SORBONNE UNIVERSITÉ

École doctorale de sciences mathématiques de Paris Centre

Mémoire

présenté pour l'obtention

du

Diplôme d'habilitation à diriger des recherches de la Sorbonne Université

Spécialité : Mathématiques

par

Omid Amini

Titre :

Geometry of graphs and applications

Soutenu le 6 juillet 2018 devant la commission d'examen :

Vladimir Berkovich	Professeur (Weizmann Institute of Science)
Philippe Biane	Directeur de recherche au CNRS (Université Paris-Est)
Jean-Michel Bismut	Professeur (Université Paris-Sud)
Charles Favre	Directeur de recherche au CNRS (École Polytechnique)
Ilia Itenberg	Professeur (Sorbonne Université)
Mattias Jonsson	Professeur (University of Michigan)
Claire Voisin	Professeure (Collège de France)

au vu des rapports de :

Vladimir Berkovich	Professeur (Weizmann Institute of Science)
Béla Bollobás	Professeur FRS (Trinity College Cambridge & U. Memphis)
Mattias Jonsson	Professeur (University of Michigan)
Ravi Vakil	Professeur (Stanford University)

Dedicated to all my teachers.

Why not go into the forest for a time, literally? Sometimes a tree tells you more than can be read in books. *Carl Gustav Jung.*

Remerciements

C'est un grand plaisir pour moi de remercier ici les nombreuses personnes qui ont soutenu et encouragé mes études de mathématiques et mes projects de recherche depuis les premiers pas dans cette voie. Je voudrais remercier tout particulièrement, par l'ordre chronologique, Moharram Iradmusa, Davoud Masoumi, Yahya Tabesh, Reza Sadeghi, Kasra Alishahi, Siavash Shahshahani, Saeed Akbari, Amir Daneshgar, Maryam Mirzakhani, Rama Cont, Roland Sénéor, Jean Lannes, François Labourie, Paul Seidel, Jean-Michel Bismut, Fabien Morel, Jean-Claude Bermond, Frédéric Havet, Stéphane Pérennes, Stéphan Thomassé et Bruce Reed pour le rôle qu'ils ont joué chacun à un moment donné dans ma formation et dans la passion que j'éprouve pour ce métier.

J'exprime ici toute ma reconnaissance à Vladimir Berkovich, Béla Bollobás, Mattias Jonsson et Ravi Vakil d'avoir aimablement accepté d'être rapporteurs de ce mémoire et pour le temps précieux qu'ils ont consacré à l'étudier. Les travaux et points de vue mathématiques de Vladimir Berkovich et Béla Bollobás ont été une source d'admiration et d'inspiration permanente, et c'est un très grand honneur pour moi de les compter parmi mes rapporteurs. Je remercie vivement Vladimir Berkovich et Mattias Jonsson pour leur participation au jury.

Philippe Biane, Jean-Michel Bismut, Charles Favre, Ilia Itenberg et Claire Voisin me font le grand honneur de participer au jury.
- Ma première experience de recherche fut sous la direction bienveillante de Jean-Michel Bismut, et une partie de ma recherche actuelle est en lien avec les thèmes et le problème qu'il m'avait suggéré de regarder à l'époque (qui fut par ailleurs brillamment résolu par Maryam Mirzakhani très peu de temps après). Les conseils précieux dont j'ai pu bénéficier pendant cette période sont gravés à jamais dans ma mémoire.
- La théorie de Hodge et ses liens avec la combinatoire forment une partie importante de mes projets de recherche actuels, et les textes et travaux de Claire Voisin sont une grande source d'inspiration pour moi.
- J'ai beaucoup appris de Philippe Biane, Charles Favre et Ilia Itenberg, que ce soit par leurs textes et cours mathématiques, par les discussions intéressantes que nous avons pu avoir ensemble, ou encore par de nombreux activités et événements mathématiques dont ils ont été à l'origine durant ces dernière années.

C'est pour dire à quel point je suis heureux de les voir tous présents à ce jury.

