List Colouring Constants of Triangle Free Graphs

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Abstract

In this paper we prove the following result about vertex list colourings, which shows in particular that a conjecture of the second author (1999, Journal of Graph Theory **31**, 149-153) is true for triangle free graphs of large maximum degree. There exists a constant K such that the following holds: Given a graph G, and a list assignment L to vertices of G, assigning a list of colours $L(v)$ to each vertex $v \in V(G)$, such that $|L(v)| = \frac{K\Delta}{\log(\Delta)}$, and for each element $c \in \bigcup L(v)$, the graph induced on vertices v, with $c \in L(v)$ is triangle free and has maximum degree at most Δ, then there exists a proper list colouring of G.

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1 Presentation of the problem

Most of the terminology and notation we use in this paper is standard and can be found in any text book on graph theory (such as [3] or [4]).

Let $G = (V, E)$ be a graph. A *list assignment* to vertices of G associates a list $L(v)$ of available colours to a vertex $v \in V$. A *proper L-colouring* of G is a proper colouring, i.e. a colouring in which neighbours have different colours, such that the colour of the vertex v belongs to $L(v)$.

Given a colour c in $L = \bigcup_{v \in V} L(v)$, we denote by G_c the graph induced on all vertices v with $c \in L(v)$, i.e.

$$G_c = G[\{v \in V \mid c \in L(v)\}]$$

By Δ we denote the integer number $\max_{c \in L} \Delta(G_c)$, and we write $\ell(v) = |L(v)|$. It was conjectured by Reed in [8] that if $\ell(v) \geq \Delta + 1$ then G admits a proper L-colouring. Bohman and Holtzman [2] provide a counter-example to this conjecture. Reed [8] and Haxell [5] prove a sufficient linear bound of type, respectively, $\ell(v) \geq 2e\Delta$ and $|L(v)| \geq 2\Delta$ for the existence of a proper L-colouring. Furthermore Reed and Sudakov [9] prove that the conjecture is asymptotically true. An intriguing open question is then

Question 1 Is there any constant C, such that for every G and L as above, with the extra condition that $|L(v)| \geq \Delta + C$, G admits a proper L-colouring?

The work of this paper is motivated by the above question restricted to triangle free graphs. We prove the following theorem which in particular provides an answer to the above question for triangle free graphs.

Theorem 1.1 Let G be a graph, and L be a list assignment as above. Suppose that each subgraph G_c is triangle free. There exists an absolute constant K such that if $|L(v)| \geq \frac{K\Delta}{\log \Delta}$ then G admits a proper L-colouring.

The general idea of the proof comes from Johansson's proof [6] that triangle free graphs have list chromatic number at most $\mathcal{O}(\frac{\Delta}{\log(\Delta)})$. It uses a randomised greedy algorithm [6], known as *semi-random method*, which at each step, consists of a) colouring a small set of vertices, b) removing from the lists of all neighbours the colours of these vertices, and c) removing all the coloured vertices. The main point of this approach is the key observation that with a good choice of parameters, the size of lists shrinks more slowly than the maximum degree of the colour classes G_c, and so after a sufficient number of iterations, each list should contain a number of colours at least twice the maximum degree of colour classes. At this stage, one can finish the colouring using Haxell's deterministic theorem [5]. The analysis of this algorithm uses classical concentration tools but also needs the polynomial method of [7], and is similar to Vu's approach in [11].

2 Proof of Theorem 1.1: Preliminaries

An *L-distribution* is the following data: For each vertex v and each $c \in L(v)$, we are given a non-negative weight $p(v,c) < 1$. Given an L-distribution $\{p(v,c)\}$, by $C(v)$ we denote the total weight of colours at v; i.e. $C(v) = \sum_{c \in L(v)} p(v,c)$. An L-distribution is called *normalised* if in addition we have $C(v) = 1$ for all $v \in V(G)$. An L-distribution proposes a natural way for generating sets of colours at a given vertex v: *For each colour $c \in L(c)$, we conduct a coin flip with probability of win equal to $p(v,c)$, and generate c if we win. So with probability $p(v,c)$, c is generated for v.* Note that $C(v)$ will be the expected number of generated colours for v. Given two adjacent vertices u and v, and a common colour $c \in L(u) \cap L(v)$, the probability that c is generated for u and v is $p(u,c)p(v,c)$. By $x_{v,u}$ we will denote the expected number of common generated colours for v and u. We have

$$x_{v,u} = \sum_{c \in L(v) \cap L(u)} p(v,c)p(u,c)$$

On the other hand, applying a simple Chebyshev inequality, we have

$$Prob(\text{ there exists at least one colour generated for both } v \textit{ and } u) \leq x_{uv}$$

To prove Theorem 1.1, we use the *Generalised Wastefull Colouring Procedure (GWCP)*. The greedy algorithm then consists of several iterations, let us say T, of the proposed procedure *GWCP*. We suppose that two global fixed parameters α and α_* are also given (the choice of α and α_* will become clear later). At each iteration, *GWCP* takes as an input a subset $V_i \subset V(G)$ (the set of uncoloured vertices), a list assignment L_i, an L_i-distribution $\{p_i(v,c)\}$ and a subset $F_i(v) \subset L(v)$ for each vertex v (the set of *forbidden* colours for v). It then provides as output a subset $V_{i+1} \subset V_i$, a new list assignment $L_{i+1} \subset L_i$, an updated L_{i+1}-distribution $\{p_{i+1}(v,c)\}$, and a new subset $F_{i+1}(v) \subset L(v)$ of forbidden colours ($F_i(v) \subset F_{i+1}(v)$). The general scheme of the procedure at the beginning of the ith iteration is described below:

Generalised Wastefull Colouring Procedure:

(i) **Generating colours** For each vertex $v \in V_i$, activate v with probability α and generate a set of colours for v relative to the L_i distribution

$\{p_i(v,c)\}$. In other words, choose the colour c for v with probability $\alpha.p_i(v,c)$.

