ of *order* r on G, or simply an r*-slope structure*, is the data of

- For any oriented edge $e = uv \in \mathbb{E}$ of G, a collection S^e of $r + 1$ integers $s^e_0 < s^e_1 < \cdots < s^e_r$, subject to the requirement that $s^{uv}_i + s^{vu}_{r-i} = 0$ for any edge $\{u, v\} \in E$.
- For any vertex v of G, a rank function ρ_v on the hypercube $\square^{d_v}_r$. If J_{ρ_v} denotes the set of jumps of ρ_v (see Definition 2.6), we denote by $S^v \subseteq \prod_{e\in\mathbb{E}_v} S^e$ the set of all points $s_{\underline{a}}$ for $\underline{a} \in J_{\rho_v}$.

  Here, for a point $\underline{a} = (a_e)_{e\in\mathbb{E}_v}$ of the hypercube, the element $s_{\underline{a}} \in \prod_{e\in\mathbb{E}_v} S^e$ denotes the point in the product which has coordinate at $e \in \mathbb{E}_v$ equal to $s^e_{a_e}$. In other words, S^v fits into the following natural commutative diagram.

$$\begin{array}{ccc} J_{\rho_v} & \hookrightarrow & \square^{d_v}_r \\ \downarrow{\scriptstyle \underline{a}\mapsto s_{\underline{a}}} & & \downarrow{\scriptstyle \underline{a}\mapsto s_{\underline{a}}} \\ S^v & \hookrightarrow & \Pi_{e\in\mathbb{E}_v} S^e \end{array}$$

We will sometimes need to separate these two kinds of data. In this case, we will denote by $\mathfrak{S}^{\mathrm{e}}$ the data of a set of prescribed slopes on each edge, and by $\mathfrak{S}^{\mathrm{v}}$ the data of a rank function for each vertex. In this paper, we will mainly use slope structures of the above form $\mathfrak{S}$ (with prescribing data on both edges and vertices). Omitting the part concerning the vertices, that is, considering only $\mathfrak{S}^{\mathrm{e}}$ with prescribed data on edges, we obtain what we call an *e-slope structure.*

4.2. **Slope structures on metric graphs.** Let now Γ be a metric graph. By an r*-slope structure* on Γ we mean an r-slope structure $\mathfrak{S}$ on a simple graph model $G = (V, E)$ of Γ, that we enrich by extending to any point of Γ as follows.

For any point x and $\nu \in \mathrm{T}_x(\Gamma)$, there exists a unique oriented edge uv of G which is parallel to ν. Define $S^\nu = S^{uv}$. Also for any point $x \in \Gamma \setminus V$ in the interior of an edge $\{u, v\}$, define ρ_x to be the standard rank function on $\square^2_r$. In particular, $S^x \subseteq S^{uv} \times S^{vu}$ can be identified with the set of all pairs (s^{uv}_i, s^{vu}_j) with $i+j \leq r$. We call the collection $\big\{\, S^x; S^\nu \;\big|\; x \in \Gamma, \nu \in \mathrm{T}_x(\Gamma)\big\}$ a *slope structure of order* r on Γ that we denote by $\mathfrak{S}_\Gamma$, or simply $\mathfrak{S}$, if there is no risk of confusion. We extend the notations $\mathfrak{S}^{\mathrm{e}}$ and $\mathfrak{S}^{\mathrm{v}}$ in the natural way. Note that a slope structure on a metric graph can arise from choices of slope structures on different graph models of Γ.

4.3. **Rational functions compatible with a slope structure.** We now define a notion of rational function on a metric graph which takes into account the choice a slope structure.

Let Γ be a metric graph and let $\mathfrak{S} = \big\{S^x; S^\nu \;\big|\; x \in \Gamma, \nu \in \mathrm{T}_x(\Gamma)\big\} = \big(\mathfrak{S}^{\mathrm{e}}, \mathfrak{S}^{\mathrm{v}}\big)$ be a slope structure of order r on Γ. Recall that we denote by $\mathrm{Rat}(\Gamma)$ the set of continuous piecewise affine functions $f : \Gamma \to \mathbb{R}$ with integral slopes.

A function f in $\mathrm{Rat}(\Gamma)$ is said to be *compatible* with $\mathfrak{S}$ if the two conditions (i) and (ii) below are verified:

(i) for any point $x \in \Gamma$ and any tangent direction $\nu \in \mathrm{T}_x(\Gamma)$, the outgoing slope of f along ν lies in S^ν.

Denote by $\delta_x(f)$ the vector in $\prod_{\nu\in\mathrm{T}_x(\Gamma)} S^\nu$ which consists of outgoing slopes of f along $\nu \in \mathrm{T}_x(\Gamma)$. Then the second condition is:

(*ii*) for any point $x \in \Gamma$, the vector $\delta_x(f)$ belongs to S^x.

Note that (*ii*) implies (*i*). We denote by $\mathrm{Rat}(\Gamma, \mathfrak{S})$, or simply $\mathrm{Rat}(\mathfrak{S})$ if there is no risk of confusion, the space of rational functions on Γ compatible with $\mathfrak{S}$. We also denote by $\mathrm{Rat}\left(\Gamma, \mathfrak{S}^{\mathrm{e}}\right)$ or $\mathrm{Rat}\left(\mathfrak{S}^{\mathrm{e}}\right)$ the space of rational functions satisfying (*i*).

If Γ is Λ-rational, we define the spaces $\mathrm{Rat}_\Lambda(\Gamma, \mathfrak{S})$, $\mathrm{Rat}_\Lambda(\Gamma, \mathfrak{S}^{\mathrm{e}})$ and $\mathrm{Rat}_\Lambda(\Gamma, \mathfrak{S}^{\mathrm{v}})$, adding the constraint that $f(v)$ is in Λ for some vertex v (equivalently, for all points of Γ).

4.4. **Divisors on a metric graph.** A divisor D on a metric graph Γ is a finite formal sum over $\mathbb{Z}$ of points of Γ, that is, $D = \sum_{i \in I \text{ finite}} n_i x_i$ with $n_i \in \mathbb{Z}$ and $x_i \in \Gamma$. The coefficient of a point x of Γ in D is denoted by $D(x)$. A divisor D is called effective, written $D \geq 0$, if $D(x) \geq 0$ for all $x \in \Gamma$. For any rational function $f \in \mathrm{Rat}(\Gamma)$, the corresponding divisor is denoted by

$$\mathrm{div}(f) \coloneqq \sum_x \mathrm{ord}_x(f)\, x, \qquad \text{where } \mathrm{ord}_x(f) \coloneqq -\sum_{\nu \in \mathrm{T}_x(\Gamma)} \mathrm{slope}_\nu(f).$$

A divisor obtained in this way is called a principal divisor. In the case Γ is Λ-rational, a divisor whose support is made up of Λ-rational points is said to be a Λ-rational divisor. Notice that the space $\mathrm{Rat}_\Lambda(\Gamma)$ defined in Section 1.11 can be redefined as the set of functions of $\mathrm{Rat}(\Gamma)$ such that $\mathrm{div}(f)$ is Λ-rational.

Note that there is a sign difference between our definition of the divisor of a rational function and that of [AB15].

We have the following elementary fact.

Proposition 4.1. *For $f, g \in \mathrm{Rat}(\Gamma)$, we have $\mathrm{div}(f) = \mathrm{div}(g)$ if, and only if, $f - g$ is constant on Γ.*

Two divisors D_1 and D_2 are called linearly equivalent if their difference $D_1 - D_2$ is principal. The Baker–Norine rank $r(D)$ of a divisor D is defined as the maximum integer among -1 and integers $r \geq 0$ such that for all points $x_1, \dots, x_r$ in Γ, the divisor $D - x_1 - \cdots - x_r$ is linearly equivalent to an effective divisor.

4.5. **Linear equivalence of slope structures.** We define a notion of linear equivalence for slope structures on a metric graph as follows.

Let $\mathfrak{S}_1 = \left\{ S_1^x; S_1^\nu \;\middle|\; x \in \Gamma, \nu \in \mathrm{T}_x(\Gamma) \right\}$ and $\mathfrak{S}_2 = \left\{ S_2^x; S_2^\nu \;\middle|\; x \in \Gamma, \nu \in \mathrm{T}_x(\Gamma) \right\}$ be two slope structures on a metric graph Γ. We say $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are linearly equivalent, and write $\mathfrak{S}_1 \simeq \mathfrak{S}_2$, if there exists a rational function f on Γ such that for any point x of Γ and any $\nu \in \mathrm{T}_x(\Gamma)$, we have $S_1^\nu = S_2^\nu - \mathrm{slope}_\nu(f)$, and $S_1^x = S_2^x - \delta_x(f)$. In this case, we write $\mathfrak{S}_1 = \mathfrak{S}_2 + \mathrm{div}(f)$. Note that if $\mathfrak{S}$ is a slope structure, then $\mathfrak{S} + \mathrm{div}(f)$ is a slope structure for every rational function f.

4.6. **Divisors endowed with a slope structure on Γ.** Expanding on the classical definition of a divisor recalled in Subsection 4.4, we now define the notion of *divisor endowed with a slope structure*, that are pairs $(D, \mathfrak{S})$ consisting of a divisor D of degree d and a slope structure $\mathfrak{S}$ of order r. To give the formal definition, we first extend the definition of linear equivalence between slope structures to all pairs $(D, \mathfrak{S})$ with D a divisor of degree d and $\mathfrak{S}$ an r-slope structure on Γ by declaring that $(D_1, \mathfrak{S}_1) \simeq (D_2, \mathfrak{S}_2)$ if there exists a rational function f on Γ such that $D_1 = D_2 + \mathrm{div}(f)$ and $\mathfrak{S}_1 = \mathfrak{S}_2 + \mathrm{div}(f)$.

Definition 4.2. A *divisor endowed with a slope structure* on Γ is the linear equivalence class of a pair $(D, \mathfrak{S})$ where D is a divisor of degree d on Γ and $\mathfrak{S}$ is an r-slope structure on Γ. ⋄

We now define the space of rational functions relative to a slope structure and a divisor.

Definition 4.3 (Space of rational functions and linear system associated to a divisor endowed with a slope structure)**.** Let $(D,\mathfrak{S})$ define a divisor endowed with a slope structure on Γ. We denote by $\mathrm{Rat}(D,\mathfrak{S})$ the space of all $f\in\mathrm{Rat}(\mathfrak{S})$ with the property that $D+\mathrm{div}(f)\geq 0$, and define the linear system $|(D,\mathfrak{S})|$ associated to $(D,\mathfrak{S})$ as the space of all effective divisors E on Γ of the form $D+\mathrm{div}(f)$ for some $f\in\mathrm{Rat}(D,\mathfrak{S})$. Note that if $D(x)>0$ for some $x\in\Gamma$ in the interior of an edge, then x essentially does not play a role in the definition of $|(D,\mathfrak{S})|$, in that we have

$$|(D,\mathfrak{S})| = |(D-(x),\mathfrak{S})| + (x).$$

We define the space $\mathrm{Rat}\left(D,\mathfrak{S}^{\mathrm{e}}\right)$ in the natural way (see Subsection 4.3). We also define $\mathrm{Rat}(D)$ to be the set of all functions $f\in\mathrm{Rat}(\Gamma)$ such that $D+\mathrm{div}(f)\geq 0$. ⋄

Remark 4.4. Note that $|(D,\mathfrak{S})|$ is independent of the choice of the pair $(D,\mathfrak{S})$ in its linear equivalence class. By an abuse of notation, we refer to both (the linear equivalence class of) $(D,\mathfrak{S})$ and $|(D,\mathfrak{S})|$ as a divisor endowed with a slope structure on Γ. ⋄

Remark 4.5. Since $\mathrm{Rat}(D)\subset\mathcal{C}^0(\Gamma,\mathbb{R})$, this space is naturally endowed with the norm $\|\cdot\|_\infty$. The corresponding topology shall be used later on to define linear series (see Definition 5.12). ⋄

Remark 4.6. We note that the slopes of all functions in $\mathrm{Rat}(D,\mathfrak{S})$ are trivially bounded in magnitude by

$$k := \max_{e\in E}\max_{1\leq i\leq r}|s_i^e|.$$

⋄

Definition 4.7. A divisor endowed with a slope structure $(D,\mathfrak{S})$ is called *effective* if $\mathrm{Rat}(D,\mathfrak{S})$ contains the null function. ⋄

Remark 4.8. This is equivalent to asking that D is effective and that we have $0\in S^\nu$ and $\underline{0}\in S^x$ for every point x and every $\nu\in\mathrm{T}_x(\Gamma)$. ⋄

We now come to the more fundamental definition of a *crude linear series*, which is the most simple concept of combinatorial linear series adapted to our context, subject to a requirement reminiscent to the rank condition on divisors on metric graphs (the rank of D is greater than or equal to r).

Definition 4.9 (Crude linear series)**.** A *crude linear series*, or *crude* $\mathfrak{g}^r_d$, is the equivalence class of a divisor endowed with a slope structure $(D,\mathfrak{S})$ on Γ subject to the following (rank) property:

($*$) For any effective divisor E on Γ of degree r, there exists a rational function $f\in\mathrm{Rat}(D,\mathfrak{S})$ such that

1. For any point $x\in\Gamma$, $\rho_x(\delta_x(f))\geq E(x)$; and in addition,
2. $D+\mathrm{div}(f)-E\geq 0$.

⋄

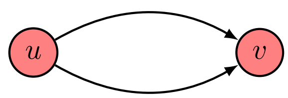
Remark 4.10. This definition generalizes to Λ-rational divisors on Λ-metric graphs. In this case, we require f to be in $\mathrm{Rat}_\Lambda(\Gamma)$. ⋄

Remark 4.11. If $\mathfrak{S}$ comes from an r-slope structure on a graph model $G=(V,E)$ of Γ, and D has support on V, then for any point $x\in\Gamma\setminus V$ lying on an edge $\{u,v\}$, the rank function ρ_x is standard and the first condition above is equivalent to $i+j\leq r-E(x)$, where s_i^{uv} and s_j^{vu} are the two slopes of f at x. In particular, since all possible slopes are integral, condition (2) $\mathrm{div}_x(f)=-s_i^{uv}-s_j^{vu}\geq E(x)$ is automatically implied by condition (1) for any point x in the interior of an edge of the model G. To sum up, for given E and f, (1) implies (2) generically (outside of vertices and of the support of D). Moreover, at a point x lying on the interior of an edge e and not in the support of D, (1) can be *strictly* stronger than (2), as long as the possible slopes on e do not form an integral interval, i.e. if there are gaps in S^e. ◇

Remark 4.12. For given E and f, (1) does not necessarily imply (2), for example at points x such that $D(x) < 0$ or such that ρ_x is not standard. ◇

Remark 4.13. The relevance of (1) will be justified in Section 8, which treats the geometric situation in which the slope structure comes from tropicalization. ◇

Example 4.14. We give a concrete example of a crude linear series for $r = 2$, $d = 4$. We consider the graph below:

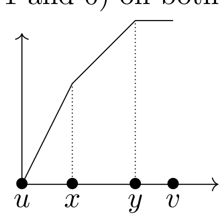


in which we allow the slopes $0 < 1 < 2$ on both edges in the direction of the arrows. The rank functions at u and v are given by the matrix

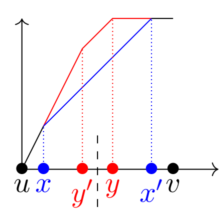
$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the other rank functions are chosen to be standard. This fully describes $\mathfrak{S}$. Consider the effective divisor $D = 4(u)$. The pair $(D, \mathfrak{S})$ is an effective crude linear series. To see this, we need to check property $(*)$ in Definition 4.9 for $E = x + y$ for points x, y in Γ. This can be done by a case analysis depending on whether x and y coincide with a vertex, or they are on the same edge of Γ. For example,

- if $x, y \notin \{u, v\}$ and x, y on the same edge. Then we can take f to behave as the following function (with slopes 2, 1 and 0) on both edges:



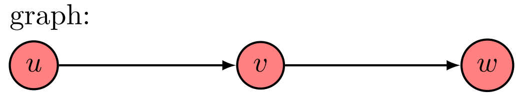
- or, if $x, y \notin \{u, v\}$ and x, y on different edges. We define x' (resp. y') to be the point of the edge containing x (resp. y) symmetrical about the middle of this edge. Then we can take f to be the function



  with values on one edge represented in blue, values on the other edge represented in red, and values common to both edges represented in black:

Using the same graph, another possible, more symmetrical, crude linear series consists in choosing $D = 2(u) + 2(v)$ and allowing slopes $-1 < 0 < 1$ on both edges. ◇

Example 4.15. Here is another simple example that will be used later on (see Remark 6.6). We consider the following graph:



We allow slopes $-2 < 0 < 2$ in the direction of the arrows and take $D = 2(u)+4(v)+2(w)$. We choose ρ to be the same as in Example 4.14 on the vertex v (it is fully determined to be standard on u and w since these vertices are of valence 1). Then, $(D, \mathfrak{S})$ is a crude linear series for $r = 2$, $d = 8$.

Without changing D, we can also consider a sub-slope structure $\mathfrak{S}'$ of $\mathfrak{S}$ of rank one by allowing slopes $0 < 2$ on the edge uv and slopes $-2 < 0$ on the edge vw. We adapt ρ to take values

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

at v. Then $(D, \mathfrak{S}')$ is an effective crude linear series for $r = 1$, $d = 8$. We can also freely divide D and the possibles slopes by two: $D = 2(v)+(u)+(w)$ and the allowed slopes become $0 < 1$ on uv and $-1 < 0$ on vw. This makes $(D, \mathfrak{S}')$ an effective crude linear series for $r = 1$, $d = 4$. ⋄

4.7. **Non-increasing property of slope vectors and a finiteness theorem.** We now formulate a useful increasing property of slope vectors for crude linear series. Let $(D, \mathfrak{S})$ be a crude linear series on Γ of rank r. Suppose that D is supported on the vertices of Γ. Let x be a point in the interior of an edge e of Γ, and let $\nu \in T_x(\Gamma)$ be the unit tangent vector corresponding to e. Let $s^\nu_0(x) < s^\nu_1(x) < \cdots < s^\nu_r(x)$ be the corresponding slopes in S^e. For each point y on e away from x in the direction ν, we still denote by ν the tangent vector in $T_y(\Gamma)$ parallel to ν, and by $s^\nu_0(y) < s^\nu_1(y) < \cdots < s^\nu_r(y)$ the corresponding slopes.