Je remercie mes collaborateurs et mes étudiants, pour les moments très agréables que nous avons consacrés à faire des mathématiques ensemble, et pour tout ce que j'ai pu apprendre grâce à eux et à leur côté.

Les travaux présentés dans ce mémoire ont été effectués pendant mon séjour au DMA durant lequel j'ai pu bénéficier d'excellentes conditions de travail, d'un cadre amical et stimulant pour faire la recherche. Je remercie la direction et tous les membres du DMA qui font de ce département ce qu'il est. J'aimerais en particulier remercier Wendelin Werner pour son soutien et ses conseils au moment de mon arrivée au DMA, Ilia Itenberg pour son aide précieuse et son suivi continu dans la réalisation de cette soutenance, Olivier Wittenberg pour son amitié et son soutien constant, ainsi que tous les collègues mathématiciens qui sont, ou qui sont passés, sous les toits du DMA, pour les moments agréables que nous avons quotidiennement passés ensemble.

Un grand merci aux personnels administratifs et scientifiques de l'école doctorale de sciences mathématiques de Paris Centre.

Je remercie ma famille et mes amis pour leur soutien et pour les moments que nous partageons à faire l'experience de la vie ensemble.

Pooran et Armand me remplissent tous les jours de bonheur :
qu'ils soient remerciés pour leur amour.

Contents

List of presented work for the habilitation

[Ami1] O. Amini, *Equidistribution of Weierstrass points on curves over non-Archimedean fields*, preprint.

[Ami2] O. Amini, *The combinatorial Chow ring of products of graphs*, preprint.

[Ami3] O. Amini, *Logarithmic tree factorials*, preprint.

[Ami4] O. Amini, *The exchange graph and variation of the ratio of the two Symanzik polynomials*, Annales de l'Institut Henri Poincaré D, to appear.

[Ami5] O. Amini, *Reduced divisors and embeddings of tropical curves*, Trans. Amer. Math. Soc. 365 (2013), 4851-4880.

[AmiBa] O. Amini, M. Baker, *Linear series on metrized complexes of algebraic curves*, Mathematische Annalen 362 (2015), 55-106.

[ABBR1] O. Amini, M. Baker, E. Brugallé, J. Rabinoff, *Lifting harmonic morphisms I: metrized complexes and Berkovich skeleta*, Research in the Mathematical Sciences 2:7 (2015), 67 p.

[ABBR2] O. Amini, M. Baker, E. Brugallé, J. Rabinoff, *Lifting harmonic morphisms II: tropical curves and metrized complexes*, Algebra and Number Theory 9 (2015), no.2, 267-315.

[ABBF] O. Amini, S. Bloch, J. Burgos, J. Fresan, *Feynman amplitudes and limits of heights*, Izvestiya: Mathematics 80 (2016), no. 5, 813-848, special volume in honor of J-P. Serre.

[AmiCa] O. Amini, L. Caporaso, *Riemann-Roch for weighted graphs and tropical curves*, Advances in Mathematics 240 (2013), 1-23.

[AmiCo] O. Amini, D. Cohen-Steiner, *A transfer principle and applications to eigenvalue estimates for graphs,* Commentarii Mathematici Helvetici 93 (2018), 203-223.

[ADGO1] O. Amini, Devroye, S. Griffiths, N. Olver, *On explosion in heavy-tailed branching random walks*, Annals of Probability 41 (2013), 1864-1899.

[ADGO2] O. Amini, L. Devroye, S. Griffiths, N. Olver, *Explosion and linear transit time in infintie trees*, Probability Theory and Related Fields 167 (2017), no.1, 325-347.

[AEH] O. Amini, L. Esperet, and J. van den Heuvel, *A unified approach to distance two colourings of graphs on surfaces*, Combinatorica 33 (2013), Issue 3, 253-296.

[AmiKool] O. Amini, J. Kool, *A spectral lower bound for the divisorial gonality of metric graphs*, International Mathematics Research Notices 8 (2016), 2423-2450.