(ii) **Updating the lists**: For each activated vertex v and each generated colour c for v, remove c from the lists of all neighbours u of v for which c is not forbidden, i.e. if $c \notin F_i(u)$. The new lists will be called L_{i+1}.

(iii) **Colouring some vertices:** For an activated vertex v with a generated set of colours $\mathcal{C}_v$, choose one arbitrary colour for v from $\mathcal{C}_v \cap (L_{i+1}(v) \setminus F_i(v))$ if it is possible, i.e. if $\mathcal{C}_v \cap (L_{i+1}(v) \setminus F_i(v)) \neq \emptyset$. Define V_{i+1} to be the set of still uncoloured vertices after this step.

(iv) **Updating the probabilities and the new sets of forbidden colours**: The way we update the weights of colours, i.e. the definition of $p_{i+1}(v,c)$, and so $F_{i+1}(v)$, is described below. .

Let us first define some more notations: $L'_i := L_i \setminus F_i$ is the set of *available* colours at v at step i. $\ell_i(v)$ will denote the cardinality of $L'_i(v)$. $G_{i,c}$ is the graph induced on vertices v with $c \in L'_i(v)$. For a vertex v with a colour $c \in L'_i(v)$, by $d_{i,c}(v)$ we will represent the degree of v in $G_{i,c}$. We initialise $d_{1,c}(v) = \Delta$ and $\ell_1(v) = \frac{K\Delta}{\log \Delta}$, for some large constant K which we will precise later.

As mentioned earlier the algorithm repeats the above procedure T times. With a careful choice of parameters, after T iterations with positive probability the minimum size of available colours $\min_{v \in V_T} \ell_T(v)$ will be at least twice the max-degree of each colour class $G_{T,c}$, and so we can finish the colouring using Haxell's theorem.

We now describe the way we update the L_i-distributions and the sets F_i: At the beginning, we define the distribution $\{p_1(v,c)\}$ to be the uniform normalised distribution on $L(v)$, i.e.:

$$p_1(v,c) = \begin{cases} \frac{1}{\ell_1(v)}, & \text{if } \ c \in L_1(v) \\ 0 & \text{otherwise} \end{cases}$$

Once again, remark that all through this paper if a colour does not belong to the original list $L(v)$, we suppose that the weight of this colour is zero.

Let us define $Keep_i(v,c)$ to be the probability that v keeps the colour c after the step **updating the lists** of the ith iteration, i.e.

$$Keep_i(v,c) = Prob(\ c \in L_{i+1}(v)).$$

For a given vertex v and given colour $c \in L_{i+1}(v)$, we will define $\{p_{i+1}(v,c)\}$ essentially as follows:

- If $\frac{p_i(v,c)}{Keep_i(v,c)} \leq \alpha_*$, define $p_{i+1}(v,c) = \frac{p_i(v,c)}{Keep_i(v,c)}$;
- If $c \in F_i(v,c)$ then we set $p_{i+1}(v,c) = \alpha_*$
- Otherwise, $\frac{p_i(v,c)}{Keep_i(v,c)} > \alpha_* > p_i(v,c)$. If c is not assigned to any neighbour of v during the ith iteration, we set $p_{i+1}(v,c) = \alpha_*$. If c is assigned to a neighbour of v, then conduct a coin flip with probability of win equal to $\frac{\frac{p_i(v,c)}{\alpha_*} - Keep_i(v,c)}{1 - Keep_i(v,c)}$, and define $p_{i+1}(v,c) = \alpha_*$ if we win, and $p_{i+1}(v,c) = 0$ otherwise.

It is not difficult to see that with this definition we have

$$\mathbb{E}(p_{i+1}(v,c)) = p_i(v,c) \tag{1}$$

Now, define $F_{i+1} = \{c \mid p_{i+1}(v,c) = \alpha_*\}$. We have $F_i(v) \subset F_{i+1}(v)$. Roughly speaking, the set $F_i(v)$ contains all colours whose weights become very *large* at some step of the algorithm. By very large we mean larger than the probability α_* that we fix as a global parameter (to be specified later) .

Note that $\{p_i(v,c)\}$ does not remain necessary a normalised distribution on $L_i(v)$, i.e. in general we can have $C_i(v) \neq 1$. On the other hand, Equation 1 implies that:

$$\mathbb{E}(C_{i+1}(v)) = C_i(v).$$

As we shall see, the variable C_i is highly concentrated around its expected value, which implies that $C_i(v) \sim 1$ [3] .