Proposition 4.16 (Non-increasing property of slope vectors)**.** *Notations as above, the collection of vectors* $(s^\nu_0, \dots, s^\nu_r)$*, as a vector-valued function on the segment corresponding to* e*, forms a coordinate-wise non-increasing sequence of vectors away from* x *in the direction indicated by* ν*.*

In other words, for each small enough $\varepsilon > 0$, denote by $y = x + \varepsilon\nu$ the point at distance ε from x in the direction of y. We have

$$s^\nu_j(y) \leq s^\nu_j(x), \qquad j = 0, 1, \dots, r.$$

Proof. We set x as a new vertex of Γ, denoting by e' the edge emanating from x in the direction ν and by e'' the other edge incident to x, oriented toward x (that is, with an orientation compatible with that of e'). To prove the proposition, it is sufficient to show that for all $0 \leq j \leq r$, $s^{e'}_j(x) \leq s^{e''}_j(x)$.

Let $p_1, \dots, p_j$ be j distinct points on e' close enough to x, in this order away from x, such that the slope structure is constant across between x and p_j. Likewise, let $q_1, \dots, q_{r-j}$ be $r - j$ distinct points on e'' close enough to x, in this order away from x, such that the slope structure is constant between x and q_{r-j}. Let

$$E \coloneqq \sum_{i=1}^{j}(p_i) + \sum_{i=1}^{r-j}(q_i).$$

By property $(*)$ in Definition 4.9, there exists $f \in \mathrm{Rat}(\mathcal{D}, \mathfrak{S})$ such that, in particular, $D + \mathrm{div}(f) - E$ is an effective divisor. The vector of outgoing slopes of f around x, $\delta_x(f)$, corresponds to some jump $\underline{a} \in J_{\rho_x}$. Since by construction D has no support between q_{r-j} an p_j, the inequality $D + \mathrm{div}(f) - E \geq 0$ implies that all the p_i's and q_i's are zeroes of f. This in turn implies that $a_{e'} \geq j$ and $a_{\overline{e''}} \geq r - j$. The fact that $\mathrm{div}(f)(x) \geq 0$ implies that $s^{e'}_{a_{e'}}(x) + s^{\overline{e''}}_{a_{\overline{e''}}}(x) \leq 0$. Finally:

$$s^{e'}_j(x) \leq s^{e'}_{a_{e'}}(x) \leq -s^{\overline{e''}}_{a_{\overline{e''}}}(x) \leq -s^{\overline{e''}}_{r-j}(x) = s^{e''}_j(x).$$

□

Theorem 4.17 (Finiteness of slopes structures relative to a fixed divisor)**.** *Let Γ be a metric graph, let D be a divisor on Γ. Let $G = (V, E)$ be a combinatorial graph underlying Γ supporting D. For each integer r, there are finitely many subdivisions $H_1, \dots, H_k$ of G, and finitely many slope structures $\mathfrak{S}_1, \dots, \mathfrak{S}_k$ of rank r defined on them, respectively, such that any crude linear series $(D, \mathfrak{S})$ has a combinatorial model H_j among $H_1, \dots, H_k$ such that $\mathfrak{S}$ coincides with $\mathfrak{S}_j$.*

Proof. The slopes appearing in $\mathfrak{S}$ are all bounded, cf. Remark 6.7. Applying Proposition 4.16, this implies that the number of graph models over which the slope structure is defined is finite, and there are only finitely many possibilities for rank functions on vertices of each of these graph models. The result then follows. □

4.8. **Partition Lemma.** In this subsection, we prove a useful result about ranked hypercubes which are linked to crude linear series.

Let ρ be a rank function on $\square_r^\delta$. The point $\underline{0}$ is the only point of $\square_r^\delta$ whose image by ρ is r (Remark 2.2). Besides, the set of jumps J_ρ of ρ contains the point $\underline{0}$ (because $\rho(\underline{e}_i) = r - 1 \geq 0$ for all i). Any jump of ρ of rank $r-1$ has only coordinates equal to zero or one (Remark 2.2), among which at least one is equal to one. For each $\underline{a} \in J_\rho$ such that $\rho(\underline{a}) = r - 1$, denote by $P_{\underline{a}}$ the subset of $\{1, \dots, \delta\}$ consisting of all the indices i with $a_i = 1$ (the support of $\underline{a}$). Denote by $\mathcal{P}_\rho$ the collection of all sets $P_{\underline{a}}$ for $\underline{a} \in J_\rho \setminus \{\underline{0}\}$ veryfing $\rho(\underline{a}) = r - 1$. We have the following proposition.

Lemma 4.18 (Partition Lemma)**.** *Using notations as above, $\mathcal{P}_\rho$ provides a partition of $\{1, \dots, \delta\}$.*

Proof. This can be verified using our Theorem 3.9 and using the properties of permutation arrays. We give a direct proof here using the supermodularity of the rank function.

We first prove that the elements of $\mathcal{P}_\rho$ are pairwise disjoint. Let $\underline{a}$ and $\underline{b}$ be two distinct elements of $J_\rho \setminus \{\underline{0}\}$ with $\rho(\underline{a}) = \rho(\underline{b}) = r - 1$. Since $\underline{a} \not\leq \underline{b}$ and $\underline{b} \not\leq \underline{a}$, we have $\rho(\underline{a} \vee \underline{b}) \leq r - 2$. Using the supermodularity property

$$\rho(\underline{a}) + \rho(\underline{a}) \leq \rho(\underline{a} \wedge \underline{b}) + \rho(\underline{a} \vee \underline{b}),$$

we get $\rho(\underline{a} \wedge \underline{b}) \geq r$ and therefore $\rho(\underline{a} \wedge \underline{b}) = r$. This forces $\underline{a} \wedge \underline{b} = \underline{0}$, from which we can conclude that $P_{\underline{a}} \cap P_{\underline{b}} = \varnothing$.

It remains to prove that the sets $P_{\underline{a}}$ cover $\{1, \dots, \delta\}$. For an $i \in \{1, \dots, \delta\}$, we need to show the existence of $\underline{a} \in J_\rho$ with $\rho(\underline{a}) = r - 1$ and $a_i = 1$. This is given by Remark A.3. □

Remark 4.19. In the case $r = 1$, in the construction above, the condition $\rho(\underline{a}) = r - 1$ is automatic. This will be crucially used in Section 7. ◇

5. Linear series on metric graphs

In this section we introduce linear series on metric graphs.

5.1. **Structure of** $\mathrm{Rat}(D, \mathfrak{S})$**.** We start by formalizing the basic structure of spaces of the form $\mathrm{Rat}(D, \mathfrak{S})$. Recall first that $(\mathbb{R}, \min, +)$ has the structure of a commutative semi-ring, where two operations min and + are the usual operations of tropical addition and multiplication, respectively, sometimes written $\oplus := \min$ and $\odot := +$.

In fact, $(\mathbb{R}, \min, +)$ is a commutative semi-field, which means, in addition, that the operation + has inverses. In the same way we define modules using commutative rings, we define semimodules using commutative semi-rings. This gives birth to the natural notion of sub-semimodule.

In the following, for $\Lambda \subseteq \mathbb{R}$, by a Λ-semimodule we mean a semimodule for the semi-field $(\Lambda, \min, +)$. The space $\mathbb{R}^\Gamma$ of functions on Γ is naturally an $\mathbb{R}$-semimodule.

The relevance of this discussion is in the following basic result which shows that the space of rational functions associated to a divisor endowed with a slope structure is an $\mathbb{R}$-semimodule.

Proposition 5.1. *Let $(D, \mathfrak{S})$ be a divisor endowed with a slope structure. For $f, g \in \mathrm{Rat}(D, \mathfrak{S})$ and $c \in \mathbb{R}$, we define the tropical operations*

$$f \oplus g := \min(f, g) \qquad \text{and} \qquad c \odot f := f + c.$$

Then $(\mathrm{Rat}(D, \mathfrak{S}), \oplus, \odot)$ is a commutative semimodule over $\mathbb{R}$.

Remark 5.2. This generalizes easily to the case of Λ-rational divisors on Λ-metric graphs. $\diamond$

Proof. The proof is largely similar to the one showing that $\mathrm{Rat}(\Gamma)$ is an $\mathbb{R}$-semimodule, see e.g. [HMY12, Lemma 4].

The only thing left to prove in order to conclude is that $\min(f, g)$ is compatible with $\mathfrak{S}^{v}$. Let v be a vertex of Γ. We can assume that $f(v) = g(v)$. We have $\delta_v(\min(f, g)) = \min(\delta_v(f), \delta_v(g))$ coordinate-wise. But if $\underline{a}$, $\underline{b}$ and $\underline{c}$ are the elements of $\square_r^{\mathrm{val}(v)}$ such that $s_{\underline{a}} = \delta_v(f)$, $s_{\underline{b}} = \delta_v(g)$ and $s_{\underline{c}} = \delta_v(\min(f, g))$, then this implies that $\underline{c} = \underline{a} \wedge \underline{b}$. The proposition follows from the fact that J_{ρ_v} is stable under $\wedge$ (Proposition 2.8). $\square$

Definition 5.3 (Restriction and extension of scalars)**.** If Γ and D are Λ-rational and if M is a sub-$\mathbb{R}$-semimodule of $(D, \mathfrak{S})$, we define M_Λ to be the sub-Λ-semimodule of $\mathrm{Rat}_\Lambda(D, \mathfrak{S})$ made up of the functions $f \in M$ which are Λ-rational on some vertex of Γ and whose divisor is Λ-rational.

Assume that Γ and D are Λ-rational. Let Λ' be another sub-semi-field of $\mathbb{R}$ such that $\Lambda \subset \Lambda'$, and M be a sub-Λ-semimodule of $\mathrm{Rat}_\Lambda(D, \mathfrak{S})$. We define $M^{\Lambda'}$ to be the sub-Λ'-semimodule of $\mathrm{Rat}_{\Lambda'}(D, \mathfrak{S})$ generated by M. $\diamond$

Note that this definition is compatible with the previously introduced notations. We now introduce two further notations that will be used throughout the paper.

Definition 5.4. Let S be a subset of $\mathbb{R}^\Gamma$ and $v \in \Gamma$. By S_v, we mean the space of all functions f of S such that $f(v) = 0$. $\diamond$

Definition 5.5. For a subset S of $\mathbb{R}^\Gamma$ and $f \in \mathbb{R}^\Gamma$, we define $S(-f) := S - f$. $\diamond$

The latter definition mimics the linear equivalence relation between divisors or slope structures (Subsection 4.6). Notice that if M is a sub-semimodule of $\mathbb{R}^\Gamma$, then this is also true of $M(-f)$.

A crucial hypothesis we could require on spaces $\mathrm{Rat}(D, \mathfrak{S})$ (or on subspaces thereof, see for example Proposition 5.19) is the finite generation property.

Definition 5.6. A semimodule $(M, \oplus, \odot)$ over a commutative semi-ring R is *finitely generated* or *of finite type* if there exist $f_1, \dots, f_n \in M$ such that for all $g \in M$, there exist $c_1, \dots, c_n \in R$ such that $g = \bigoplus_{1 \le i \le n} c_i \odot f_i$. $\diamond$

Using cut sets and extremal points, Haase, Musiker and Yu showed in [HMY12] that $\mathrm{Rat}(D)$ is a finitely generated $\mathbb{R}$-semimodule. We will see in the sequel that this is not necessarily the case for spaces of the form $\mathrm{Rat}(D, \mathfrak{S})$ (see Remark 6.6).

The above finiteness condition on a sub-module of $\mathrm{Rat}(D, \mathfrak{S})$ will imply its closedness (Proposition 5.19) and thus will be useful to define *reduced divisors* (see Subsection 6.3). This is the property used in this paper. We will show in the sequel that when D and $\mathfrak{S}$ come from geometry, the closedness condition holds (see Theorem 8.3).

Remark 5.7. Note that, if necessary, all the functions f_i in a generating set for M can be changed by additive constants so that they all vanish at a given point v of Γ. This simply changes the values of the constants c_i in Definition 5.6. $\diamond$

Definition 5.8. A sub-semimodule M of $\mathbb{R}^\Gamma$ is said to be *effective* if it contains the null function. ◇

Remark 5.9. This definition extends Definition 4.7 where $M = \mathrm{Rat}(D, \mathfrak{S})$. Note that if $f \in M$ then $M(-f)$ is effective. ◇

In finite-dimensional linear algebra, the dimension d of a vector space can be characterized using either minimal generating sets (of cardinal d) or dependence equations (the least number such that every family of n vectors is dependent is equal to $d+1$). For spaces of tropical functions, however, there is *a priori* no direct link between the two corresponding notions. Definition 5.6 adapts the notion of finiteness for a space using tropical generating sets.

We will need another notion of finiteness based on tropical independence from [JP14, JP16].

Definition 5.10 (Tropical rank)**.** Let M be a sub-semimodule of $\mathbb{R}^\Gamma$. We call the *tropical rank* of M the least integer r such that for all functions $f_0, \dots, f_{r+1} \in M$, there exist $c_0, \dots, c_{r+1} \in \mathbb{R}$ such that for all $x \in \Gamma$, the minimum in

$$\min_{0 \le i \le r+1} (f_i(x) + c_i)$$

is attained at least twice, i.e. for at least two indices i. ◇

Remark 5.11. The tropical rank can be defined the same way (imposing $c_i \in \Lambda$) for a sub-Λ-semimodule. ◇

5.2. **Linear series: definition.** We now define the notion of *linear series* on metric graphs.

Definition 5.12. Let $(D, \mathfrak{S})$ be a crude linear series. A *linear series*, or $\mathfrak{g}^r_d$, associated to $(D, \mathfrak{S})$ is a sub-semimodule M of $\mathrm{Rat}(D, \mathfrak{S})$ which is closed in $(\mathrm{Rat}(D), \|\cdot\|_\infty)$ and satisfies the following property, adapted from condition $(*)$ in Definition 4.9:

$(**)$ For any effective divisor E on Γ of degree r, there exists a rational function $f \in M$ such that

1. For any point $x \in \Gamma$, $\rho_x(\delta_x(f)) \ge E(x)$; and in addition,
2. $D + \mathrm{div}(f) - E \ge 0$.

A *refined linear series*, or *refined* $\mathfrak{g}^r_d$, associated to $(D, \mathfrak{S})$ is a linear series with tropical rank r.

When Γ and D are Λ-rational, we also define a Λ-linear series (or Λ-$\mathfrak{g}^r_d$) accordingly, as a sub-Λ-semi module M of $\mathrm{Rat}(D, \mathfrak{S})$ such that $M^{\mathbb{R}}$ is closed in $\mathrm{Rat}(D)$ and satisfies Condition $(**)$. ◇

Definition 5.13 (Strongly refined linear series)**.** A *strongly refined linear series*, or *strongly refined* $\mathfrak{g}^r_d$, associated to $(D, \mathfrak{S})$ is a linear series of tropical rank r which in addition verifies the following stronger version of $(**)$:

$(***)$ For any effective divisor E on Γ of degree $s \le r$, there exists a refined linear series of rank $r-s$ M_E associated to $(D, \mathfrak{S}_E)$ with $\mathfrak{S}_E$ a slope substructure of $\mathfrak{S}$ of rank $r-s$ such that for any function $f \in M_E$, we have

1. For any point $x \in \Gamma$, $\rho_x(\delta_x(f)) \ge E(x)$; and in addition,
2. $D + \mathrm{div}(f) - E \ge 0$.

◇

As the proof of the specialization theorem 8.2 shows, tropicalizations of linear series on curves are strongly refined.

Remark 5.14. Taking E to be concentrated on one point x, that is, $E = r(x)$, we deduce from property $(**)$ in Definition 5.12 the existence of a function $f \in M$ such that in particular $\rho_x(\delta_x(f)) \ge r$, which implies that f takes all minimal slopes around x, that is, for all $\nu \in \mathrm{T}_x(\Gamma)$, $\mathrm{slope}_\nu(f) = s^\nu_0$. ◇

Remark 5.15. It is easy to see that any refined linear series of rank one is automatically strongly refined. ◇

Remark 5.16 (Relative rank)**.** Property (2) in Definition 5.12 implies that the Baker–Norine rank $r(D)$ is greater than or equal to r. ◇

Remark 5.17 (Comparison with the tropical rank)**.** If M is a linear series associated to $(D, \mathfrak{S})$, then for M to be refined, it is equivalent to impose that the tropical rank of M is at most r. In fact, the tropical rank is always greater or equal to the rank defined by property $(**)$. To see this, we observe that at any point of Γ, in any direction, any of the $r+1$ allowed slopes is taken by a function in M. That is, there are functions $f_0, \dots, f_r$ defined on a segment I around the point, at a given direction, with f_j affine linear, all taking different slopes on I. (We prove in Theorem 5.27 below a stronger version of the above property.) To see that this implies that the tropical rank of M is always at least r, observe that for all $c_0, \dots, c_r \in \mathbb{R}$, the minimum in $\min_{0 \le i \le r}(f_i + c_i)$ is attained at least twice only at a finite number of points of I, therefore not on all I. ◇

We now define linear systems associated to linear series.

Definition 5.18 (Linear system associated to a $\mathfrak{g}^r_d$)**.** The same way we defined the linear system $|(D, \mathfrak{S})|$ associated to a divisor endowed with a slope structure $(D, \mathfrak{S})$ (Definition 4.3), we define for a linear series M the linear system $|M|$ as the space of all effective divisors E on Γ of the form $D + \mathrm{div}(f)$ for $f \in M$. Note that $|M| \subset |(D, \mathfrak{S})|$. ◇

5.3. **Basic properties of sub-semimodules of $\mathbf{Rat}(\boldsymbol{D}, \boldsymbol{\mathfrak{S}})$.** Since the definition of linear series, and, as we will see in the next section, the existence of reduced divisors require the sub-semimodule M of $\mathrm{Rat}(D, \mathfrak{S})$ to be closed in $\mathrm{Rat}(D)$, it is interesting to explore the links between this closedness condition and other properties on semimodules. We start by the following basic result.