How about the random variables $x_{u,v}$ at the $(i+1)$th iteration, which we denote by $x_{i+1,u,v}$? How their values evolve by time [4] ? Remember that

$$x_{i+1,u,v} = \sum_c p_{i+1}(u,c)p_{i+1}(v,c)$$

The first remark concerning $x_{i+1,u,v}$ goes as follows: as the graphs G_c is triangle free, it follows that the variables $p_{i+1}(u,c)$ and $p_{i+1}(v,c)$ are independent and so we have $\mathbb{E}(x_{i+1,u,v}) = \sum_c \mathbb{E}(p_i(u,c))\mathbb{E}(p_{i+1}(v,c)) = \sum_c p_i(u,c)p_i(v,c) = x_{i,u,v}$. It turns out that these variables are also highly concentrated around their expected value, which roughly permits us to conclude that with positive probability, we can ensure $x_{i,u,v} \sim x_{1,u,v} = \sum_c p_1(u,c)p_1(v,c) = \frac{|L_u \cap L_v|}{\ell_1^2}$.

[3] Remember that this in particular means that for an activated vertex the expected number of generated colours will be ~ 1.

[4] Remark that in the ith iteration, $x_{i+1,u,v}$ is a random variable which depends on the random choices we make at this step, while we have already from previous iterations the values of $p_i(v,c)$ and so $x_{i,u,v}$.

Before going through the details, we now provide an intuitive explanation of why the above algorithm should work and provide a proof of Theorem 1.1. The first remark is that with high probability, if a vertex is activated it will get a colour. As the probability of activating a vertex is α, this means that the degree sequence shrinks exponentially, more precisely we have $d_{i+1,c}(v) \leq (1-\alpha)^i \Delta$.

For each i and each $v \in V_{i-1}$, we now define a new random variable $t_{i,v}$ as follows:

$$t_{i,v} = \begin{cases} 1 \text{ if } v \text{ remains uncoloured after the } (i-1)\text{th iteration, i.e. } v \in V_i; \\ 0 \text{ if } v \text{ gets a colour, i.e. } v \notin V_i. \end{cases}$$

It is clear [5] that

$$d_{i,c}(v) \leq \sum_{u \in N_{G_{i-1,c}(v)}} t_{i,u}. \tag{2}$$

For a given vertex v, let us define the random variable $x_{i,v}$ as follows:

$$x_{i,v} = \sum_{u \in N_{G_{i-1}}(v)} x_{i,v,u} t_{i,u}. \tag{3}$$

Remark that because $x_{i,u,v} = \sum_c p_i(v,c)p_i(u,c)$ counts the expected number of common colours in generating colours for u and v with respect to L_i-distribution $\{p_i(v,c)\}$, and the random variables $t_{i,u}$ represents the presence or non presence of a given vertex u in G_i, the random variable $x_{i,v}$ has this simple interpretation:
$x_{i,v}$ counts the expected number of common colours between v and one of its neighbours in G_i with respect to the distribution L_i.

Remark the similarity between the two equations 2 and 3 above: Equation 3 can be seen as a weighted version of 2. As we said $x_{i,u,v}$ is highly concentrated and so $x_{i,u,v} \sim \frac{|L(u)\cap L(v)|}{\ell_1^2} \leq \frac{1}{\ell_1}$. Intuitively $\mathbb{E}t_{i,u} \sim 1-\alpha$ [6], which implies

$$\mathbb{E}(d_{i,c}(v)) \leq (1-\alpha)d_{i-1,c}(v).$$

$$\mathbb{E}(x_{i,v}) \sim \sum_{u \in N_{G_{i-1}}(v)} x_{i-1,u,v}\mathbb{E}(t_{i,u}) \sim (1-\alpha)(\sum p_{i-1}(u,c)p_{i-1}(v,c)) \sim (1-\alpha)x_{i-1,v}.$$

The above equality then simply states that in some sense the changes in $x_{i,v}$ (resp. $d_{i,c}$), from one step to the next one, should be as $x_{i,v} \sim (1-\alpha)x_{i-1,v}$

[5] we have an inequality here because it is possible that u loses the colour c.

[6] Remark that $x_{i,u,v}$ and $t_{i,u}$ are not independent, but it turns out that these two random variables are not to much correlated.

(resp. $(1-\alpha)d_{i-1,c}$) [7] .

It is clear that at the beginning we have

$$x_{1,v} = \sum_c \sum_u \frac{1}{\ell_1^2} = \frac{\Delta \times \ell_1}{\ell_1^2} = \frac{\Delta}{\ell_1} = \frac{\log(\Delta)}{K}.$$

The remaining part of this intuitive proof is similar to Johansson's proof in [6]. Let us introduce the entropy function

$$H_i(v) = -\sum_c p_i(v,c)\log(p_i(v,c)).$$

Remark that $e^{H_i(v)} = \prod_c \ p_i(v,c)^{-p_i(v,c)}$. It turns out that $e^{H_i(v)}$ provides a lower bound for the number of colours in $L_i(v)$. The intuitive idea is the following: because of the concentration phenomena, we should have $C_i = \sum_c p_i(v,c) \sim 1$. With the extra hypothesis that the distribution $\{p_i(v,c)\}$ is almost uniform, i.e. $p_i(v,c) \sim \frac{1}{\ell_i}$, we infer that

$$e^{H_i(v)} = (\frac{1}{\ell_i})^{-\sum p_i(v,c)} \sim \ell_i.$$

which is what we claimed [8] .

Now the main fact is that writing the changes in entropy step by step, we can see that the random variables $x_{i,v}$ enters to the picture very naturally: indeed, we have

$$Keep_{i-1}(v,c) \sim \prod_{u \in N_{G_{i-1,c}(v)}} (1-\alpha.p_{i-1}(u,c)) \sim e^{\alpha.(\sum p_{i-1}(u,c))}.$$

and roughly we have $p_i(v,c) = \frac{p_{i-1}(v,c)}{Keep_{i-1}(v,c)}.$, which implies

$$H_i(v) - H_{i-1}(v) = -\alpha x_{i,v} - (p_i(v,c) - p_{i-1}(v,c))\log(p_i(v,c)).$$

It turns out that we can ignore the second term on the right:

$$H_i(v) - H_{i-1}(v) \sim -\alpha x_{i,v}$$

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[7] Unfortunately $x_{i,v}$ (resp. $\sum_{u \in N_{G_{i-1,c}}(v)} t_{i,u}$) can have big Lipschitz coefficients, which roughly means that writing $x_{i,v}$ (resp. $\sum_{u \in N_{G_{i-1,c}}(v)} t_{i,u}$) as a function of the corresponding binary variables, changing the outcome of one variable can make a big change in the outcome of $x_{i,v}$. This in particular forbids us to use the classical concentration tools, i.e. Talagrand or Azuma's inequalities, to state that with positive probability this is satisfied. We will see further a way to encounter the problem. For now on, let us suppose that this is true, and continue our arguments.

[8] Of course here we ignore completely the set $F_i(v)$, as we will see the size of $F_i(v)$ will be small enough.

and so

$$H_i(v) - H_{i-1}(v) \sim -\alpha(1-\alpha)^{i-1}\frac{\Delta}{\ell_1}.$$

Summing up over all i and using $\ell_1 = \frac{K\Delta}{\log(\Delta)}$, we infer

$$H_i(v) \geq H_1(v) - \alpha(\sum_{i\geq 1}(1-\alpha)^{i-1})\Delta \geq \log(\Delta) - \frac{\log(\Delta)}{K} = \log(\Delta^{1-\frac{1}{K}}).$$

This proves that after T steps, we can intuitively ensure to have $\geq \Delta^{1-\frac{1}{K}}$ colours in each list $L_T(v)$.

On the other hand the degree sequence will decrease at least as $(1-\alpha)^i\Delta$, and so after T steps we will have $d_{T,c}(v) \leq (1-\alpha)^T\Delta \sim e^{-\alpha T}\Delta$. If we choose T and α in such a way to have $\alpha T \geq \frac{\log(\Delta)}{500}$, the above arguments implie $d_{T,c}(v) \leq e^{-\frac{\log(\Delta)}{500}}.\Delta = \Delta^{1-\frac{1}{500}}$. Now, if $K \geq 1000$ we will have $l_T(v) \geq \Delta^{1-\frac{1}{K}} \geq 2\Delta^{1-\frac{1}{500}} \geq 2.d_{T,c}(v)$. This finishes the outline of the proof.

Remark 2.1 There are two main issues to handle in the above lines: as one can easily see the random variables $d_{i,c}(v)$ and $x_{i,v}$ behave totally different of the variables $C_i(v)$ and $x_{i,u,v}$. This is mainly because of the existence of four cycles. More precisely, if some other vertex v' is connected to a lot of vertices in $N(v)$, changing a colour associated to v', can make a big change in $d_{i,c}(v)$ and $x_{i,v}$. On the other hand, the two random variables, $d_i(v)$ and $x_{i,v}$ are very similar in definition. This will permit us to apply the polynomial method of [7] to both of these variables, to obtain upper bounds on $d_{i,c}(v)$ and $x_{i,v}$. These upper bounds then can be put together to finish to proof of Theorem 1.1 in the very same way as the intuitive proof given above.

3 Details

Let $\alpha_* = \Delta^{-\frac{49}{50}}$, $\alpha = \Delta^{-\frac{1}{20}}$, $K = 1000$ and so $\ell_1 = \frac{1000\Delta}{\log(\Delta)}$, and $T = \Delta^{\frac{1}{20}}\log(\Delta)/20$. Let us introduce the following property:

Property P(i): for each uncoloured vertex v and neighbour $u \in N_{G_{i-1}}(v)$.

$$C_i(v) = 1 + \mathcal{O}(\frac{1}{\Delta^{\frac{1}{60}}}).$$

$$x_{i,u,v} \leq \frac{L(u)\cap L(v)}{\ell_1^2} + \frac{1}{\ell_1\Delta^{\frac{1}{60}}}.$$

$$x_{i,v} \leq (1-\frac{\alpha}{2})^{i-1}\frac{\log(\Delta)}{K}.$$

$$d_{i,c}(v) \leq \Delta(1-\frac{3\alpha}{5})^{i-1}.$$

$$H_i(v) - H_{i-1}(v) \geq -2\alpha x_{i,v} + o(\frac{1}{\Delta^{\frac{1}{20}}}).$$