Proposition 5.19 (Finiteness implies closedness)**.** *Let M be a sub-semimodule of $\mathrm{Rat}(D, \mathfrak{S})$ for crude linear series $(D, \mathfrak{S})$ on Γ. We assume that M is finitely generated. Then M is closed in $\mathrm{Rat}(D)$.*

Remark 5.20. This shows that the results of Sections 6 and 7 apply to finitely generated sub-semimodules of spaces of rational functions. ◇

Proof. Let (f_n) be a sequence of functions of M converging for $\|\cdot\|_\infty$ to a function f in $\mathrm{Rat}(D)$. Assume that M is generated by some functions $h_1, \dots, h_r$, and write for all n, $f_n = \min_{1 \le i \le r}(h_i + c_i^n)$ for $c_i^n \in \mathbb{R}$. We can suppose that all h_i are zero at some vertex v (Remark 5.7). It follows that the sequences $(c_i^n)_n$ are bounded. By diagonal extraction, we can assume $(c_i^n)_n$ converges to some $c_i \in \mathbb{R}$, implying that $f_n \xrightarrow{\|\cdot\|_\infty} \min_{1 \le i \le r}(h_i + c_i)$, and thus $f \in M$. □

We now show that two natural notions of closedness are in fact equivalent. Note that any effective divisor D of degree d that we write $D = x_1 + \dots + x_d$ can be seen as a point $[x_1, \dots, x_d]$ in the d-th symmetric product of the metric graph $\mathrm{Sym}^d(\Gamma)$, which is still a compact metric space. This enables us to define a map

$$\varphi : \mathrm{Rat}(D) \longrightarrow \mathrm{Sym}^d(\Gamma)$$

by $\varphi(f) \coloneqq D + \mathrm{div}(f)$. We thus have $|M| = \varphi(M)$.

Proposition 5.21. *M is closed in $\mathrm{Rat}(D)$ if, and only if, $|M|$ is closed in $\mathrm{Sym}^d(\Gamma)$.*

Proof. $\Longrightarrow$ If M is closed then so is M_v, and $|M| = \varphi(M_v)$. We recall that $M_v \subset B$ (see Remark 6.8) which is compact, so M_v is compact. Since φ is continuous, $|M|$ is compact and thus closed in $\mathrm{Sym}^d(\Gamma)$.

$\Longleftarrow$ Let (f_n) be a sequence of functions of M converging to some $f \in \mathrm{Rat}(D)$. Then by continuity $\varphi(f_n) \longrightarrow \varphi(f)$. By closedness $\varphi(f) \in |M|$, so there exists some $g \in M$ such that $\varphi(f) = \varphi(g)$. Since $\mathrm{div}(f) = \mathrm{div}(g)$, f and g differ by some constant (see Proposition 4.1), so in fact $f \in M$. □

We now give a result linking the closedness property and extremal generators. We recall that if M is a subset of a semimodule, then $x \in M$ is said to be *extremal* if $x = y \oplus z$ with $y, z \in M$ implies $y = x$ or $z = x$. We also recall that if M is a *finitely generated* sub-semimodule of $\mathbb{R}^\Gamma$, then it is generated by its extremals, of which there is a finite number up to tropical scaling (see, for example, Proposition 8 in [HMY12]).

Proposition 5.22. *Let M be a sub-semimodule of $\mathrm{Rat}(D, \mathfrak{S})$ for some crude linear series $(D, \mathfrak{S})$ on Γ. Assume that M is closed in $\mathrm{Rat}(D)$. Then M is generated by its extremals.*

The proof relies on the following lemmas, which have their own significance.

Lemma 5.23. *Using the notations of Proposition 5.22 and under the same hypotheses, let $f \in M$ and $v \in \Gamma$. Then there exists a function $g \in M$ which is extremal in M and such that $g \geq f$ and $g(v) = f(v)$.*

Proof. Without loss of generality, we assume that $f(v) = 0$. Let P be the (non-empty) set of all functions $h \in M$ such that $h(v) = f(v)$ and $h \geq f$. P is closed in M_v, which is compact. Let $(h_s)_{s \in S}$ be a chain in P (i.e., a totally ordered subset). Since P is bounded, the function $h \coloneqq \sup_s h_s$ is well-defined. By an argument similar to that used in the proof of Lemma 6.4, and using the fact that $\{h_s,\, s \in S\}$ is totally ordered, h can be written as the limit of some sequence $(h_n)_n$ of functions of P. Since P is closed, $h \in P$.

We have shown that every chain in P has an upper bound, so by Zorn's lemma, P admits a maximal element g. Since g is maximal, it is extremal, which concludes. □

Lemma 5.24. *Using the notations of Proposition 5.22, let $f \in M$. Then there exists an integer n and points $x_1, \dots, x_n \in \Gamma$ such that for all $g \in M$, we have $g = f$ if, and only if, $g(x_i) = f(x_i)$ for all i.*

Proof. We take a model G of Γ such that the supports of D and $\mathrm{div}(f)$ are included in the set of vertices. Then we take all vertices of G, and add an extra point strictly between every pair of adjacent vertices: this gives a set of points $x_1, \dots, x_n$. Let now $g \in M$ such that for all i, $g(x_i) = f(x_i)$, so f and g coincide at every vertex and at some point in the interior of every edge. Let now $e = x_i x_j$ be an edge, containing the marked point x_k. Since the interior of e contains no point of the support of $\mathrm{div}(f)$, we know that f is linear on e. Since the interior of e contains no point of the support of D, the slopes of g along e are decreasing. Combined with the fact that $g(x_i) = f(x_i)$ and $g(x_j) = f(x_j)$, we get that $g \geq f$ on e. Since $g(x_k) = f(x_k)$, we have in fact $g = f$ on e, and this is true on every edge of Γ, so $g = f$. □

Proof of Proposition 5.22. Let f be an element of M. For every $x \in \Gamma$, Lemma 5.23 provides an extremal $g^x \in M$ such that $g^x \geq f$ and $g^x(x) = f(x)$. We apply this to every point x_i given by Lemma 5.24, which yields extremal functions $g^{x_1}, \dots, g^{x_n} \in M$ such that for all i, $g^{x_i} \geq f$ and $g^{x_i}(x_i) = f(x_i)$. Define $g \coloneqq \min_i g^{x_i}$. For all i, we have $g(x_i) = f(x_i)$, and thus $g = f$, which shows that f is generated by extremals. □

We end this section with two questions about the topological properties of linear series.

Question 5.25. *Let M be a refined linear series in $\mathrm{Rat}(D, \mathfrak{S})$. Is M finitely generated?*

Question 5.26. *Let M be a linear series in $\mathrm{Rat}(D, \mathfrak{S})$. Is there any connection between M being finitely generated and M being of finite tropical rank? Is there any connection between M having a finite number of extremals and M being of finite tropical rank?*

5.4. **Realization of jumps by linear series.** We will now show that given a linear series, at any point, every jump of the rank function is realized by some function belonging to the linear series.

Theorem 5.27 (Realization theorem for jumps)**.** *Let M be a linear series associated to a pair $(D, \mathfrak{S})$. Let $v \in \Gamma$ and let $\underline{a} \in J_{\rho_v}$ be a jump. Then, there exists $f \in M$ such that $\delta_v(\rho_v(f)) = \underline{a}$.*

This theorem implies the following result.

Corollary 5.28. *Keeping the notations of Theorem 5.27, $\mathfrak{S}$ can be entirely retrieved from M.*

Remark 5.29. Since $\mathfrak{S}$ is entirely determined by M, we could wonder whether this is the case for D. This cannot be true as such, because if $M \subset \mathrm{Rat}(D, \mathfrak{S})$, then $M \subset \mathrm{Rat}(D', \mathfrak{S})$ for any divisor $D' \geq D$.

However, there is a unique base-point free choice, that is, a minimal choice for D. Simply define $D_{\min}(x)$ as minus the maximum order of poles of functions of M at x if such functions with a pole at x exist, and $D_{\min}(x) = 0$ else. This is a well-defined divisor on Γ because we already know M is associated to some crude linear series. Then $D_{\min}$ is the minimal divisor D such that $M \subset \mathrm{Rat}(D, \mathfrak{S})$. $\diamond$

Now let $\delta = d_v$. We start with a lemma.

Lemma 5.30. *Let $\underline{p} \in \square_r^\delta$ be any point in the ranked hypercube at v such that $\rho_v\left(\underline{p}\right) \geq 0$. Then there exists a unique jump $\underline{a} \in J_{\rho_v}$ of rank $\rho_v\left(\underline{p}\right)$ with $\underline{a} \geq \underline{p}$.*

Proof. We reason by contradiction to show the existence. If no such jump exists, we can construct an increasing path starting at $\underline{x}$ of constant rank (see the proof of Lemma A.1). This process necessarily ends up at the point $(r, \dots, r)$. We know that $\rho_v((r, \dots, r)) = \rho_v\left(\underline{p}\right) = 0$. Then $(r, \dots, r)$ is a jump, a contradiction.

We now show the uniqueness. Suppose there are two different jumps $\underline{a}$ and $\underline{b}$ such that $\underline{a} \geq \underline{p}$, $\underline{b} \geq \underline{p}$, $\rho_v(\underline{a}) = \rho_v(\underline{b}) = \rho_v\left(\underline{p}\right)$. Since $\underline{a} \neq \underline{b}$, then $\underline{a} \wedge \underline{b}$ is different from at least one among $\underline{a}$ and $\underline{b}$, say $\underline{a}$. Fact 2.7 yields that $\rho_v(\underline{a} \wedge \underline{b}) > \rho_v(\underline{a})$, which is impossible because $\underline{p} \leq \underline{a} \wedge \underline{b} \leq \underline{a}$ implies $\rho_v(\underline{a}) = \rho_v\left(\underline{p}\right) \geq \rho_v(\underline{a} \wedge \underline{b}) \geq \rho_v(\underline{a})$. $\square$

We now come to the proof of the theorem.

Proof of Theorem 5.27. We start by defining an increasing path in $\square_r^\delta$ that starts at $\underline{0}$, stays below $\underline{a}$ and separates the δ directions. More precisely, the path will first only move along the first direction (chosen arbitrarily), then only along the second direction, and so on, until direction δ. Let, for all $1 \leq i \leq \delta$, e_i be the edge incident to v corresponding to the direction i. For convenience, we also define, for all $1 \leq i \leq \delta$, $e'_i := e_{n-i+1}$, just reversing the arbitrary order of the edges around v.

Suppose that we have already built the path along directions $1, \dots, i-1$ with $1 \leq i \leq \delta$. The path thus currently ends at the point $\sum_{k=1}^{i-1} t_k \, \underline{e}_k$, $t_k \in [r]$. We will now let the path continue in the direction i by adding several times the unit vector $\underline{e}_i$. More precisely, we keep adding $\underline{e}_i$ until the following situation arises. Suppose the path is currently at the point $\underline{y} = \sum_{k=1}^{i-1} t_k \, \underline{e}_k + s_i \, \underline{e}_i$ with $\underline{y} \leq \underline{a}$. Suppose that for every $n \geq 1$, either $\underline{y} + n \, \underline{e}_i \not\leq \underline{a}$ or $\rho_v\left(\underline{y} + n \, \underline{e}_i\right) = \rho_v\left(\underline{y}\right)$. We say that $\underline{y}$ is the largest *fall* in the direction i above $\underline{y}$ such that $\underline{y} \leq \underline{a}$. Then, we stop at $\underline{y}$ for direction i and move on to the next direction; otherwise, we continue the path in the direction i by going to the next fall in the direction i, which is the point $\underline{y} + n_0 \, \underline{e}_i$, where n_0 is the least $n \geq 1$ such that $\underline{y} + n \, \underline{e}_i \leq \underline{a}$ and $\rho_v\left(\underline{y} + n \, \underline{e}_i\right) = \rho_v\left(\underline{y}\right) - 1$. This way, we have built an increasing path starting at $\underline{0}$, staying below $\underline{a}$ and moving successively in the different directions, without repetition. This path consists of falls in the right directions (except $\underline{0}$ which is not).

The ending point of the path is a fall in the last direction indexed δ of the form

$$\underline{z} = \sum_{k=1}^{\delta} t_k \, \underline{e}_k.$$

For every $1 \leq i \leq \delta$, let ℓ_i be the rank drop of the path in the direction i:

$$\ell_i \coloneqq \rho_v \left(\sum_{k=1}^{i-1} t_k \, \underline{e}_k \right) - \rho_v \left(\sum_{k=1}^{i} t_k \, \underline{e}_k \right).$$

Then, $\sum_{i=1}^{\delta} \ell_i = r - \rho_v(\underline{z})$.

We show that in fact $\rho_v(\underline{z}) = \rho_v(\underline{a})$ using the properties of the increasing path. Let $\underline{w} \in \square_r^\delta$ be such that $\underline{z} \leq \underline{w} \leq \underline{a}$ and let $1 \leq i \leq \delta$ be such that $\underline{w} + \underline{e}_i \leq \underline{a}$. By construction, we know that

$$\rho_v \left(\sum_{k=1}^{i} t_k \, \underline{e}_k + \underline{e}_i \right) = \rho_v \left(\sum_{k=1}^{i} t_k \, \underline{e}_k \right).$$

This equality together with supermodularity implies that $\rho_v(\underline{w} + \underline{e}_i) = \rho_v(\underline{w})$. Applying this fact recursively yields $\rho_v(\underline{z}) = \rho_v(\underline{a})$. (In fact, ρ_v is constant on $\{\underline{w} \in \square_r^\delta, \, \underline{z} \leq \underline{w} \leq \underline{a}\}$.)

What precedes shows that $\rho_v(\underline{a}) + \sum_{i=1}^{\delta} \ell_i = r$, which motivates the following definition.

For every $1 \leq i \leq \delta$, let $p_1^i, \ldots, p_{\ell_i}^i$ be distinct points on the edge e_i'. Let

$$E \coloneqq \rho_v(\underline{a})(v) + \sum_{i=1}^{\delta} \sum_{j=1}^{\ell_i} \left(p_j^i\right).$$

E is an effective divisor of degree r. By the definition of linear series, there exists $f \in M$ such that, in particular, $\rho_v(\delta_v(f)) \geq \rho_v(\underline{a})$ and $\mathrm{div}(f) + D - E \geq 0$. Let $\underline{b} \coloneqq \delta_v(f) \in \square_r^\delta$.

Note that $\underline{b}$ is a jump of ρ_v. The first property tells us that $\rho_v(\underline{b}) \geq \rho_v(\underline{a}) = \rho_v(\underline{z})$. We will now show that the second property, applied to a sequence of effective divisors starting at E, implies that $\underline{b} \geq \underline{z}$. To this end, we will make all marked points converge to v one after the other, defining a non-increasing sequence of distinct jumps.

Let $\left(p_{1,n}^1\right)_n$ be a sequence of points of e_1' starting at p_1^1, converging to v. Replacing p_1^1 by $p_{1,n}^1$ yields an effective divisor $E_{1,n}^1$ defined by

$$E_{1,n}^1 \coloneqq \rho_v(\underline{a})(v) + \left(p_{1,n}^1\right) + \sum_{j=2}^{\ell_1} \left(p_j^1\right) + \sum_{i=2}^{\delta} \sum_{j=1}^{\ell_i} \left(p_j^i\right).$$

The definition of linear series implies that for all n, there exists a function $f_{1,n}^1 \in M$ with $f_{1,n}^1(v) = 0$ such that $\mathrm{div}\left(f_{1,n}^1\right) + D - E_{1,n}^1 \geq 0$. Replacing $f_{1,n}^1$ by $\min\left(f, f_{1,n}^1\right)$ ensures that for all n, $\delta_v\left(f_{1,n}^1\right) \leq \delta_v(f)$. (Note that the set of functions satisfying (1) and (2) in Definition 5.12 is stable by min.) Moreover, we can extract a subsequence so that the sequence $\left(f_{1,n}^1\right)$ converges to some $f_1^1 \in M$.

By Remark 6.9, for all n,

$$\delta_v\left(f_1^1\right) \leq \delta_v\left(f_{1,n}^1\right) \leq \delta_v(f).$$

Besides, the fact that $\left(\delta_v\left(f_1^1\right)\right)_1 < \left(\delta_v(f)\right)_1$ implies that $\delta_v\left(f_1^1\right) \neq \delta_v(f)$.

What precedes yields a jump $\underline{b}_1^1 \coloneqq \delta_v\left(f_1^1\right) \leq \delta_v(f) = \underline{b}$ which is different from $\underline{b}$ because $\left(\underline{b}_1^1\right)_1 > b_1$.

We then repeat the same process, taking a sequence $\left(p^1_{2,n}\right)_n$ of points of e'_1 starting at p^1_2, defining

$$E^1_{2,n} \coloneqq (\rho_v(\underline{a})+1)(v) + \left(p^1_{2,n}\right) + \sum_{j=3}^{\ell_1} \left(p^1_j\right) + \sum_{i=2}^{\delta}\sum_{j=1}^{\ell_i} \left(p^i_j\right),$$

which, after taking the minimum with f^1_1 and extracting, yields a jump $\underline{b}^1_2 \leq \underline{b}^1_1$ different from $\underline{b}^1_1$ (because $\left(\delta_v\left(\underline{b}^1_2\right)\right)_1 < \left(\delta_v\left(\underline{b}^1_1\right)\right)_1$).