Below, we prove by induction on i for $i \leq T$, that if all the P(j)'s are verified for $j < i$, then with positive probability, $\mathbf{P(i)}$ is also satisfied. This implies that with positive probability we can ensure that each $\mathbf{P(i)}$ is verified for $i \leq T$. Before, let us explain how to use this result to obtain the proof of Theorem 1.1:

First, we observe that

$$H_T(v) - H_1(v) = \sum_{i=1}^{i=T-1}(H_{i+1}(v) - H_i(v)) \geq -2\sum_i \alpha x_{i,v} + o(\Delta^{\frac{1}{19}})).$$

$$H_T(v) \geq H_1(v) - 2\sum_i \alpha(1-\frac{\alpha}{2})^{i-1}\frac{\log(\Delta)}{K} - o(1).$$

Remark that $H_1(v) = -\sum p_1(v,c)\log(p_1(v,c)) = (1-o(1))\log(\Delta)$, so we have

$$H_T(v) \geq (1-o(1))\log(\Delta) - 2\alpha\sum_{i=1}^{i=\infty}(1-\frac{\alpha}{2})^{i-1}\frac{\log(\Delta)}{K} - o(1)$$

so

$$H_T(v) \geq (1-o(1))\log(\Delta) - 4\log(\Delta)/K - o(1) \geq \log(\Delta^{1-\frac{5}{K}}).$$

for Δ large enough. On the other hand by the definition of entropy and the fact that all the probabilities are bounded by α_*, we have

$$|L_T(v)| \geq \frac{|H_T(v)|}{-\alpha_*\log(\alpha_*)} \geq \frac{\log(\Delta^{1-\frac{5}{K}})}{\frac{49}{50}.\Delta^{-\frac{49}{50}}\log(\Delta)} \sim \Delta^{\frac{49}{50}}$$

Now we want to prove that $\ell_T(v)$, the number of available colours at v, is at least a constant proportion of $|L_T(v)|$. For this we will bound $|F_T(v)|$ as follows:

By $\mathbf{P(i)}$, we have $C_T(v)-C_1(v) = \mathcal{O}(\Delta^{-\frac{1}{60}})$. We use the fact that $\{p_1(v,c)\}$ is uniform to infer that

$$H_1(v) = -\sum p_T(v,c)\log(p_1(v,c)) + \mathcal{O}(\Delta^{-\frac{1}{60}}\log(\Delta)).$$

We also have $H_T(v) = -\sum p_T(v,c)\log(p_T(v,c))$. Putting these two equations together we obtain

$$H_1(v,c) - H_T(v,c) = \sum p_T(v,c)\log(\frac{p_T(v,c)}{p_1(v,c)}) + \mathcal{O}(\Delta^{-\frac{1}{60}}\log(\Delta)).$$

Each colour $c \in F_T(v)$ contributes $\alpha_* \log(\alpha_* \ell_1)$ to the above sum. On the other hand, if $p_T(v,c) \neq 0$ then $p_T(v,c) \geq p_1(v,c)$ which implies that the contribution of the other terms to the above sum is non negative. This implies

$$\alpha_* \log(\alpha_* \ell_1)|F_T(v)| \leq H_1(v) - H_T(v) + \mathcal{O}(\Delta^{-\frac{1}{60}}\log(\Delta)).$$

Using the initialisation $\ell_1 = \frac{K\Delta}{\log(\Delta)}$, we obtain

$$|F_T(v)| \leq \frac{H_1(v) - H_T(v) + \mathcal{O}(\Delta^{-\frac{1}{60}}\log(\Delta))}{\Delta^{-\frac{49}{50}}(\frac{\log(\Delta)}{50})}.$$

Recall that $H_1(v) - H_T(v) \leq \log(\Delta^{\frac{5}{K}})$, and so

$$|F_T(v)| \leq \Delta^{\frac{49}{50}} \times 5 \times 50/K \leq \Delta^{\frac{49}{50}}/4. \tag{4}$$

But we just observed that

$$|L_T(v)| \geq \Delta^{\frac{49}{50}}.$$

which in turn shows that

$$|F_T(v)| \leq |L_T(v)|/3.$$

and so

$$\ell_T(v) \geq |L_T(v)| - |F_T(v)| \geq \frac{\Delta^{\frac{49}{50}}}{2}.$$

Now we obtain some upper bounds for the degree sequences. By Property **P(i)**, we have

$$d_{T,c}(v) \leq (1 - \frac{3\alpha}{5})^{T-1}\Delta \sim \Delta e^{-\frac{3(T-1)\alpha}{5}}.$$

But $T = \Delta^{\frac{1}{20}}\log(\Delta)/20$ and $\alpha = \Delta^{-\frac{1}{20}}$ which implies that

$$d_{T,c}(v) \leq (1 - \frac{3\alpha}{5})^{T-1}\Delta \sim \Delta^{1-\frac{3}{100}}.$$

This in turn implies that for large Δ

$$2d_{T,c}(v) \leq \Delta^{1-\frac{3}{100}} \leq \frac{\Delta^{1-\frac{1}{50}}}{2} \leq \ell_T(v).$$

and so we can finish the colouring using Haxell's theorem.