We repeat the same process over and over for all marked points p^1_j on e'_1, and then for all marked points on the other edges, yielding a decreasing path of jumps

$$\underline{b} \gneq \underline{b}^1_1 \gneq \underline{b}^1_2 \gneq \cdots \gneq \underline{b}^1_{\ell_1} \gneq \cdots \gneq \underline{b}^\delta_1 \gneq \cdots \gneq \underline{b}^\delta_{\ell_\delta}$$

such that two consecutive jumps are different. Using Fact 2.7 again, we know that the sequence of corresponding ranks $\left(\rho_v\left(\underline{b}^i_j\right)\right)$ is increasing. As a consequence,

$$\rho_v\left(\underline{b}^\delta_{\ell_\delta}\right) \geq \rho_v\left(\underline{b}\right) + \sum_{i=1}^{\delta} \ell_i \geq r,$$

and thus, $\underline{b}^\delta_{\ell_\delta} = \underline{0}$ and all the rank differences are exactly one.

Reversing the order of the jumps, we get an increasing path of jumps starting at $\underline{0}$ and ending at $\underline{b}$, whereas we defined beforehand an increasing path starting at $\underline{0}$ and ending at $\underline{z}$. To show that $\underline{b} \geq \underline{z}$, we will prove that that at each step, the path leading to $\underline{b}$ remains greater than or equal to the path leading to $\underline{z}$. (Here, a step corresponds to a fall for the path leading to $\underline{z}$ and to a jump for the path leading to $\underline{b}$.)

The inequality is true at the beginning because the starting point of both paths is $\underline{0}$. Suppose that the inequality is true at some step j_0 with $0 \leq j_0 < r - \rho_v(\underline{a})$. We denote by $\underline{z}_{j_0}$ the current fall and by $\underline{b}_{j_0}$ the current jump. The inequality reads $\underline{b}_{j_0} \geq \underline{z}_{j_0}$. We suppose that the next fall $\underline{z}_{j_0+1}$ differs from $\underline{z}_{j_0}$ (only) in the direction i_0. Let

$$\underline{c}_{j_0} \coloneqq \underline{z}_{j_0} + \left(\left(\underline{b}_{j_0}\right)_{i_0} - \left(\underline{z}_{j_0}\right)_{i_0}\right) \underline{e}_{i_0}$$

be the point obtained by starting at $\underline{z}_{j_0}$ and moving in the direction i_0 as much as possible without overtaking $\underline{b}_{j_0}$ along this axis.

Since both paths are parametrized by the same integers ℓ_i, we know that the next jump $\underline{b}_{j_0+1}$ will also differ from $\underline{b}_{j_0}$ (at least) in the direction i_0. The latter statement implies that $\underline{b}_{j_0+1} \geq \underline{b}_{j_0} + \underline{e}_{i_0}$. Since $\underline{b}_{j_0}$ is a jump, by supermodularity and using $\underline{b}_{j_0} \geq \underline{c}_{j_0}$, we get that

$$\rho_v\left(\underline{c}_{j_0} + \underline{e}_{i_0}\right) = \rho_v\left(\underline{c}_{j_0}\right) - 1$$

so $\underline{c}_{j_0} \geq \underline{z}_{j_0}$ is a fall in the direction i which coincides with $\underline{z}_{j_0}$ in all directions but i_0. Therefore,

$$\underline{z}_{j_0+1} \leq \underline{c}_{j_0} + \underline{e}_{i_0} \leq \underline{b}_{j_0} + \underline{e}_{i_0} \leq \underline{b}_{j_0+1}.$$

So the iteration is true and the inequality propagates, yielding $\underline{b} \geq \underline{z}$, as desired.

Now we know that $\underline{b}$ is a jump of ρ_v such that $\rho_v\left(\underline{b}\right) \geq \rho(\underline{z})$ and $\underline{b} \geq \underline{z}$. The uniqueness in Lemma 5.30 implies that $\underline{b} = \underline{a}$, which finishes the proof. □

6. Reduced divisors in linear series

The aim of this section is to provide an extension of the machinery of reduced divisors and the surrounding results to the linear series.

6.1. **Reduced divisors in the chip-firing context.** We first recall the definition of reduced divisors in the "chip-firing" context, without the presence of slope structures. A *cut* X in a metric graph Γ is a compact connected subset of Γ. The set of boundary points of X, denoted by ∂X, is the set of all points of X which are in the closure of the complement of X in Γ. For any point $x \in \partial X$, we denote by $\mathrm{out}_X(x)$ the set of all out-going branches at x from X, and by $\mathrm{outval}_X(x)$ the number of such branches.

Definition 6.1 (Reduced divisors in the classical context)**.** Let x be a point of Γ. A divisor D on Γ is x-reduced if the following two conditions are met:

(1) For all $y \in \Gamma \setminus \{x\}$, $D(y) \geq 0$, and
(2) For all $X \subset \Gamma \setminus \{x\}$, there exists a point $y \in \partial X$ such that $D(y) < \mathrm{outval}_X(y)$.

◇

It has been proven that for any divisor D and point x, there exists a unique x-reduced divisor denoted by D_x linearly equivalent to D (see [BN07, Ami13] for graphs and metric graphs, respectively). In terms of the chip-firing game, the x-reduced divisor is obtained from D by firing all possible chips the closest possible to x.

6.2. **Tools for reduced divisors.** In our context involving slope structures, we will now develop some tools useful to define the notion of reduced divisors. We adapt our previous convention and for any space of rational functions and any point x of Γ, we add a subscript v to indicate that we consider only functions that vanish at v.

Definition 6.2. Let $(D, \mathfrak{S})$ be a crude linear series on Γ and v a point of Γ. Let $M \subset \mathrm{Rat}(D, \mathfrak{S})$ be a linear series. We define the rational function f_v^M (denoted f_v when the context is clear) by

$$f_v^M(x) \coloneqq \inf_{g \in M} [g(x) - g(v)] = \inf_{f \in M_v} f(x)$$

for any point x of Γ. (In particular, $f_v(v) = 0$.) ◇

Remark 6.3. Remark 5.14 implies that f_v^M takes minimal slopes around v, that is, for all $\nu \in \mathrm{T}_v(\Gamma)$, $\mathrm{slope}_\nu\left(f_v^M\right) = s_0^\nu$. ◇

The following lemma shows that the function f_v is still a rational function and even belongs to the semimodule M.

Lemma 6.4. *Let M be a $\mathfrak{g}_d^r$ on Γ associated to some $(D, \mathfrak{S})$. Then the function f_v^M is well-defined and belongs to M_v.*

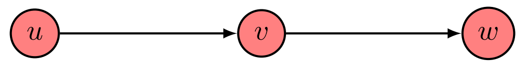
Note that by this lemma, since the infimum is reached, we can write $f_v = \min_{f \in M_v} f(x)$.

Proof. The first claim does not require that M is closed in $(\mathrm{Rat}(D), \|\cdot\|_\infty)$ and boils down to proving that the set $\{f(x),\ f \in M_v\}$ is bounded for a given $x \in \Gamma$. This is however clear from the rough estimate $|f(x)| \leq k \,\mathrm{Diam}(\Gamma)$, with k a bound for (the absolute values of) the slopes appearing in $\mathfrak{S}$ and $\mathrm{Diam}(\Gamma)$ the diameter of Γ as a compact metric space. This shows that f_v is well-defined.

To prove the second claim, we show that f_v can be written as the uniform infimum of a (decreasing) sequence of functions in M_v, essentially following the proof of Dini's theorem. Since the slopes of functions in M_v are universally bounded, this set is uniformly Lipschitz, so f_v is itself Lipschitz and therefore continuous. To conclude, we use the closedness of M in $\mathrm{Rat}(D)$, which yields $f_v \in M$. □

Remark 6.5. We saw in Proposition 5.19 that if M is finitely generated, then it is closed in $\mathrm{Rat}(D)$, so Lemma 6.4 applies. In this case, we can apply a simple argument similar to that of Proposition 5.19 to prove the result. ◇

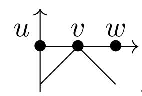
Remark 6.6 (Counterexample without the closedness condition)**.** The closedness condition on M is crucial here. As a counter-example to Lemma 6.4 when we drop this condition, consider the graph and the slope structure of rank 1 defined at the very end of Example 4.15.



We allow slopes $0 < 1$ on the edge uv and slopes $-1 < 0$ on the edge vw. We set $D = 2(v) + (u) + (w)$ and we adapt ρ to take values

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

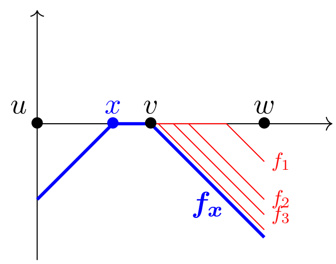
at v, which makes $(D, \mathfrak{S})$ a crude linear series for $r = 1$, $d = 4$. We take the whole space of rational functions $\mathrm{Rat}(D, \mathfrak{S})$, which is not *a priori* a $\mathfrak{g}^1_4$. The function f_v



,

taking slopes 1 and -1, does belong to $\mathrm{Rat}(D, \mathfrak{S})$.

But if we choose any $x \neq v$, then $f_x \notin \mathrm{Rat}(D, \mathfrak{S})$.



Indeed, all the functions f_i represented in thin red belong to $\mathrm{Rat}(D, \mathfrak{S})_x$ and f_x is their infimum, because it uses the smallest possible slopes starting at v with value 0. But f_x does not belong to $\mathrm{Rat}(D, \mathfrak{S})$, because it does not take symmetrical values around v, which is mandatory provided the expression of ρ_v. In particular, this show that $\mathrm{Rat}(D, \mathfrak{S})$ is not closed in $\mathrm{Rat}(D)$, and so it is not a $\mathfrak{g}^1_4$. ◇

Remark 6.7 (Bounds on slopes)**.** In the proof of Lemma 6.4, we have used a universal bound for the slopes of functions of $\mathrm{Rat}(D, \mathfrak{S})$ using the finiteness of the set of slopes prescribed by $\mathfrak{S}^{\mathrm{e}}$. In fact, as shown in [GK08, Lemma 1.8], there is a universal bound for these slopes even without a slope structure: for D any effective divisor, the slopes of functions in $\mathrm{Rat}(D)$ are uniformly bounded. ◇

Remark 6.8 (Topological discussion)**.** Let k be a universal bound for the slopes in $\mathfrak{S}$. Let us denote by C the space $\big(\mathcal{C}^0(\Gamma, \mathbb{R})_v, \|\cdot\|_\infty\big)$ and by B the subspace formed by the functions of C whose slopes are bounded by k. Since B is closed in C, bounded and equicontinuous, it is compact by the Arzelà–Ascoli theorem. Proposition 5.19 shows in particular that the space

$$A \coloneqq (\mathrm{Rat}(D, \mathfrak{S})_v, \|\cdot\|_\infty) \subset B$$

is compact as long as it is finitely generated as a semimodule. Furthermore, if M is a sub-semimodule of $\mathrm{Rat}(D, \mathfrak{S})$ which is closed in $\mathrm{Rat}(D)$ (in particular, if M is a $\mathfrak{g}^r_d$), then M_v is again compact. But Remark 6.6 shows that when the finiteness condition is dropped, A is not necessarily compact (it is not necessarily closed in B). However, it is possible to show that the space

$$A^{\mathrm{e}} \coloneqq \left(\mathrm{Rat}\left(D, \mathfrak{S}^{\mathrm{e}}\right)_v, \|\cdot\|_\infty\right) \subset B$$

is closed in B (and thus compact) even if A is not finitely generated. ◇

Remark 6.9. In Remark 6.8, we proved better than just $\mathrm{div}(f)(v) \geq \mathrm{div}(f_n)(v)$. In fact, we proved that

$$\delta_v(f) = \delta_v(f_n) + \left(s^{e_k}_{\alpha_k} - s^{e_k}_{\beta_k}\right)_k,$$

and thus in particular that $\delta_v(f) \leq \delta_v(f_n)$.

Therefore, if $(f_n)_n$ is a sequence of functions of M such that for all n, f_n satisfies conditions (1) and (2) in Definition 5.12 (linear series) for a given divisor $E \geq 0$, then the same is true of f. ⋄

6.3. **Definition of the reduced divisor.** We now define reduced divisors adapted to our context of slope structures.

Definition 6.10 (Reduced divisor)**.** The (effective) divisor defined by $D^M_v := D + \mathrm{div}\left(f^M_v\right)$ (or simply D_v when M is contextually clear, since there is no ambiguity with the notation introduced in Definition 5.4) is called the *v-reduced divisor linearly equivalent to D with respect to M*. Similarly, we denote the new slope structure $\mathfrak{S} + \mathrm{div} f_v$ by $\mathfrak{S}_v$ (again, no ambiguity). The effective pair $(D_v, \mathfrak{S}_v)$ is then equivalent to the pair $(D, \mathfrak{S})$. Finally, f_v gives rise to a modification of M, which is denoted by $M(-f_v)$ according to Definition 5.5 and which is effective. ⋄

Remark 6.11. We call D_v *the* v-reduced divisor linearly equivalent to D with respect to M because we define it in a concrete and computational way. But we will see in Theorem 6.14 that D_v is in fact the *only* effective divisor linearly equivalent to D that satisfies an interesting property involving unsaturated cuts (see Definition 6.13). The uniqueness of the reduced divisor is analogous to what happens in the classical setting. In fact, just by looking at the definition of f_v, we can already give a simple property that makes the reduced divisor unique: there exists no function $f \in M(-f_v) \setminus \{0\}$ such that $f(v) = 0$ and $f(x) \leq 0$ for all $x \in \Gamma$. ⋄

6.4. **Coefficient at the base-point.** We have the following useful result.

Proposition 6.12. *Let M be a $\mathfrak{g}^r_d$ associated to some crude linear series $(D, \mathfrak{S})$. For any point $x \in \Gamma$, we have*

$$D_x(x) = D(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} s^\nu_0.$$

In addition, this quantity is greater than or equal to r.

Proof. We apply Definition 5.12, considering the effective divisor $E = r(x)$. So there exists a rational function $f \in M$ satisfying conditions (1) and (2). (1) tells us that $\rho_x(\delta_x(f)) = r$. This implies that $\delta_x(f) = \prod_{\nu \in \mathrm{T}_x(\Gamma)} s^\nu_0 \in S_x$. We can also assume that $f(x) = 0$. It follows that f takes its smallest possible values (compatible with the prescribed slopes and the value at x) in a sufficiently small neighborhood of x in Γ. Thus, f_x and f must coincide in a small neighborhood of x in Γ, which implies that $\delta_x(f_x) = \Pi_{\nu \in \mathrm{T}_x(\Gamma)} s^\nu_0$. We conclude that $D_x(x) = D(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} s^\nu_0$. The final claim then follows from condition (2). □

6.5. **Unsaturated cuts.** In the rest of this section we provide another characterization of reduced divisors in terms of saturated cuts in a metric graph with respect to a $\mathfrak{g}^r_d$.

Definition 6.13 (Unsaturated cut)**.** Let M be an effective $\mathfrak{g}^r_d$ on Γ associated to some $(D, \mathfrak{S})$, and let v be a point of Γ. Consider a cut X in Γ and assume that

- the point v does not belong to X, and

- for any point $x \in \partial X$, there exists an element $s = \prod_{\nu \in T_x(\Gamma)} s^\nu \in S^x$ such that $s^\nu \geq 0$ for all $\nu \in \mathrm{T}_x(\Gamma)$ with equality $s^\nu = 0$ if and only if $\nu \in T_x(\Gamma) \setminus \mathrm{out}_X(x)$, and in addition,
$$D(x) - \sum_{\nu \in T_x(\Gamma)} s^\nu \geq 0.$$

Then define the function f which is linear of slope s^ν on a small segment I_e on each adjacent out-going branch $e \in \mathrm{out}_X(x)$, takes value $-\varepsilon$ on X for a sufficiently small ε, and is zero everywhere else. Since $\underline{0} \in S^x$ for any $x \in \Gamma$ and thanks to the inequality above, f belongs to $\mathrm{Rat}(D, \mathfrak{S})$ and is non-zero.

We say that X is *unsaturated with respect to v and M* if the function f defined above belongs to M. Otherwise, X is called *saturated*. ◇

The following theorem gives another characterization of the reduced divisors.

Theorem 6.14 (Characterization of reduced divisors by unsaturated cuts)**.** *Let M be an effective $\mathfrak{g}^r_d$ relative to $(D, \mathfrak{S})$ on Γ. Then D is v-reduced, that is, $D = D_v^M$, if and only if there is no unsaturated cut with respect to v and M.*

Proof. We prove the equivalence of the negations.

First we assume that there is an unsaturated cut X with respect to v and M and show that D is not reduced. By definition, this means that for any point $x \in \partial X$, there exists $s \in S^x$ such that $s^\nu > 0$ for any $\nu \in \mathrm{out}_X(x)$ and $s^\nu = 0$ for any $\nu \in \mathrm{T}_x(\Gamma) \setminus \mathrm{out}_X(x)$, and $D(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} s^\nu \geq 0$. These slopes give rise to a function $f \in \mathrm{Rat}(D, \mathfrak{S})$ as in Definition 6.13, and since X is unsaturated, $f \in M$. This shows that D is not v-reduced (see Remark 6.11).

We now prove the other direction. We assume that there exists a function $f \in M \setminus \{0\}$ such that $f(v) = 0$ and $f(x) \leq 0$ for any $x \in \Gamma$ (i.e. D is not v-reduced), and prove that there exists an unsaturated cut with respect to v and M.

Let X be the set of points of Γ where f takes its minimum value. Note that $X \neq \varnothing$ is compact and $v \notin X$ (because of the existence of f). It is now easy to see that X is unsaturated: for any point $x \in \partial X$, the vector $s = \delta_x(f) \in S^x$ is so that $s^e \geq 0$ for any $\nu \in \mathrm{T}_x(\Gamma)$, with equality $s^\nu = 0$ if and only if $e \in \mathrm{T}_x(\Gamma) \setminus \mathrm{out}_X(x)$, and $D(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} s^\nu = D(x) + \mathrm{div}_x(f) \geq 0$ since $f \in M \subset \mathrm{Rat}(D, \mathfrak{S})$. This proves the theorem. ☐

6.6. **Behavior of reduced divisors with respect to the base point.** We now provide an explicit description of the reduced divisors under an infinitesimal change of the base point. This will be used in the next section to prove the continuity of the reduced divisor map.

Let M be an effective $\mathfrak{g}^r_d$ associated to $(D, \mathfrak{S})$, v be a point of Γ and $\nu \in \mathrm{T}_v(\Gamma)$, and e the out-going branch of Γ parallel to ν, all fixed for the remainder of this subsection. We give an explicit description of D_u for points u in a small segment I^e on e in Γ with an end-point equal to v.

Up to changing $(D, \mathfrak{S})$ in its linear equivalence class and adapting M accordingly, we can assume that D is v-reduced. This boils down to replacing $(D, \mathfrak{S})$ with the linearly equivalent $(D_v, \mathfrak{S}_v)$ and M with $M(-f_v)$, which is effective. There exists a small enough segment on e adjacent to v which does not contain any point in the support of D apart from v. For a point u on this segment, we have $D(u) = 0$ and $D_u(u) > 0$ and thus $D \neq D_u$. By uniqueness of the u-reduced divisor, we infer that D is not u-reduced. It thus follows by Theorem 6.14 that there exist unsaturated cuts with respect to u and M. Since D is v-reduced, we infer that any such cut Y contains v. In addition, e must belong to $\mathrm{out}_Y(v)$, and so $v \in \partial Y$, since otherwise the cut's boundary would contain a point of e between v and u which is impossible by the second condition in the Definition 6.13 of unsaturated cuts and the assumption made on the support of D. We have proved

Claim 6.15. *For any unsaturated cut Y with respect to u and M, we have $v \in \partial Y$.*

Since D is v-reduced, we have $s_0^\nu = 0$ for any $\nu \in \mathrm{T}_v(\Gamma)$ (see, for example, Definition 6.2). It follows that $s_r^\nu > 0$, and by Proposition 6.12, and the definition of slope structures, the coefficient of D_u at u is precisely equal to $s_r^\nu + s_0^\nu = s_r^\nu$ ($-s_r^\nu$ is the smallest possible slope in the direction of v, and $-s_0^\nu = 0$ is the smallest possible slope in the other direction). We now claim

Claim 6.16. *There exists an unsaturated cut Y with respect to u and M and an element $\underline{s} \in S^v$ such that*

- *We have $s^{\nu'} \geq 0$ with equality if and only if $\nu' \in \mathrm{T}_v(\Gamma) \setminus \mathrm{out}_Y(v)$ (the property required by the definition of unsaturated cuts);*
- *The coordinate s^ν of $\underline{s}$ is equal to s_r^ν.*

Proof. Let f be a rational function in M such that $D + \mathrm{div}(f) = D_u$ (for example, $f = f_u$ as defined in Definition 6.2). Note that the slope of f on I^e is equal to s_r^ν, and v is a local minimum of f (since $s_0^{\nu'} = 0$ for any $\nu' \in \mathrm{T}_v(\Gamma)$). Consider the set Y of all points of Γ which are a local minimum of Γ. It is straightforward to check that Y is an unsaturated cut with respect to u and M, which verifies the conditions of the claim. □

Now consider the family $\mathcal{X}$ of all the unsaturated cuts Y with respect to u and M such that Y verifies the properties of Claim 6.16. Let $X \coloneqq \cup_{Y \in \mathcal{X}} Y$.

Fact 6.17. We notice that if we choose another point u' on this segment (i.e. close enough to v that the segment $[v, u']$ does not contain any point in the support of D apart from v), a cut Y is unsaturated with respect to u' if, and only if, it is unsaturated with respect to u. This shows that $\mathcal{X}$ (and thus X) does not change when we choose u to be even closer to v, a fact that will be used in the proof of Theorem 6.19. ◇

We now claim that

Claim 6.18. *For any point u on a sufficiently small segment I^e on e, the cut X itself belongs to $\mathcal{X}$.*

Proof. To prove that X is unsaturated, let x be a point on the boundary of X. For any cut Y in $\mathcal{X}$ which contains x, since Y is unsaturated, there exists an element $\underline{s}_Y \in S^x$ such that $s_Y^{\nu'} > 0$ for any $\nu' \in \mathrm{out}_X(x)$, and $s^{\nu'} = 0$ for any $\nu' \in \mathrm{T}_x(\Gamma) \setminus \mathrm{out}_Y(x)$, and $D(x) - \sum_{\nu' \in E_x} s_Y^{\nu'} \geq 0$. We also know by hypothesis that the function built from these slope values (see Definition 6.13) is in M.

Let $\underline{s} = \wedge \underline{s}_Y \in S^x$. We obviously have $s^{\nu'} \geq 0$ for any $\nu' \in \mathrm{T}_x(\Gamma)$, with equality $s^{\nu'} = 0$ if and only if $\nu' \in \mathrm{T}_x \setminus \mathrm{out}_X(x)$, and $D(x) - \sum s^{\nu'} \geq 0$. Moreover, the function defined from these slopes is the minimum of the functions defined for each unsaturated cut Y, so it is still in M which is stable by min.

This shows that X is unsaturated. To prove that $X \in \mathcal{X}$, note that for any cut $Y \in \mathcal{X}$, there exists $\underline{s}_Y \in S^v$ such that $s_Y^\nu = s_r^\nu$ for all $\nu \in T_v(\Gamma)$. Then the point $\underline{s} = \wedge \underline{s}_Y \in S^v$ still verifies $s^\nu = s_r^\nu$ for all $\nu \in T_v(\Gamma)$, which shows that X is a cut verifying the properties of Claim 6.16, and thus that $X \in \mathcal{X}$. □

For any point $x \in \partial X$ different from v, let $\underline{s}_x$ be the minimum point of S^x which has $s^{\nu'} \geq 0$ for any $\nu' \in \mathrm{T}_x(\Gamma)$, with equality if and only if $\nu \in \mathrm{T}_x(\Gamma) \setminus \mathrm{out}_X(x)$. Such an element exists by the definition of unsaturated cuts and because S^x is closed under $\wedge$ (see Proposition 2.8). The associated function belongs to M. In addition, let $\underline{s}_v \in S^v$ be the minimum point of S^v with $s^\nu = s_r^\nu$, and $s^{\nu'} \geq 0$ for any $\nu' \in \mathrm{T}_v(\Gamma)$, with equality if and only if $\nu' \in \mathrm{T}_v(\Gamma) \setminus \mathrm{out}_X(v)$ and such that the associated function belongs to M. The collection of all these points in S^x, for any $x \in \partial X$, defines a function f^ε, which takes value $-\varepsilon$ on X, is linear of slope $s^{\nu'}$ on a segment of length $\frac{\varepsilon}{s^{\nu'}}$ for any $\nu' \in \mathrm{out}_X(x)$, for any point $x \in \partial X$, and is zero elsewhere. It is trivial by construction that $f \in \mathrm{Rat}(\mathfrak{S})$, and the fact that f is compatible with D is due to the second condition in the definition of unsaturated cuts. In fact f even belongs to M following this definition. For any point u on I^e, for a sufficiently small segment I^e, let δ be the distance between u and v on I^e, define $\varepsilon := s_r^{\nu}\delta$, and let $f^u := f^{\varepsilon}$, which is well-defined as long as δ is smaller than the lengths of all the outgoing edges starting at a point $x \in \partial X$.

With these notations, we can formulate the main theorem of this section.

Theorem 6.19. *The u-reduced divisor with respect to M is $D_u = D + \mathrm{div}(f^u)$.*

Proof. Let f_u be the rational function in M which defines the u-reduced divisor D_u, and which takes the value zero at u (see Definition 6.2). We will prove that in fact $f_u = f^u$. Note that f_u verifies the following properties.

- f_u is linear on the segment $[v, u]$ with slope s_r^{ν}.

This follows from the fact already observed that the coefficient of D_u at u is s_r^{ν}, and s_r^{ν} is the maximum possible slope along the segment I^e. The fact that D_u is effective then implies that f_u has constant slope along $[v, u]$.

- f_u takes it minimum value at v, and its maximum value at u.

The fact that f_u takes its maximum value at u comes from the definition of the u-reduced divisor map, given that $0 \in M$. To show that f_u takes its minimum value at v, note that $D_u + \mathrm{div}(-f_u) = D$, now since the constant functions belong to $M(-f_u)$, and since D is v-reduced, $-f_u$ takes its maximum value at v, and the claim follows.

- Let X_0 be the set of all points where f_u takes its minimum value. Then $X_0 = X$.

Obviously X_0 is an unsaturated cut with respect to u and M which in addition verifies the property of Claim 6.16. So $X_0 \in \mathcal{X}$. This shows by Claim 6.18 that $X_0 \subset X$. To prove the equality, note that f^u takes value $-\varepsilon = f(v)$ on X. On any point $x \in X \setminus X_0$, we would have $-\varepsilon = f^u(x) < f_u(x)$, which would be impossible by the definition of the u-redu ced divisor as $f_u = \min_g g$ for $g \in M$ with $g(u) = 0$. This shows that $X_0 = X$.

- Let $x \in \partial X$. There is a small enough neighborhood of x in Γ such that f_u and f^u coincide on that neighborhood.

Note that $f_u = f^u$ on X (and thus at the point x) by the previous claim. Consider the point $\underline{s} \in S^x$ which represents the slopes of f^u along all the branches adjacent to x in $\mathrm{T}_x(\Gamma)$. Since f_u and f^u coincide at x, by the definition of f^u (we use here the choice of $\underline{s}_x$ and $\underline{s}_v$ as minimal) and the fact that $f_u \leq f^u$ we have $\delta_x(f_u) = \underline{s}$, which is precisely the claim. In fact,

- Up to taking a smaller δ, f_u and f^u coincide on all segments of length δ around x.

Consider an out-going branch e' of Γ from X in $\mathrm{out}_X(x)$. Let $s^{\nu'}$ be the slope of f_u along e', and let $I^{e'}$ be a segment of length $\frac{\varepsilon}{s^{\nu'}} = \delta$ on e' with x as end-point. By the previous case, we know that f_u and f^u coincide on a neighborhood of x, so we can take a smaller δ such that they coincide on $I^{e'}$. Since the graph model is finite, we can assume this for all x and e' at the same time, and this does not change X (see Fact 6.17); it does change f_u and f^u (their minimum value $-\varepsilon$ is modified), but does not alter their properties established this far in the proof, so we can simply resume our line of reasoning where it was and conclude that $f_u = f^u$ on any segment $I^{e'}$.

- $f_u = f^u$ everywhere.

To finish the proof of the theorem, let X' be the closure of the complement of $X \cup \bigcup_{x \in \partial X, e' \in \mathrm{out}_X(x)} I^{e'}$. We need to show that $f_u = f^u$ on X'. In other words, we need to show that $f_u = 0$ on X'. Suppose this is not the case, and consider the minimum locus Y of f_u on X'. Note that Y lies in the interior of X', i.e., $Y \cap \partial X' = \varnothing$, and $v \notin Y$. This shows that Y is an unsaturated cut with respect to v and M, which contradicts Theorem 6.14 given that D is v-reduced. This finally establishes the theorem. □

Remark 6.20. Note that the proof of Theorem 6.19 does not use the fact that M is closed, but this does not contradict Remark 6.6. Indeed, Lemma 6.4, assuming closedness, states that D_v exists for all v, whereas Theorem 6.19 only implies that if D_v does exist for some v, then D_u exists for all u in a neighborhood of v. In other words, the set of v for which D_v exists is an open subset of Γ. ⋄

6.7. **Continuity of the map to $|M|$ defined by reduced divisors.** Theorem 6.19 has the following direct consequence.

Let M be a $\mathfrak{g}^r_d$ associated to a crude linear series $(D, \mathfrak{S})$ on a metric graph. Reduced divisors with respect to points of Γ define a map from Γ to the linear system $|M|$. More precisely, define the map

$$\mathrm{Red} : \Gamma \to |M| \subset |(D, \mathfrak{S})| \subset \mathrm{Sym}^d(\Gamma)$$

by sending a point v of Γ to D_v, the v-reduced divisor linearly equivalent to D with respect to M.

Theorem 6.21. *Let M be a $\mathfrak{g}^r_d$ associated to the crude linear series $(D, \mathfrak{S})$. The map $\mathrm{Red} : \Gamma \to |M| \subset \mathrm{Sym}^d(\Gamma)$ is continuous and non-contracting.*

Proof. By Proposition 6.12, the coefficient of D_v at v is precisely $D(v) - \sum_{e \in E_v} s_0^e > 0$ for any point v of Γ. This obviously shows that the map cannot be constant on any segment of positive length, which implies that Red is non-contracting. The continuity follows from Theorem 6.19 of the previous subsection, since the map Red is obviously continuous on small segments I^e for an adjacent branch e at $v \in \Gamma$. □

Remark 6.22. The above theorem is a generalization of [Ami13, Theorem 3]. Note that the map Red is also affine linear with respect to the natural affine structures on Γ and $\mathrm{Sym}^d(\Gamma)$ as described in [Ami13, Section 2]. Thus, the image of Red is a metric graph. In the case $r = 1$, we will see later on that this image is in fact a metric tree (Proposition 7.16). ⋄

7. Classification of $\mathfrak{g}^1_d$'s

In this section, we consider the case $r = 1$, and prove, roughly speaking, that the data of a $\mathfrak{g}^1_d$ on Γ is equivalent to the data of a finite harmonic map to a tree (see Theorem 7.6 below for a precise statement). Then we formulate a smoothing theorem for our combinatorial $\mathfrak{g}^1_d$'s. In this regard, $\mathfrak{g}^1_d$'s on metric graphs are remarkably well-behaved, and our theorem can be regarded as a broad generalization of Eisenbud-Harris smoothing theorem for their limit $\mathfrak{g}^1_d$'s.

Let $(D, \mathfrak{S})$ be a crude linear series and M be a $\mathfrak{g}^1_d$ for $(D, \mathfrak{S})$. By Remarks 4.4 and 5.9, we can assume that $(D, \mathfrak{S})$ and M are effective. This implies that for any point x and any outgoing tangent direction $\nu \in T_x(\Gamma)$ at x, one of the two terms s_0^ν or s_1^ν is equal to zero.

Assume that $(D, \mathfrak{S})$ is defined on a model $G = (V, E)$ of Γ.

Definition 7.1 (Orientation associated to a $\mathfrak{g}^1_d$)**.** Notation as above, we define an orientation of G in such a way that the edge $\{u, v\}$ gets orientation uv if $s_0^{uv} = 0 < s_1^{uv}$. ⋄

Let ρ be a rank function on $\square^\delta_1$. The point $\underline{0}$ is the only point of $\square^\delta_r$ whose image by ρ is 1 (Remark 2.2). Besides, the set of jumps J_ρ of ρ contains the point $\underline{0}$ (because $\rho(\underline{e}_i) = r - 1 = 0$ for all i) and any other point $\underline{a} \neq \underline{0}$ has at least one coordinate equal to one. For each $\underline{a} \in J_\rho$, denote by $P_{\underline{a}}$ the subset of $\{1, \dots, \delta\}$ consisting of all the indices i with $a_i = 1$ (the support of $\underline{a}$). Denote by $\mathcal{P}_\rho$ the collection of all sets $P_{\underline{a}}$ for $\underline{a} \in J_\rho \setminus \{\underline{0}\}$. Recall (Lemma 4.18 and Remark 4.19) that $\mathcal{P}_\rho$ provides a partition of $\{1, \dots, \delta\}$.

This construction provides a direct proof of the realizability of rank functions of rank one, which also follows easily from the fact that such a rank function is geometric in the sense of Section 2 (see the results in [ABBR15a] and [ABBR15b]).

Proposition 7.2. *Any such rank function on $\square_1^\delta$ is realizable, and the field can be chosen to be of characteristic zero.*

Proof. Chose κ to be any infinite field of characteristic zero. Let $P_\rho = \{P_1, \dots, P_s\}$ be the partition of $\{1, \dots, \delta\}$ as previously described, each P_i being a subset of $\{1, \dots, \delta\}$. In the plane κ^2, let $L_1, \dots, L_s$ be distinct lines. Now let, for each $1 \le i \le \delta$, $F_i^1 \coloneqq L_{\tau(i)}$, where $1 \le \tau(i) \le \delta$ is the integer such that $i \in P_{\tau(i)}$. This in turn defines δ complete flags in κ^2, completing the lines with κ^2 and (0). Then it is easy to check that the collection of flags $F_1^\bullet, \dots, F_\delta^\bullet$ is a realization of the rank function ρ. □

For any point p of Γ (resp. G) of valence d_p, consider the rank function ρ_p on $\square_1^{d_p}$, and define $\mathcal{P}_p$ as the partition of $T_p(\Gamma)$ (resp. $\mathbb{E}_p$) given by the previous proposition. Note that S^p consists of the point $(s_0^\nu)_{\nu \in T_p}$, and vectors $(s_{a_\nu}^\nu)$ for $\underline{a} \in \square_1^{d_p}$ with $P_{\underline{a}} \in \mathcal{P}_p$.

Proposition 7.3. *Let v be a vertex of G, and denote by $I_v \subset \mathbb{E}_p$ the set of all the edges vw with $s_0^{vw} < 0$. Then we have $I_v \in \mathcal{P}_v$.*

Proof. We know by hypothesis that $0 \in M$. Let $\underline{a}$ be the element of $\square_1^{d_v}$ such that $\delta_v(0) = s_{\underline{a}}$. Then if vw is an edge incident to v, since exactly one of the possible slopes along vw is zero, we have

$$s_0^{vw} < 0 \Leftrightarrow a_{vw} = 1,$$

which concludes. □

A consequence of this proposition is the following corollary.