Now, let us suppose that at the beginning of the ith iteration, all the properties $\mathbf{P(j)}$ for $j < i$ are verified, we prove that with positive probability P(i) is also true.

Concentration results I: Bounding $C_i - C_{i-1}$ and $x_{i,u,v} - x_{i-1,u,v}$

To ensure the property $\mathbf{P(i)}$ for C_i and $x_{i,u,v}$ we use a variant of Azuma's inequality. Here the way it works:

Lemma 3.1 *Let $P(T_1, \ldots, T_\ell)$ be a random variable determined by ℓ trials $T_1, \ldots, T_\ell$. Suppose that P is c-Lipschitz, i.e. changing the value of one coordinate in P changes the value of P by at most c. Then we have*

$$Prob(|P - \mathbb{E}(P)| \geq t) \leq e^{\frac{-t^2}{\ell c^2}}.$$

Each of our random variables can be seen as a function depending on the outcome of some trials: these trials are simply given by choosing one colour or by conducting a coin flip. We have the following basic facts (the proofs are omitted):

Basic Fact I: The random variable C_i is α_*-Lipschitz.
Basic Fact II: The random variable $x_{i,u,v}$ is α_*^2-Lipschitz.

Lemma 3.2 • *$\mathbb{E}(C_i(v)) = C_{i-1}(v)$;*

- *$\mathbb{E}(x_{i,u,v}) = x_{i-1,u,v}$.*

Lemma 3.3 • *$Prob(|C_i(v) - \mathbb{E}(C_i(v))| \geq \frac{1}{\Delta^{\frac{1}{15}}}) \leq \Delta^{-100}$;*

- *$Prob(|x_{i,u,v} - \mathbb{E}(x_{i,u,v})| \geq \frac{1}{\ell_1 \Delta^{\frac{1}{15}}}) \leq \Delta^{-100}$.*

Proof.

- Apply Lemma 3.1 to C_i, we have

$$Prob(|C_i(v) - \mathbb{E}(C_i(v))| \geq \frac{1}{\Delta^{\frac{1}{15}}})) \leq e^{\frac{-\Delta^{\frac{-2}{15}}}{2\ell_1 \Delta^{-2\times\frac{49}{50}}}}.$$

  We have $\ell_1 = 1000\Delta/\log(\Delta)$ and so

$$Prob(|C_i(v) - \mathbb{E}(C_i(v))| \geq \frac{1}{\Delta^{\frac{1}{15}}})) \leq e^{-\frac{\log(\Delta)}{1000}\Delta^{\frac{48}{50}-\frac{2}{15}}} \leq \Delta^{-100}.$$

  for Δ large enough.

- Apply Lemma 3.1 to $x_{i,u,v}$, we have

$$Prob(|x_{i,u,v} - \mathbb{E}(x_{i,u,v})| \geq \frac{1}{\ell_1 \Delta^{\frac{1}{15}}}) \leq e^{\frac{-\ell_1^{-2} \Delta^{\frac{-2}{15}}}{\ell_1 \alpha_*^4}}.$$

$$Prob(|x_{i,u,v} - \mathbb{E}(x_{i,u,v})| \geq \frac{1}{\ell_1 \Delta^{\frac{1}{15}}}) \leq \frac{1}{e^{\ell_1^3 \cdot (\Delta^{\frac{-49}{50}})^4 . \Delta^{\frac{2}{15}}}}.$$

$$= \Delta^{\frac{-\log(\Delta)^2 . \Delta^{\frac{47}{50}}}{10^6}} \leq \Delta^{-100}$$

  for Δ large enough.

$\square$

Concentration results II: Bounding $x_{i,v}$ and $d_{i,c}(v)$

We prove here how to bound $x_{i,v}$, the same method applies for $d_{i,c}(v)$ (actually for $\sum_{u \in N_{G_{i-1,c}}(v)} t_{i,u}$ which bounds $d_{i,c}(v)$).

Let us introduce the binary variables $b_i(u, c)$ as follows:

- $b_i(u, c) = 1$ if the colour c is chosen for u (so with probability $\alpha p_{i-1}(u, c)$ we have $b_i(u, c) = 1$),
- $b_i(u, c) = 0$ otherwise.

We first obtain a polynomial upper bound for the random variable $t_{i,u}$ in terms of the binary variables $b_i(u, c)$. The first observation is that $t_{i,u} = 1$ if one of the following three cases arise:

(i) *No colour is chosen for u*: the random variable $b_i(u, c)$ represents the generation of colour c for u. It is then clear that the contribution of this case to $t_{i,u}$ is at most $\prod_c (1 - b_i(u, c))$.

(ii) *All the colours generated for u appear among the colours generated for some neighbours*: in this case, for each generated colour c, one of the neighbours of u should also have c among its generated colours. The contribution of c is bounded by $b_i(u, c)$ (for generating c at u) times $\sum_{w \in N_{G_{i-1,c}}(u)} b_i(w, c)$ (for generating c at one of the neighbours). So the total contribution is at most

$$\sum_{c \in L_{i-1}(u)} b_i(u, c) (\sum_{w \in N_{G_{i-1,c}}(u)} b_i(w, c)).$$

(iii) *The colour c generated for u is forbidden, i.e. $c \in F_{i-1}(u)$*: the contribution of this case to $t_{i,u}$ is at most $\sum_{c \in F_{i-1}(u)} b_i(u, c)$.