Corollary 7.4. *Let p be a point of Γ. For each $P \in \mathcal{P}_p$, all the coordinates $s_i^{e_i}$ with $i \in P$ have the same sign. In addition, there exists at most one $P \in \mathcal{P}_p$ such that all $s_i^{e_i}$ for $i \in P$ have negative sign (and this is I_p).*

Proposition - Definition 7.5. *Let T be a metric tree. Then there exists a unique $\mathfrak{g}_1^1$ on T up to linear equivalence. In fact, it is a refined $\mathfrak{g}_1^1$.*

Proof. Let N be an effective $\mathfrak{g}_1^1$ associated to some crude linear series $(D, \mathfrak{S})$ on T. Then, we can assume that $D = v$ for some vertex v of the tree. For any point y in the tree, let $f_{v \to y}$ to be the unique function taking value zero at v with $\mathrm{div}(f_{v\to y}) + v = y$. Since N has rank one and $N \subseteq \mathrm{Rat}(D)$, $f_{v\to y}$ must belong to N. This shows that $\mathrm{Rat}(D) = N$ and from this, the proposition follows.

Note that, orienting the edges of T away from v, $\mathfrak{S}^{\mathrm{e}}$ is fully determined to allow slopes $0 < 1$ on each oriented edge, and that $\mathfrak{S}^{\mathrm{v}}$ is standard at each vertex of T. We moreover have $N = \mathrm{Rat}(D, \mathfrak{S}) = \mathrm{Rat}(D) = \{f_{x\to y} + c \,|\, y \in T,\, c \in \mathbb{R}\}$. Finally, it is easy to check that $\mathrm{Rat}(D)$ has tropical rank one. □

The following is the main theorem of this section.

Theorem 7.6. *Let M be an effective refined $\mathfrak{g}_d^1$ associated to a crude linear series $(D, \mathfrak{S})$ on Γ. We suppose that D is x_0-reduced for some point $x_0 \in \Gamma$. Then, we have*

- *The image of the map $\mathrm{Red} : \Gamma \to |M| \subset \mathrm{Sym}^d(\Gamma)$ is a metric tree T, and Red is a finite morphism. Moreover, we have $M = \mathrm{Red}^*(N)$, where N is the semimodule on T defined in Proposition–Definition 7.5 using the point $\mathrm{Red}(x_0) \in T$.*
- *There exist a tropical modification $\alpha : \widetilde{\Gamma} \longrightarrow \Gamma$ of Γ and a finite harmonic morphism $\varphi : \widetilde{\Gamma} \to T$ such that $\varphi_{|\Gamma} = \mathrm{Red}$.*

Conversely, let $\psi : \Gamma \to T'$ be any finite harmonic morphism to a metric tree T' on which we put the semimodule N using the point $\psi(x_0)$. Then $\psi^*(N)$ is an effective refined $\mathfrak{g}_d^1$ on Γ for some d, which is the degree of ψ.

Remark 7.7. The theorem shows that the combinatorial $\mathfrak{g}_d^1$'s are the precise analogues of algebraic geometric linear series of rank one. In the context of linear series on metrized complexes introduced in [AB15], a smoothing theorem in rank one was previously obtained by Luo and Manjunath in [LM18]. However, in that context, there are obstructions to smoothing. ⋄

As a corollary, using [ABBR15a, Theorem 7.7] and [ABBR15b, Theorem 3.11], we get the following smoothing theorem. See [ABBR15a] and [ABBR15b] for terminology.

Theorem 7.8 (Smoothing theorem for $\mathfrak{g}_d^1$'s)**.** *Any refined $\mathfrak{g}_d^1$ $M \subset \mathrm{Rat}(D, \mathfrak{S})$ on Γ is smoothable.*

That is, there exists a smoothing X of the metrized complex $\mathfrak{C}$ defined by Γ and the collection of marked curves $(C_v, \{x_e^v\}_{e \in \mathbb{E}_v})$, $v \in V(G)$, a divisor E on X, and a vector subspace $H \subset H^0(\mathcal{O}(E))$ of rank one on X such that M is the specialization of the $\mathfrak{g}_d^1$ (E, H) from X to Γ.

Remark 7.9. This crucially uses the fact that rank functions of rank one are geometric in the terminology of Section 2 (see [ABBR15a] and [ABBR15b]). This is applied to every ρ_v with $v \in V$ for $G = (V, E)$ a model a Γ on which $(D, \mathfrak{S})$ is defined. ⋄

The rest of this section is devoted to the proof of Theorem 7.6.

7.1. **Proof of Theorem 7.6.** To this end, we will define an equivalence relation and a partial preorder on the points of Γ, show that the equivalence classes correspond to the map Red and show that the quotient $\Gamma/\sim$ is a metric tree.

Let M be an effective refined $\mathfrak{g}_d^1$ associated to a crude linear series $(D, \mathfrak{S})$ on a metric graph Γ. We define an equivalence relation $\sim_M$ on the set of points of Γ as follows. For two points $x, y \in \Gamma$, we write $x \sim_M y$ (or simply $x \sim y$) if for all $f \in M$ we have $f(x) = f(y)$. We also define the corresponding partial order by writing $x \leq_M y$, or simply $x \leq y$, when for all $f \in M$ we have $f(x) \leq f(y)$. Note that if x and x' belong to the same edge e, then x and x' are comparable for $\leq_M$: in this case, the comparability is given by the orientation of e. We also write $x <_M y$, or simply $x < y$, the corresponding strict partial preorder, when we have $x \leq y$ and there exists some $f \in M$ such that $f(x) < f(y)$. The statement $x <_M y$ is equivalent to $x \leq_M y$ and $x \nsim_M y$.

It is immediate that $\leq_M$ is indeed reflexive and transitive, and $<_M$ irreflexive, transitive and asymmetric. Also, $\leq_M$, resp. $<_M$, induces a well-defined partial order, resp. a well-defined strict partial order, on the quotient $\Gamma/\sim$.

We first show that $\sim$ is a well-behaved equivalence relation.

Proposition 7.10. *For all $x \in \Gamma$, the class of x under $\sim$ is finite.*

Proof. Let x and x' belong to a common edge e. Suppose without loss of generality that $x \leq x'$. Then by Remark 5.14 there exists $f \in M$ which takes all minimal slopes around x' (for example $f = f_{x'}$), in particular in the direction of x in which the minimal slope is negative. Thus $f(x) < f(x')$ and $x \nsim y$. This shows that we can only have $x \sim x'$ if x and x' share no edge, from which the result follows. □

This shows in particular that Red is a finite morphism.

Remark 7.11. A consequence of Proposition 7.10 is that, at each point, each jump is realized, a particular case of Theorem 5.27. ⋄

We now show that the equivalence relation corresponds to the map Red.

Proposition 7.12. *Let $x, y \in \Gamma$. Then $x \sim y \Longleftrightarrow \mathrm{Red}(x) = \mathrm{Red}(y)$.*

Proof. $\Longrightarrow$ If $x \sim y$, then the sets M_x and M_y coincide, which shows, according to Definition 6.2, that $f_x = f_y$ and therefore $\mathrm{Red}(x) = \mathrm{Red}(y)$.

$\Longleftarrow$ If $\mathrm{Red}(x) = \mathrm{Red}(y)$, by Proposition 4.1, there exists a constant c such that $f_{x'} = f_x + c$. Suppose that $c > 0$. Then $f_{x'}(x) = f_x(x) + c = c > 0$, which is impossible because for all v, $f_v \le 0$. So we have in fact $f_x = f_{x'}$. Now let g be any function of M such that $g(x) = 0$. We need to show that $g(x') = 0$. First we note that

$$g(x') \ge f_x(x') = f_{x'}(x') = 0.$$

Second, let $h \coloneqq g - g(x')$. h belongs to M and verifies $h(x') = 0$, so we know that $h \ge f_{x'} = f_x$ and thus $-g(x') = h(x) \ge f_x(x) = 0$, which concludes. ☐

The map Red is equal to the projection map $\Gamma \to \Gamma/\sim$. We denote $\Gamma/\sim$ by T, and the goal is now to show that T is a metric tree. This amounts to showing that T has no cycle (T is already a metric graph), and the proof will use the condition that the tropical rank is one.

Lemma 7.13. *Let $\overline{x} \in T$. Then $\overline{x}$ has indegree at most one in T.*

Proof. Suppose on the contrary that $\overline{x}$ has indegree at least two, that is, using the orientation of G defined by $\mathfrak{S}$, that two edges $\overline{e}$ and $\overline{e}'$ incident to x are oriented toward x. There are two cases.

(i) $\overline{e}$ and $\overline{e}'$ originate from two edges of G, e and e' respectively which are incident to the same preimage x of $\overline{x}$. We show that this cannot happen since e and e' are in fact glued together by Red in the following natural way: there exist some $\lambda, \varepsilon > 0$ such that
(a) ε and $\lambda \cdot \varepsilon$ are smaller than $\ell(e), \ell(e')$, and
(b) For $y \in e$ at distance ε from x, for $y' \in e'$ at distance $\lambda \cdot \varepsilon$ from x,
then $y \sim y'$.

Let now g be any function of M. Without loss of generality, we assume that $g(x) = 0$. We know that f_x takes all minimal slopes around x, and in particular takes negative slopes on e and e' away from x. These two slopes can be different, but since there are both negative, they identify e and e' like described above, with a dilation factor $\lambda_0 = \frac{s_0^e}{s_0^{e'}}$. The null function 0 also identifies e and e' with the same dilation factor (in fact, for any dilation factor).

Since M is assumed to have tropical rank one, there exist $c, d \in \mathbb{R}$ such that for all $z \in \Gamma$, the minimum in

$$\min(0, f_x(z) + c, g(z) + d)$$

is attained at least twice.

Let us first assume that c is negative. This implies that, for y on e or e', $f_x(y) + c < 0$ and thus $g(y) + d = f_x(y) + c$. So $g = f_x$ on e and e' (evaluate at x) and thus identifies e and e' with dilation factor λ_0.

Now let us assume that c is positive. Then for y on e or e' *close to* x, $f_x(y) + c$ is still positive and thus $g(y) + d = 0$. In fact $d = 0$ and g is null close to x on e and e', so g identifies e and e' with dilation factor λ_0.

The last case is when $c = 0$. For y on e different from x, we have $f(y) = g(y) + d$ and so by continuity g identifies e and e' with dilation factor λ_0.

We have thus shown that in all cases, g identifies the two edges oriented toward x. Since this is true for all $g \in M$, the map Red glues e and e' together, and $\overline{x}$ cannot have indegree at least two.

(ii) $\overline{e}$ and $\overline{e}'$ originate from two edges of G, e and e' respectively, which are incident respectively to two different preimages x and x' of $\overline{x}$. Since by definition $f_x = f_{x'}$ and $g(x) = g(x')$ for all $g \in M$, we can use exactly the same argument, *mutadis mutandis.*

□

Fact 7.14. We have the following fact: the edges incident to some $x \in \Gamma$ that are glued by Red are exactly those belonging to the same set in the partition $\mathcal{P}_x$.

$\Longleftarrow$ Take $g \in M$. Then $\delta_x(g)$ is a jump in S^x, so, using Lemma 4.18 and Proposition 7.3, we are in one of the three following cases:

(i) g is constant around x.
(ii) g takes negative slopes away from x on all incoming edges, and zero slope on all outgoing edges.
(iii) g takes negative slopes away from x on all incoming edges, positive slopes on some set $S \in \mathcal{P}_x$ of outgoing edges and slope zero on all other (outgoing) edges.

In each one of these cases, g identifies the edges of each set of $\mathcal{P}_x$. Since this is true for all g, then Red glues together e and e'.

$\Longrightarrow$ If e and e' are two edges that do not belong to the same set of $\mathcal{P}_x$, then we can assume without loss of generality that e is an outgoing edge (see Corollary 7.4). Moreover, there is a jump $\underline{a} \in \square_1^{d_x}$ such that $e \in P_{\underline{a}}$ and $e' \notin P_{\underline{a}}$. The jump $\underline{a}$ is realized by some function $f \in M$ (Remark 7.11). There are two cases:

(a) e' is also an outgoing edge. Then f has a positive slope on e (away from x) and zero slope on e'.
(b) e' is an incoming edge. Then f has a positive slope on e and negative slope on e'.

In both cases, f does not identify e and e', so Red does not glue these edges.

Finally, every edge incident to x in Γ is sent to an actual edge incident to $\overline{x}$ in T because Red is affine linear and non-contracting (Theorem 6.21). ⋄

Remark 7.15. Fact 7.14 has an interesting consequence. At a point $x \in \Gamma$, we know that the set of all incoming edges constitutes a set of the partition $\mathcal{P}_x$ (Proposition 7.3). Then applying the first part of Remark 7.14 gives that all incoming edges at x are glued together by the map Red, which yields case (i) in Lemma 7.13 automatically. However, case (ii) really requires the argument involving tropical rank developed in the proof. ⋄

We now come to the desired result.

Proposition 7.16. *T has no cycles and so is a metric tree.*

Proof. T can have no *oriented* cycle. Indeed, Proposition 7.10 states that if x and y are the vertices of an edge oriented from x to y, then $x < y$ (using the strict partial order induced on T). So in an oriented cycle, we would get a strict inequality of the form $x < x$, which is absurd.

Moreover, T can have no *other kind of cycle*. Indeed, a cycle which is not an oriented cycle goes through some vertex with indegree at least two, which is impossible (see Lemma 7.13). □

We thus conclude that T is an acyclic metric graph, that is, a metric tree. Note that the metric on T is the one induced from Γ and the gluing. It is the only metric such that the slopes of the functions $T \to \mathbb{R}$ factored from functions $\Gamma \to \mathbb{R}$ in $\mathrm{Rat}(D, \mathfrak{S})$ are in the set $\{-1, 0, 1\}$. Roughly speaking, this amounts to giving an edge e of T a length equal to the product of the length of e' times the non-zero possible slope on e', for e' any edge of Γ send to e by Red. Equivalently, the metric is such that the relative slope of Red on an edge is the non-zero possible slope on this edge.

We now want to show the following claim of Theorem 7.6: $M = \mathrm{Red}^*(N)$, where N is associated to $\mathrm{Red}(x_0) \in T$.

Proof. $\subset$ Let $f \in M$. By definition of Red, f is constant on the equivalence classes for $\simeq_M$, so it can be written in the form $f = g \circ \varphi$, with g a function $T \to \mathbb{R}$. It is straightforward that g is continuous and affine linear (see Proposition 7.10 which implies that φ is a local homeomorphism on its image). By definition of the metric on T, g has slopes zero and one compatible with $\mathfrak{S}^{\mathrm{e}}$. To show that g is compatible with D and $\mathfrak{S}^{\mathrm{v}}$, we simply look at the possible sets of slopes of f around a point $y \in \Gamma$ in the cases (i) and (ii) explored in the proof of Proposition–Definition 7.5.

$\supset$ The other way around, let $g \in N$ and $f := g \circ \varphi$. We want to show that $f \in M$. Since g belongs to N, it is of the form $g = f_y = f_{\mathrm{Red}(x_0)\to y} \in \mathrm{Rat}(\mathrm{Red}(x_0), \mathfrak{S}')$ with $y \in T$ and $\mathfrak{S}'$ the only slope structure on T (following the notations of the proof of Proposition–Definition 7.5). Let $w \in \mathrm{Red}^{-1}(y)$. All the functions $f_{x_0 \to w}$ for such w are equal thanks to Proposition 7.12. The fact that f belongs to M then comes from the equality $f = f_w$ which is implied by Remark 5.14. □

To finish the first part of the theorem, we observe that any finite morphism from a metric graph Γ to a metric tree T can be resolved to a finite harmonic morphism by a tropical modification of Γ, see [ABBR15b].

Now we show the second part of Theorem 7.6: if $\psi : \Gamma \to T'$ is any finite morphism to a metric tree T' on which we put the semimodule N using the point $\psi(x_0)$, then $\psi^*(N)$ is an effective refined $\mathfrak{g}^1_d$ on Γ.

Proof. We first have to define the pullback of $\mathfrak{S}'$ by ψ on Γ. The (effective) divisor D is taken to be $\psi^{-1}(\psi(x_0))$. Its degree is denoted by d, and it is the degree of ψ. At some point $x \in \Gamma$, in the direction $\nu \in \mathrm{T}_x(\Gamma)$, the non-zero possible slope is defined to be the relative slope of ψ in the direction ν: this defines $\mathfrak{S}^{\mathrm{e}}$. We now define $\mathfrak{S}^{\mathrm{v}}$ around x by saying that the jumps of ρ_x are exactly the vector $\underline{0}$, and the vectors having ones for all edges belonging to a certain complete set of edges identified by ψ, and zero on all other edges, which entirely defines ρ_x. We have defined the pair $(D, \mathfrak{S})$.

Let us now show that $\psi^*(N)$ is a sub-semimodule of $\mathrm{Rat}(D, \mathfrak{S})$. Firstly, it is stable by the two tropical operations since N is. Secondly, we show that $\psi^*(N) \subset \mathrm{Rat}(D, \mathfrak{S})$. If f is a function of $\psi^*(N)$, we can write it $f = g \circ \psi$ with $g \in N$. It is automatic by the construction of $\mathfrak{S}$ on Γ that f is compatible with $\mathfrak{S}^{\mathrm{e}}$ and $\mathfrak{S}^{\mathrm{v}}$. The fact that $D + \mathrm{div}(f) \geq 0$ comes from the harmonicity of ψ.

We now check property $(*)$ of Definition 4.9 to show that $(D, \mathfrak{S})$ is a crude linear series. Let $y \in \Gamma$ and $E = y$. Then, the function

$$f_{\psi(x_0)\to\psi(y)} \circ \psi$$

has the required properties (1) and (2). This function a priori belongs to $\psi^*(N)$, but we just showed that $\psi^*(N) \subset \mathrm{Rat}(D, \mathfrak{S})$. Therefore, $(D, \mathfrak{S})$ is a crude linear series. Moreover, we also checked property $(**)$ of Definition 5.12.

Finally, the closedness of $\psi^*(N)$ in $\mathrm{Rat}(D, \mathfrak{S})$ and the fact that $\psi^*(N)$ has tropical rank one follow from the same properties for N. This finishes the proof that $\psi^*(N)$ is an effective refined $\mathfrak{g}^1_d$. □

This finishes the classification of refined $\mathfrak{g}^1_d$'s on metric graph.