This finally provides us with the following general upper bound

$$t_{i,u} \leq \prod_c (1 - b_i(u,c)) + \sum_{c \in L_{i-1}(u)} b_i(u,c)(\sum_{w \in N_{G_{i-1,c}}(u)} b_i(w,c)) + \sum_{c \in F_{i-1}(u)} b_i(u,c).$$

Which implies that

$$x_{i,v} \leq \sum_{u \in N_{G_{i-1}}(v)} x_{i,u,v}[\prod_c (1 - b_i(u,c)) + \sum_{c;\ \{u,w\} \in E(G_{i-1,c})} b_i(u,c) b_i(w,c)].$$

$$+ \sum_{c \in F_{i-1}(u)} x_{i,u,v} b_i(u,c).$$

On the other hand, using the fact that each $b_i(u,c)$ is a binary random variable, we have

$$\prod_c (1 - b_i(u,c)) \leq 1 - \sum_c b_i(u,c) + \sum_{c_1,c_2} b_i(u,c_1) b_i(u,c_2).$$

Which implies that we have

$$x_{i,v} \leq \sum_{u \in N(v)} x_{i,u,v} - \sum_c x_{i,v,u} b_i(u,c) + \sum_{u \in N(v); c_1,c_2} x_{i,v,u} b_i(u,c_1) b_i(u,c_2) +$$

$$\sum_{c;\ w \in N_{G_{i-1,c}}(u)} x_{i,v,u} b_i(u,c) b_i(w,c) + \sum_{c \in F_{i-1}(u)} x_{i,v,u} b_i(u,c).$$

$$= A_0 - A_1 + A_2 + A_3 + A_4$$

We first calculate the expectation of the terms in the right hand side:

$$\mathbb{E}(\sum_u x_{i,u,v}) \sim \sum x_{i-1,u,v} = x_{i-1,v}.$$

Now using the concentration results for C_i and $x_{i,u,v}$, we obtain

$$\mathbb{E}(\sum_c x_{i,v,u} b_i(u,c)) \sim x_{i-1,v} \alpha C_i(u) \sim \alpha x_{i-1,v}.$$

$$\mathbb{E}(\sum_{u;c_1,c_2} x_{i,v,u} b_i(u,c_1) b_i(u,c_2)) \leq x_{i-1,v} \ell_1^2 \alpha^2 \alpha_*^2.$$

but we have $\ell_1^2 \alpha \alpha_*^2 = (\frac{\Delta}{\log(\Delta)})^2 \times \Delta^{\frac{-1}{20}} \times \Delta^{-2 \times \frac{49}{50}} = o(1)$, and so

$$\mathbb{E}(\sum_{u;c_1,c_2} x_{i,v,u} b_i(u,c_1) b_i(u,c_2)) = o(\alpha x_{i-1,v}).$$

For the last term we have

$$\mathbb{E}(\sum_{c \in F_{i-1}(u)} x_{i,v,u} b_i(u,c)) \leq x_{i-1,v} \alpha \alpha_* |F_{i-1}(u)|.$$

As we suppose that the properties $\mathbf{P(j)}$ are verified for all $j \leq i$, a similar method we used to obtain Equation 4 can be engaged to show that

$$|F_{i-1}(u)| \leq \frac{\Delta^{\frac{49}{50}}}{4} = \alpha_*^{-1}/4.$$

and so

$$\mathbb{E}(\sum_{c \in F_{i-1}(u)} x_{i,v,u} b_i(u,c)) \leq \alpha x_{i-1,v}/4.$$

Putting all these inequalities together, if we can ensure that $A_i \sim \mathbb{E}(A_i)$ with a high positive probability ($\geq 1 - \Delta^{-100}$), then we should have

$$x_{i,v} \leq \mathbb{E}(A_0 - A_1 + A_2 + A_3 + A_4) \leq (1 - \frac{\alpha}{2}) x_{i-1,v}.$$

which proves the desired claim of $\mathbf{P(i)}$.

Let us now prove that with high probability (at least $1 - \Delta^{-100}$) we have $|A_i - \mathbb{E}(A_i)| = o(\alpha x_{i-1,v})$. To this end, we need the following lemma which is known as *polynomial method*; for more details see the first chapter of [10]:

Lemma 3.4 Let $P = P(X_1, X_2, \ldots, X_n)$ be a polynomial in n variables and of degree k, such that every coefficient of P is non-negative. Let $b_1, b_2 \ldots, b_n$ be n binary random variables. For $d \in \mathbb{N}$ define

$$\mathbb{E}_{\geq d}(P(b_1, \ldots, b_n)) = \max_{(d_1, \ldots, d_n): \sum d_i \geq d} \mathbb{E}((\frac{\partial}{\partial X_1})^{d_1} \ldots (\frac{\partial}{\partial X_n})^{d_n}(P)(b_1, \ldots, b_n)).$$

There exists a constant C_k, depending only on k, such that we have

$$Prob[|P(b_1, \ldots, b_n) - \mathbb{E}(P(b_1, \ldots, b_n))| \geq C_k \lambda^k \sqrt{\mathbb{E}(P)\mathbb{E}_{\geq 1}(P)}] = \mathcal{O}(e^{-\lambda}).$$

In order to apply the above lemma, we show that the expected value of all the partial derivatives of A_i's is of order $\mathcal{O}(1)$.

(i) $\mathbb{E}(\frac{\partial A_1}{\partial b_i(u,c)}) = \mathbb{E}(x_{i,u,v}) = \mathcal{O}(\frac{1}{\ell_1})$;

(ii) $\mathbb{E}(\frac{\partial A_2}{\partial b_i(u,c)}) = \mathbb{E}(\sum_{w \in N_{G_{i-1,c}(u)}} x_{i,u,v} b_i(u,c')) \sim x_{i-1,u,v} C_i(u) = \mathcal{O}(\frac{1}{\ell_1})$;

(iii) $\mathbb{E}(\frac{\partial^2 A_2}{\partial b_i(u,c) \partial b_i(u',c')}) \leq x_{i-1,u,v}$;

(iv) $\mathbb{E}(\frac{\partial A_3}{\partial b_i(u,c)}) = \mathbb{E}(\sum_{w \in N_{G_{i-1,c}(u)}} x_{i,u,v} b_i(w,c)) \leq \frac{1}{\ell_1} \Delta \alpha \alpha_* = \mathcal{O}(\frac{1}{\ell_1})$;

(v) $\mathbb{E}(\frac{\partial^2 A_3}{\partial b_i(u,c) \partial b_i(u',c')}) \leq x_{i-1,u,v}$;

(vi) $\mathbb{E}(\frac{\partial A_4}{\partial b_i(u,c)}) \leq \mathbb{E}(x_{i,u,v}) \leq \frac{1}{\ell_1}$.

From the above estimates it is clear that $\mathbb{E}_1(x_{i,v}) = \mathcal{O}(1)$. Now applying the inequality of Lemma 3.4 we have:

Lemma 3.5 For $k = \deg(A_i)$ (so $k = 1$ or 2 depending on i), we have

$$Prob(|A_i - \mathbb{E}(A_i)| \geq c_k(\log(\Delta))^{2k}\sqrt{\mathbb{E}(A_i)}) \leq e^{-(\log(\Delta))^2} \leq \Delta^{-100}.$$

for Δ big enough

The proof of the last condition in $\mathbf{P(i)}$ concerning the entropies is omitted.

Having all these concentration results 3.3 and 3.5, the only remaining point is to put them all together, to prove that with positive probability all the random variables of Lemma 3.3 and 3.5, will remain close to their expectations. To prove this, it is sufficient to see that the dependency graph of all the bad events, i.e. the events defined by deviation of our random variables from their expected values, has degree strictly less than Δ^{10} (the degree of original graph is at most Δ^2, each two events at distance at least 3 are independent, a simple calculations show that the degree is bounded by Δ^{10}). As the probability of each of these bad events is at most Δ^{-100}, applying the Lovász Local Lemma, we conclude that with positive probability all these bad events do not happen, and so at iteration i, the property $\mathbf{P(i)}$ is verified.

The Lovász Local Lemma (LLL): Consider a set $\mathcal{E}$ of (typically bad) events such that for each event $A \in \mathcal{E}$

(i) $P(A) \leq p < 1$;

(ii) A is mutually independent of a set of all but at most d of the other events in $\mathcal{E}$.

If $4pd \leq 1$ then with positive probability, non of the events in $\mathcal{E}$ occur.
Remark that in over application of the LLL, $p = \Delta^{-100}$ and $d \leq \Delta^{-10}$, so for large enough Δ we have $4pd \leq 1$.

References

[1] K. Azuma, *Weighted Sums of Certain Dependent Random Variables*, Tohuku Math. Journal, **19**, 535–567, 1967.

[2] T. Bohman and R. Holtzman, *On a List Colouring of Conjecture of Reed*, Journal of Graph Theory, **41**, n. 2, 106–109, 2002

[3] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications.* Macmillan, London and Elsevier, New York, 1976.

[4] R. Diestel, *Graph Theory*, 3rd edition, Springer-Verlag, Berlin and Heidelberg, 2005.

[5] P. Haxell, *A Note on Vertex List Colouring*, Combinatorics, Probability and Computing, **10**, n. 4, 345–347, 2001

[6] A. Johansson, *Asymptotic choice number for triangle free graphs*, DIMACS Technical Report 91-95, 1996.

[7] J. H. Kim and V. H. Vu, *Concentrations of Multivariate Polynomials and its Applications*, Combinatorica, **20**, no.3, 417–434, 2000.

[8] B. Reed, *The List Colouring Constants*, Journal of Graph Theory, **31**, 149–153, 1999

[9] B. Reed, and B. Sudakov, *Asymptotically the List Colouring Constants are 1*, Journal of Combinatorial Theory, Series B, **86**, 27–37, 2002.

[10] T. Tao, V. H. Vu, *Additive combinatorics*, Cambridge Studies in Advanced Mathematics, **105**, Cambridge University Press, 2006.

[11] V. H. Vu, *A general upper bound on the list chromatic number of locally sparse graphs*, Combinatorics, Probability and Computing, **11**, n. 1, 103-111, 2002.