8. Limit linear series on the skeleton of a Berkovich curve

Let now $\mathbb{K}$ be an algebraically closed field with a non-trivial non-Archimedean valuation val and C be a smooth proper curve over $\mathbb{K}$. We assume that $\mathbb{K}$ is complete with respect to val and we denote by κ the residue field of $\mathbb{K}$, which is also algebraically closed.

8.1. **Limit linear series defined by divisors.** Let $\mathcal{D}$ be divisor of degree d and rank r on C, and $(\mathcal{O}(\mathcal{D}), H)$ be a $\mathfrak{g}^r_d$ on C. We identify H with a subspace of $\mathbb{K}(C)$ of dimension $r+1$. Let Γ be a skeleton of C^{an}. In this section, we define two notions of reduction (or specialization) of $(\mathcal{O}(\mathcal{D}), H)$ to Γ. The first one will be a crude linear series on Γ that we will call the *crude limit linear series on* Γ *induced by* $(\mathcal{O}(\mathcal{D}), H)$. The second one will be a refined linear series, that is, a refined $\mathfrak{g}^r_d$, on Γ, that we will call the *limit linear series* (or *limit* $\mathfrak{g}^r_d$) *on* Γ *induced by* $(\mathcal{O}(\mathcal{D}), H)$. First we recall the following basic fact from [AB15] (Lemma 4.3). For the notations, see [AB15, Subsections 4.2 and 4.4].

Lemma 8.1. *Let C be a smooth proper curve over $\mathbb{K}$, and $x \in C^{\mathrm{an}}$ a point of type II. The κ-vector space H_x defined by the reduction to $\widetilde{\mathcal{H}(x)}$ of an $(r+1)$-dimensional $\mathbb{K}$-subspace $H \subset \mathbb{K}(C)$ has dimension $r+1$.*

By the slope formula, the reduction $F \coloneqq \mathrm{trop}(f) = -\log(|f|)$ of any function $f \in H$ to Γ is a piecewise affine function on Γ with integer slopes. Let first x be a type II point of Γ, and ν a tangent direction in $\mathrm{T}_x(\Gamma)$. Denote by C_x the corresponding smooth projective curve over κ, which has function field $\kappa(C_x) = \widetilde{\mathcal{H}(x)}$, and let p^ν_x be the point of $C_x(\kappa)$ which corresponds to ν.

By Lemma 8.1, the dimension of $H_x \subset \kappa(\mathfrak{C}_x)$ is $r+1$. The orders of vanishing of reductions $\widetilde{f} \in H_x$ of elements $f \in H$ at p^ν_x define a sequence of integers $s^\nu_0 < s^\nu_1 < \cdots < s^\nu_r$ (see Proposition 2.15). Denote by $S^\nu = \{s^\nu_i\}$. In addition, the collection of points $p^\nu_x \in C_x(\kappa)$ for $\nu \in \mathrm{T}_x(\Gamma)$ defines a geometric rank function ρ_x associated to the corresponding filtrations on $\widetilde{H}$ as in Section 2.3. We define S^x as the set of jumps of ρ_x. We have the following theorem which can be regarded as a refinement of [AB15, Theorem 5.9].

Theorem 8.2 (Specialization of linear series yielding a crude linear series on skeleta)**.** *Notations as above, let $(\mathcal{O}(\mathcal{D}), H)$, $H \subseteq H^0(C, \mathcal{O}(\mathcal{D})) \subset \mathbb{K}(C)$, be a g^r_d on C. Let Γ be a skeleton of C^{an}. There exists a semistable vertex set V for C such that $\Sigma(C, V) = \Gamma$, and such that the slopes of rational functions f in H along edges in Γ define a well-defined crude linear series $(D, \mathfrak{S})$ on Γ, with $D = \tau_*(\mathcal{D})$, described as above. We call this a* crude limit linear series on Γ.

Proof. We already defined S^x and S^ν for type II points of $\Gamma \subset C^{\mathrm{an}}$ and $\nu \in \mathrm{T}_x(\Gamma)$. The main point of the above theorem is that the definitions can be extended to all points of Γ, and that the collection $\mathfrak{S} = \{S^x; S^\nu\}_{x \in \Gamma, \nu \in \mathrm{T}_x(\Gamma)}$ is induced from a simple graph model of Γ (or equivalently, from a semistable vertex set of C^{an}). To show this, let x be a point of type II, and let $\nu \in \mathrm{T}_x(\Gamma)$ be a tangent direction at x in Γ. Let $f_0, \dots, f_r$ be a basis of H such that the reductions $f_{0,x}, \dots, f_{r,x}$ to $\widetilde{H(x)}$ yield orders of vanishings $s^\nu_0, \dots, s^\nu_r$ at the point p^ν_x. By the slope formula, the slope of $F_j = \mathrm{trop}(f_j)$ along ν at x coincides with s^ν_j. There thus exists a half segment $I^\nu_x = [x, y^\nu]$ on the edge supporting the point x and tangent direction ν such that the slope of functions F_j along ν at any point of I^ν_x is s^ν_j. By Lemma 8.1, these are all the possible slopes along ν in I^ν_x of tropicalizations of functions in H. We can thus extend the definition of $S^\bullet$ to any point of type III on I^ν_x in this segment, by taking these slopes and by declaring the rank function to be standard. Applying now the compactness of Γ, we deduce a finite covering of Γ by segments I^ν_x, from which we deduce the first statement in the theorem.

The fact that $\tau_*(\mathrm{div}(f)) = \mathrm{div}(\mathrm{trop}(f))$, which is a consequence of the slope formula, then shows that $(\tau_*(\mathcal{D}), \mathfrak{S})$ has rank r (defined in Definition 4.9). This is analogous to the proof of the specialization theorem for metrized complexes in [AB15], we thus omit the details. This shows that $(D, \mathfrak{S})$ is a crude linear series on Γ. ∎

We should say that in particular, for two linearly equivalent divisors $\mathcal{D} \sim \mathcal{D}'$ on C, and H a subspace of the space of global sections of the corresponding line bundles $\mathcal{O}(\mathcal{D}) \simeq \mathcal{O}(\mathcal{D}')$ of projective dimension r, the above theorem ensures that the two pairs $(D, \mathfrak{S})$ and $(D', \mathfrak{S}')$ are linearly equivalent.

Note that in particular, if $(D, \mathfrak{S})$ is a limit crude linear series on Γ induced by a $\mathfrak{g}^r_d$ $(\mathcal{D}, H)$ on C, for $H \subseteq H^0(C, \mathcal{O}(\mathcal{D}))$, then, by Proposition 6.12, for any point x of Γ, we have

$$D(x) - \sum_{\nu \in \mathrm{T}_x(\Gamma)} s_0^\nu \geq r\,. \tag{4}$$

Our aim now is to refine the above theorem by defining limit linear series which are refined linear series on Γ.

This allows in particular to show that the results stated in Sections 6 and 7 apply to linear series obtained from geometry.

We take $(\mathcal{O}(\mathcal{D}), H)$ with $H \subset H^0(C, \mathcal{O}(\mathcal{D})) \subset \mathbb{K}(C)$ a $\mathfrak{g}^r_d$ on C, and V a well-chosen vertex for C, as in Theorem 8.2. This yields a crude linear series $(D, \mathfrak{S})$ on Γ. We now define

$$M \coloneqq \mathrm{trop}(H) = \{\mathrm{trop}(f), f \in H \setminus \{0\}\}.$$

It is easy to see by construction, applying the slope formula, that $M \subset \mathrm{Rat}(D, \mathfrak{S})$.

Theorem 8.3 (Specialization of linear series yielding a refined linear series on the skeleta)**.** *If the value group of* val *is* $\mathbb{R}$*, then* M *is a refined* $\mathfrak{g}^r_d$ *on* Γ*.*

Remark 8.4. More generally, if Γ and D are Λ-rational and if the value group of val is Λ, then M is a refined Λ-$\mathfrak{g}^r_d$ on Γ. (Note that the value group of val is automatically divisible if $\mathbb{K}$ is algebraically closed, which is the assumption we make in our setup.) ⋄

Proof. We first show that M is a sub-semimodule of $\mathrm{Rat}(D, \mathfrak{S})$. We will assume without loss of generality that the support of D is included in the set of vertices V. Let F and G be functions of M, and let $\lambda \in R$. By definition, we can write $F = \mathrm{trop}(f)$ and $G = \mathrm{trop}(g)$ with $f, g \in H \setminus \{0\}$. Since val is assumed to be surjective, we write $\lambda = \mathrm{val}(\alpha)$ with $\alpha \in \mathbb{K}$. We write $h \coloneqq \alpha f + g$ and $H \coloneqq \mathrm{trop}(h)$. The goal is to show that $\min(F + \lambda, G) = H$, replacing α if necessary with $\alpha \cdot \beta$ for $\beta \in \mathbb{K}$ of valuation zero, so that the valuation remains λ.

Let $x \in \Gamma$ such that $F(x) + \lambda \neq G(x)$. Then, by the enhanced non-Archimedean triangular inequality (that is, if $\mathrm{val}(a) \neq \mathrm{val}(b)$, then $\mathrm{val}(a + b) = \min(\mathrm{val}(a), \mathrm{val}(b))$), we get automatically that $H(x) = \min(F(x) + \lambda, G(x))$.

Let now $\Gamma_0 \coloneqq \{x \in \Gamma, F(x) + \lambda = G(x)\}$. Since F and G are piecewise affine linear, Γ_0 can be written as a union of finitely many segments of Γ (a segment of Γ is a segment of an edge of G; it can be reduced to just a point). These segments can be refined so that both F and G are affine linear on each of these segments. Let now I be any of them, whose extremities we denote x and y, on an edge e of Γ. Let ν be the tangent direction in $T_x(\Gamma)$ pointing toward y, and p^ν_x be the point of $C_x(\kappa)$ corresponding to ν. By the slope formula and the fact that $F + \lambda = G$ on all I, we have

$$\mathrm{ord}_{p^\nu_x}\left(\widetilde{f}\right) = \mathrm{slope}_\nu(F) = \mathrm{slope}_\nu(G) = \mathrm{ord}_{p^\nu_x}\left(\widetilde{g}\right).$$

Up to multiplying α by some element β of $\mathbb{K}^\times$ of valuation zero, we can ensure that

$$\mathrm{ord}_{p^\nu_x}\left(\widetilde{\alpha f + g}\right) = \mathrm{ord}_{p^\nu_x}\left(\widetilde{f}\right) = \mathrm{ord}_{p^\nu_x}\left(\widetilde{g}\right) = \mathrm{ord}_{p^\nu_x}\left(\widetilde{\alpha}\widetilde{f} + \widetilde{g}\right).$$

Note that only one value for the reduction $\widetilde{\alpha}$ in κ is forbidden. Using again the slope formula for $\widetilde{\alpha f + g}$ yields $\mathrm{ord}_{p^\nu_x}\left(\widetilde{\alpha}\widetilde{f} + \widetilde{g}\right) = \mathrm{slope}_\nu(H)$, so that

$$\mathrm{slope}_\nu(H) = \mathrm{slope}_\nu(F) = \mathrm{slope}_\nu(G).$$

We can do the same at the other extremity y of I, and ensure that locally, starting at either extremity of I, H has the same slope as F and G. Since F and G are linear on I, D has no support in the interior of I, and H coincides with $F + \lambda$ and G on the extremities of I, we must have $H = F + \lambda = G$ on the full interval I.

Since this can be done for each of the finitely many segments composing Γ_0, forbidding at most one value for the reduction $\tilde{\alpha} \in \kappa$ each time, and since κ is algebraically closed (and thus infinite), there is some $\alpha \in \mathbb{K}$ such that $\mathrm{val}(\alpha) = \lambda$ and

$$\mathrm{trop}(\alpha f + g) = \min(F + \lambda, G)$$

on all Γ. We have shown that $\min(F + \lambda, G) \in M$, so M is a sub-semimodule of $\mathrm{Rat}(D, \mathfrak{S})$.

Secondly, we show that M is closed in $(\mathrm{Rat}(D), \|\cdot\|_\infty)$. Observe that

$$|\mathcal{D}| \coloneqq \{\mathcal{D} + \mathrm{div}(f),\ f \in H \setminus \{0\}\}$$

is a closed subset of $\mathrm{Sym}^d(C^{\mathrm{an}})$, the d-symmetrical product of C^{an}, which is compact.

Considering the continuous specialization map

$$\tau_* : \mathrm{Sym}^d(C^{\mathrm{an}}) \longrightarrow \mathrm{Sym}^d(\Gamma),$$

we infer that $\tau_*(|\mathcal{D}|)$ is also compact. Thanks to the slope formula, this space is equal to $\{D + \mathrm{div}(F),\ F \in M\}$. This implies that M is closed in $\mathrm{Rat}(D, \mathfrak{S})$.

Thirdly, we have to show that property $(**)$ in Definition 5.12 is verified. This is a refinement of the fact, obtained in the proof of Theorem 8.2, that the pair $(D, \mathfrak{S})$ has rank r according to Definition 4.9 (property $(*)$). This property is automatically verified, since the functions used in verifying $(*)$ all come from geometry.

Finally, we need to show that M has tropical rank r. Keeping in mind Remark 5.17, we have to prove that the tropical rank is at most r. Let $F_0, \dots, F_{r+1} \in M$. For each i, we write $F_i = \mathrm{trop}(f_i)$ with $f_i \in H \setminus \{0\}$. Since $\dim(H) = r + 1$, there exist some $\lambda_i \in \mathbb{K}$ such that $\sum_i \lambda_i f_i = 0$. This shows that for all $x \in \Gamma$, the minimum in

$$\min_{0 \le i \le r+1} (F_i(x) + \mathrm{val}(\lambda_i))$$

is attained at least twice. Therefore the tropical rank of M is $\le r$. $\square$

8.2. **Limit linear series defined by pluri-canonical sheaves.** In this section we discuss the tropicalization of subspaces of global sections of pluri-canonical sheaves, and explain how it fits to the theory presented in the previous sections.

Let C be a smooth proper curve over an algebraically closed complete non-trivially valued non-Archimedean field $\mathbb{K}$. We assume that the residue field κ of $\mathbb{K}$ has characteristic zero. We denote by ω_C the canonical sheaf of C, and by $\omega_C^{\otimes n}$ its n-th power, for $n \in \mathbb{N}$.

Definition 8.5 (Pluri-canonical linear series)**.** By a *pluri-canonical linear series* of rank r and order n, we mean a vector subspace $H \subseteq H^0(C, \omega_C^{\otimes n})$ of rank r, i.e., of dimension $r+1$. $\diamond$

We follow the notations of the previous section and denote by Γ a skeleton of C^{an} with combinatorial model $G = (V, E)$. The vertex set is included in the set of points of type II, and for each $v \in V$, the corresponding curve over the residue field κ is denoted by C_v. Note that Γ is an augmented metric graph: the genus function g is given by sending points x of type II to the genus of the corresponding curve C_x, extended by zero over all the points of Γ.

8.2.1. *Temkin metrization and tropicalization.* Let $\|\cdot\|$ be the Kähler norm on the sheaf of differentials ω_C defined by Temkin in [Tem16]. It induces a norm on each $\omega_C^{\otimes n}$, $n \in \mathbb{N}$, that we continue to denote by $\|\cdot\|$.

Definition 8.6 (Tropicalization of pluri-canonical forms)**.** For each non-zero section α of $\omega_C^{\otimes n}$, the tropicalization of α is the function

$$\mathrm{trop}(\alpha)\colon \Gamma \longrightarrow \mathbb{R}, \qquad x \longmapsto -\log \|\alpha\|_x.$$

$\diamond$

Let $K = K_\Gamma$ be the canonical divisor of the augmented metric graph Γ, given explicitly by

$$K = \sum_{x \in \Gamma} (2g(x) - 2 + d_x)\,(x).$$

Theorem 8.7 (Slope formula)**.** *The tropicalization $F = \mathrm{trop}(\alpha)$ of a pluri-canonical differential form of order n is a piecewise affine function with integral slopes. It verifies the slope formula*

$$\mathrm{div}(F) + n\,K = \tau_*(\mathrm{div}(\alpha))$$

with $\mathrm{div}(\alpha)$ the divisor of zeroes of α, defined by $\mathrm{div}(\alpha) := \sum_{x \in C} \mathrm{ord}_x(\alpha)$, $\tau \colon C^{\mathrm{an}} \longrightarrow \Gamma$ the retraction map to Γ, and τ_* the induced map on the level of divisors.

Proof. The first part is a consequence of [Tem16, Theorem 8.2.4]. The second is proved independently in [Ami14, KRZ16, BN16, BT20]. □

By (a generalization of) the slope formula for the reduction of (pluri-)canonical differential forms of order n, the scaled reduction $\widetilde{\alpha}$ of a pluri-canonical form α at a point $x \in C^{\mathrm{an}}$ of type II is a pluri-canonical meromoprhic form on C_x, which is well-defined up to scaling by a scalar in κ^*. In addition, for any tangent direction ν at x, the order of vanishing of $\widetilde{\alpha}$ at the point p_x^ν of C_x is given by $-n - \mathrm{slope}_\nu F(x)$, see e.g. [BT20, Lemma 3.3.2] for the case $n = 1$.

Let $H \subseteq H^0(C, \omega_C^{\otimes n})$ be a pluri-canonical linear series of rank r and order n. Let $M := \mathrm{trop}(H) = \{\mathrm{trop}(\alpha) \mid \alpha \in H \setminus \{0\}\}$, and denote by H_x the vector space of pluri-canonical meromorphic forms of the same order on C_x generated by the reductions $\widetilde{\alpha}$ of $\alpha \in H$ at x. As in the case of the reduction of functions, the dimension of $H_x \subset \kappa(C_x)$ is $r+1$ (see Lemma 8.1), and the orders of vanishing of reductions $\widetilde{\alpha} \in H_x$ of elements $\alpha \in H$ at p_x^ν define a sequence of integers $s_0^\nu < s_1^\nu < \cdots < s_r^\nu$. We define $S^\nu = \{s_i^\nu\}$, and denote by ρ_x the geometric rank function associated to the collection of points $p_x^\nu \in C_x(\kappa)$ for $\nu \in \mathrm{T}_x(\Gamma)$. Moreover, we define S^x as the set of jumps of ρ_x. We have the following theorem.

Theorem 8.8 (Specialization of pluri-canonical linear series)**.** *Let $H \subseteq H^0(C, \omega_C^{\otimes n})$ be a pluri-canonical linear series of rank r and order n. Let $M := \mathrm{trop}(H) = \{\mathrm{trop}(\alpha) \mid \alpha \in H\}$. Then $M \subseteq \mathrm{Rat}(nK, \mathfrak{S})$. Moreover, M is a refined $\mathfrak{g}_d^r$ on Γ, for $d = n(2g-2)$.*

Proof. The proof is similar to that of Theorem 8.3 given in the previous section. □

Definition 8.9 (Pluri-canonical limit linear series)**.** Notations as above, the semimodule

$$M_n^{\mathrm{can}} := \mathrm{trop}\left(H^0(C, \omega_C^{\otimes n})\right) \subseteq \mathrm{Rat}(nK, \mathfrak{S})$$

is called the *tropical semimodule of pluri-canonical differential forms* on Γ induced by C. The slope structure $\mathfrak{S}$ is called the *pluri-canonical slope structure of order n on* Γ (induced by C). For any pluri-canonical linear series $H \subseteq H^0(C, \omega_C)$ of rank r and order n, the sub-semimodule M of M_n^{can} defined by the specialization of H is called the *limit pluri-canonical linear series* $\mathfrak{g}^r_{n(2g-2)}$, of rank r and order n, on Γ induced by H. ⋄

8.2.2. *Combinatorial pluri-canonical types.* For each pluri-canonical slope structure on Γ of order n, its combinatorial type is the pair $(G, \mathfrak{S})$ consisting of the minimal graph model G of Γ over which $\mathfrak{S}$ is defined.

Theorem 8.10 (Finiteness of pluri-canonical slope structures)**.** *There are only finitely many combinatorial types for pluri-canonical slope structures of order n on augmented metric graphs Γ of the same combinatorial type.*

Proof. This follows directly from Theorem 4.17. □

Let $G = (V, E)$ be a given graph that we assume to be augmented with a genus function. A slope structure $\mathfrak{S}$ on G is called *pluri-canonical* if there exists a length function $l \colon E \to \mathbb{R}$ on the edges of G such that $\mathfrak{S}$ defines a pluri-canonical slope structure on Γ.

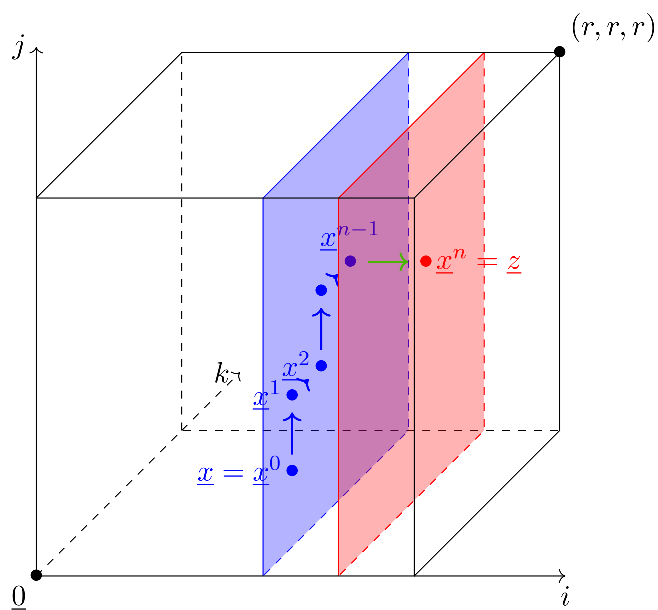
Problem 8.11. *Provide a classification of all the pluri-canonical slope structures on a given augmented graph G.*

Appendix A.

In this section, we prove Theorem 3.9. We will need the following two lemmas.

Lemma A.1. *Let ρ be a rank function on $\square_r^\delta$. Let $1 \leq i \leq \delta$ and $0 \leq t < r$. Consider an element $\underline{x} \in L_t^i$ such that there are no jumps $\underline{y} \in L_t^i$ verifying $\underline{y} \geq \underline{x}$. Then there exists a point $\underline{z} \in L_{t+1}^i$ such that $\underline{z} \geq \underline{x}$ and $\rho(\underline{z}) = \rho(\underline{x})$.*

Proof. Let $\underline{x}^0 = \underline{x}$. Since $\underline{x}^0$ is itself not a jump, there is some $1 \leq k^0 \leq \delta$ such that $\underline{x}^1 := \underline{x}^0 + \underline{e}_{k^0}$ belongs to $\square_r^\delta$ and $\rho(\underline{x}^0 + \underline{e}_{k^0}) = \rho(\underline{x}^0)$. If $\underline{x}^1 \notin L_t^i$, we stop here; if $\underline{x}^1 \in L_t^i$, we carry on and define a point $\underline{x}^2 := \underline{x}^1 + \underline{e}_{k^1} \in \square_r^\delta$ having the same image by ρ. This defines inductively a finite sequence $\underline{x} = \underline{x}^0, \underline{x}^1, \dots, \underline{x}^n$ whose elements are in L_t^i (except $\underline{x}^n$) and whose image by ρ is constant, equal to $\rho(\underline{x})$ (including $\underline{x}^n$). But by construction $\underline{z} := \underline{x}^n$ verifies $\underline{z} \in L_{t+1}^i$ and $\underline{z} \geq \underline{x}$. $\square$



Remark A.2. For $t = r$, following the same line of reasoning, we get a finite sequence $\underline{x} = \underline{x}^0, \underline{x}^1, \dots, \underline{x}^{n-1} = (r, \dots, r)$ with $\rho(\underline{x}^i) = \rho(\underline{x})$ for all $0 \leq i \leq n-1$. In particular, $\rho((r, \dots, r)) = \rho(\underline{x})$. $\diamond$

Remark A.3. Lemma A.1 and Remark A.2 together imply that for all $1 \leq i \leq \delta$ and $0 \leq t \leq r$, the layer L_t^i contains at least one jump of rank $r - t$ (choose $\underline{x} := t\,\underline{e}_i$ and reason by contradiction). $\diamond$

Lemma A.4. *Let $\underline{x}$ be an element of the dot array P_ρ and $1 \leq i \leq \delta$. Then*

$$\mathrm{rank}_i(P_\rho[\underline{x}]) = \rho(\underline{x}) + 1.$$

Proof. To prove this lemma properly, we have to distinguish three cases:

(i) L_r^i contains no jump.
(ii) $(r, \dots, r)$ is a jump (and therefore is the only one in L_r^i according to Fact 2.9).
(iii) L_r^i contains some jumps different from $(r, \dots, r)$, which is thus not a jump itself.

We will treat case (i) first. To this end, we will define two different routes consisting of finite sequences of points starting at $\underline{x}$, only adding unit vectors $\underline{e}_k$ along the way (sometimes several ones at a time) to finally reach $(r, \ldots, r)$. The first route will try to minimize the drop in values taken by ρ, whereas the second one will try to maximize it.

We start the first route at $\underline{x}^0 = \underline{x}$, which is a point of $L^i_{x_i}$. There are two cases.

- If this layer contains no jump $\underline{y}$ such that $\underline{y} \geq \underline{x}$, using Lemma A.1, we find $\underline{x}^1 \in L^i_{x_i+1}$ such that $\rho(\underline{x}^1) = \rho(\underline{x}^0)$.
- If $L^i_{x_i}$ contains jumps, we just define $\underline{x}^1 = \underline{x}^0 + \underline{e}_i$.

Wrapping us the two cases above gives a point $\underline{x}^1 \in L^i_{x_i+1}$ with $\underline{x}^1 \geq \underline{x}^0$ and the following fact: $\rho(\underline{x}^1) = \rho(\underline{x}^0)$ if $L^i_{x_i}$ contains no jump greater than or equal to $\underline{x}^0$; $\rho(\underline{x}^1) \geq \rho(\underline{x}^0)$ in the other case.

We then continue this process, during which the image by ρ of the successive points remains the same when we leave a layer without jumps greater than or equal to the current point, and loses at most one when we leave a layer with jumps greater than or equal to the current point, until we reach L^i_r with the point $\underline{x}^{r-x_i}$. At this moment, we apply Remark A.2 which yields $\rho(\underline{x}^{r-x_i}) = \rho((r, \ldots, r))$. From this we deduce that $\rho(\underline{x}) - \rho((r, \ldots, r)) \leq \mathrm{rank}_i(P_\rho[\underline{x}])$ and thus that

$$\mathrm{rank}_i(P_\rho[\underline{x}]) \geq \rho(\underline{x}) + 1.$$

To show the reverse inequality, we define a second route from $\underline{x}$ to $(r, \ldots, r)$. For $0 \leq s \leq r - x_i$, we define $\underline{x}^s = \underline{x} + s\,\underline{e}_i$. Let now consider some $0 \leq s < r - x_i$. We know that $\rho(\underline{x}^s) - \rho(\underline{x}^{s+1}) \geq 0$. Moreover, assume that L^i_s contains a jump of ρ $\underline{y}$ greater than or equal to $\underline{x}$. Then by definition $\rho\left(\underline{y}\right) - \rho\left(\underline{y} + \underline{e}_i\right) = 1$. But $\underline{y} \geq \underline{x}^s$ and $y_i \geq x^s_i$ so by $(*)^1_1$, $\rho(\underline{x}^s) - \rho(\underline{x}^{s+1}) = 1$. We have shown that ρ can only decrease along the route, and that it systematically decreases by one when leaving a layer containing a jump of ρ belonging to $P_\rho[\underline{x}]$. Then we complete the route to reach $(r, \ldots, r)$. The existence of this route proves that $\rho(\underline{x}) - \rho((r, \ldots, r)) \geq \mathrm{rank}_i(P_\rho[\underline{x}])$ and thus that

$$\mathrm{rank}_i(P_\rho[\underline{x}]) \leq \rho(\underline{x}) + 1.$$

We have shown that $\mathrm{rank}_i(P_\rho[\underline{x}]) = \rho(\underline{x}) + 1$ in case (i), which is the simplest case. In fact, the proof for cases (ii) and (iii) is very similar, and only requires some attention about the way to complete the route after the layer L^i_r is reached for the first time (also note that $\rho((r, \ldots, r)) = 0$ in case (ii)). □

We now come to the proof of the theorem.

Proof. The proof we give here is a more detailed version of the proof given for Proposition 4.3 in [EL00a]. We first show the first part of the theorem. We thus consider P a permutation array and define

$$\rho_P(\underline{a}) := \mathrm{rank}(P[\underline{a}]) - 1$$

for all $\underline{a} \in [r]^\delta$. It is obvious by definition that ρ_P takes values in $[r] \cup \{-1\}$. Since $\underline{a} \leq \underline{b}$ implies $P[\underline{a}] \supset P[\underline{b}]$, it is also easy to see that ρ_P is a decreasing function.

Now let $1 \leq i \leq \delta$. We have to show that $\rho_P(t\,\underline{e}_i) = r - t$ for all $0 \leq t \leq r$. By definition, we have, for any t,

$$\mathrm{rank}(P[t\,\underline{e}_i]) \leq r - t + 1.$$

The reverse inequality is shown by induction on t. The case $t = 0$ is true because $\mathrm{rank}(P) = r+1$. Then if we assume that $\mathrm{rank}(P[t\,\underline{e}_i]) \geq r-t+1$, we must show that $\mathrm{rank}(P[(t+1)\,\underline{e}_i]) \geq r - t$. In fact, we show a more general fact (akin to Proposition 2.5): for all $\underline{a} \in [r]^\delta$ such that $\underline{a} + \underline{e}_i \in [r]^\delta$, $\mathrm{rank}(P[\underline{a} + \underline{e}_i]) \geq \mathrm{rank}(P[\underline{a}]) - 1$. Indeed, let $j = i$ and denote $\mathrm{rank}(P[\underline{a}])$ by r. Then by definition of $\mathrm{rank}_i(P[\underline{a}]) = \mathrm{rank}(P[\underline{a}])$ and using the convention with substraction, there exist some integers $0 \leq s_1 < \cdots < s_r \leq r - a_i$ such that for all $1 \leq k \leq r$, there exists at least one dot in $P[\underline{a}]$ whose first coordinate is s_k, and these are the only integers between 0 and $r - a_i$ with this property. $s_1 = 0$ implies that $\mathrm{rank}_i(P[\underline{a} + \underline{e}_i]) = r - 1$ and $s_1 > 0$ implies that $\mathrm{rank}_i(P[\underline{a}]) = r$. In each case, we conclude, using the fact that $\mathrm{rank}_j(P[\underline{a} + \underline{e}_i])$ does not depend on j, that $\mathrm{rank}(P[\underline{a} + \underline{e}_i]) \geq \mathrm{rank}(P[\underline{a}]) - 1$.

To finish, we show that ρ_P is supermodular. Thanks to Theorem 3.6, it is sufficient to show that ρ_P satisfies $(*)^1_1$. We thus take an integer $1 \leq i \leq \delta$ and elements $\underline{x} \leq \underline{y} \in \square^\delta_r$ such that $\underline{x} + \underline{e}_i \in \square^\delta_r$ and $x_i = y_i$. We assume that $\rho\left(\underline{y}\right) - \rho\left(\underline{y} + \underline{e}_i\right) = 1$ and show that $\rho(\underline{x}) - \rho(\underline{x} + \underline{e}_i) = 1$. The hypothesis implies that $L^i_{x_i}$ contains a dotted point greater than or equal to $\underline{y}$. This point is in particular greater than or equal to $\underline{x}$, which yields the desired result.

Let then $\underline{x}$ be a dot in P and $1 \leq i \leq d$ such that $\underline{x} + \underline{e}_i \in \square^\delta_r$. We have

$$\rho_P(\underline{x} + \underline{e}_i) = \mathrm{rank}(P[\underline{x} + \underline{e}_i]) - 1 = \mathrm{rank}_i(P[\underline{x} + \underline{e}_i]) - 1.$$

But the last quantity is equal to

$$\mathrm{rank}_i(P[\underline{x}]) - 2 = \mathrm{rank}(P[\underline{x}]) - 2 = \rho_P(\underline{x}) - 1$$

thanks to the general fact proven earlier in this proof. So $\underline{x}$ is a jump of ρ_P. Then any redundant position of P is a jump of ρ_P because the set of jumps of ρ_P is closed under $\wedge$ (Proposition 2.8).

Conversely, let $\underline{x}$ be a jump of ρ_P. If $\underline{x} = (r, \ldots, r)$, then the fact that $\rho_P(\underline{x}) = 0$ means precisely that $\underline{x}$ is dotted (in P). In the other cases, let $1 \leq i_1 < \cdots < i_s \leq \delta$ be the (non-empty) set of indices k such that $\underline{x} + \underline{e}_k \in \square^\delta_r$. For all $1 \leq j \leq s$, $\rho_P(\underline{x} + \underline{e}_{i_j}) = \rho_P(\underline{x}) - 1$, which ensures that there exists some dotted $\underline{x}^j \in P[\underline{x}]$ whose i_j-th coordinate is the same as that of $\underline{x}$. We get that

$$\underline{x} = \bigwedge_{1 \leq j \leq s} \underline{x}^j.$$

If $s = 1$ or if some $\underline{x}^j$ is equal to $\underline{x}$, then x is a dot in P; otherwise, x is a redundant position. In a nutshell, $\underline{x}$ is a dot position in $P \cup R(P)$.

We now show that if ρ is a rank function on $\square^\delta_r$, then P_ρ is a permutation array. We first notice that by construction, P_ρ has no redundant dots.

Let us now show that P_ρ is totally rankable. We have to prove that for every $\underline{x} \in P_\rho$ and $1 \leq i, j \leq \delta$, $\mathrm{rank}_i(P_\rho[\underline{x}]) = \mathrm{rank}_j(P_\rho[\underline{x}])$. This is a direct consequence of Lemma A.4, which shows that $\mathrm{rank}_i(P_\rho[\underline{x}])$ can be expressed solely in terms of ρ.

To conclude, we show that the rank of P_ρ is $r + 1$. We can compute this rank in the first dimension, and thus have to show that for every $0 \leq s \leq r$, L^1_s contains a dotted point. For the sake of a contradiction, we assume otherwise and consider $\underline{y}^0 = s\,\underline{e}_1 \in L^1_s$. We know that $\rho(\underline{y}^0) = r - s$. Let us distinguish two cases.

- If $s < r$. We then apply Lemma A.1 which yields an element $\underline{z} \in L^t_{t+1}$ such that $\rho(\underline{z}) = \rho(\underline{x})$. This is a contradiction because on the one hand $\rho(\underline{z}) = \rho(\underline{x}) = r - s$ and on the other hand $\underline{z} \geq (s + 1)\,\underline{e}_1$ so $\rho(\underline{z}) \leq r - s - 1$.
- If $s = r$, the line of reasoning is similar, but involves Remark A.2 instead. We get that $\rho((r, \ldots, r)) = \rho(\underline{x}) = 0$. That makes $(r, \ldots, r)$ a jump, which is a contradiction.

□

Remark A.5. Theorem 3.9 shows in particular that the functions $P \mapsto \rho_P$ and $\rho \mapsto P_\rho$ are inverse of each other. Note that $P \mapsto \rho_P$ remains well-defined if we allow P to contain redundant dots. In fact, Eriksson and Linusson showed in [EL00a] that two permutation arrays with redundant dots allowed and with the same dimension and rank, P and P', give the same rank function (are *rank equivalent* in the terminology of [EL00a]) if, and only if,

$$P \setminus R(P) \subset P' \subset P \cup R(P).$$

This shows that each equivalence class contains a unique minimal representative, which can be computed by undotting all redundant positions (making it an actual permutation array). ⋄

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