[AmiMan] O. Amini, M. Manjunath, *Riemann-Roch for sub-lattices of the root lattice A_n*, Electron. J. Combin. 17 (2010), #R124, 50p.

Préface en français

Ce texte rassemble une partie de mes travaux de recherche réalisés depuis mon entrée au CNRS, entre 2008 et 2017. Le fil conducteur de tous ces travaux est la géométrie des graphes. Les graphes sont des objets simples qui apparaissent dans de nombreux domaines, et qui malgré leur simplicité d'apparence remarquable possèdent une mathématique très riche. Dans ce texte, nous pŕesentons des travaux qui touchent à des questions diverses de géométrie algébrique et arithmétique, d'analyse archimédienne et non-archimédienne, de probabilité, de physique mathématique et de combinatoire.

Au chapitre 1, nous commençons par une introduction générale à la géométrie algébrique des graphes finis et métriques, en révisant une grande partie des théorèmes fondamentaux qui les concernent. Ce chapitre contient aussi une partie des résultats de [**Ami5, Ami6, AmiMan**]. C'est un chapitre long où on a décidé, une fois n'est pas de coutume, de donner suffisamment de détails sur un certain nombre de preuves et d'outils qui vont être utilisés dans la suite de ce texte. Ainsi, nous espérons que cela permettra au lecteur de lire chacun des autres chapitres de manière plus ou moins indépendante.

Le chapitre 2 présente les résultats de [**AmiBa, AmiCa**], obtenus en collaboration avec Matt Baker et Lucia Caporaso. Dans [**AmiBa**] nous introduisons et étudions les complexes métriques de courbes. Cela permet de proposer une définition commode de séries linéaires limites, et d'expliquer comment la raffiner en une définition purement combinatoire [**Ami8**]. Ce chapitre contient aussi un survol de la structure des courbes analytiques au sens de Berkovich.

Le chapitre 3 présente les résultats d'une série de deux articles [**ABBR1, ABBR2**] obtenus en collaboration avec Matt Baker, Erwan Brugallé et Joe Rabinoff. L'objectif est de montrer comment les graphes et les complexes métriques de courbes permettent de contrôler la géométrie des morphismes de courbes algébriques. Nous discutons le problème de réduction semistable simultanée et des problèmes de relèvement de morphismes harmoniques.

Le chapitre 4 concerne les résultats de l'article [**Ami1**]. Nous utilisons les séries linéaires combinatoires du chapitre 2 pour démontrer un théorème d'équirépartition concernant les points de Weierstrass des courbes algébriques sur les corps non-Archimédiens.

Dans le chapitre 5, nous présentons les résultats de deux articles [**ABBF**] et [**Ami4**], le premier obtenu en collaboration avec Spencer Bloch, José Burgos Gil et Javier Fresàn. Cela concerne des problèmes mathématiques liés aux amplitudes de Feynman. On montre un théorème de limite pour l'accouplement de hauteur entre zéro-cycles de degré zéro sur une famille de surfaces de Riemann, sur une base quelconque, vers l'accouplement de hauteur entre zéro-cycle de degré zéro sur un graphe métrique. La preuve pose des problèmes combinatoires intéressants sur un graphe qui contrôle les propriétes d'échange entres les fôrets couvrantes d'un graphe.

Le chapitre 6 est consacré à l'étude entreprise dans [**Ami2**] d'un anneau de Chow associé à un produit de graphes, qui contrôle la partie combinatoire de l'anneau de Chow d'un produit de courbes semistables. Les résultats obtenus permettent de trouver des formules analytiques sur la partie non-archimédienne de l'accouplement entre les diviseurs dans un produit de courbes sur un corps non-archimédiens, géneralisant ainsi la formule analytique étudiée dans le chapitre précédent pour l'accouplement entre deux zéro-cycles dans une courbe.

Le chapitre 7 contient une présentation des résultats sur les propriétés spectrales des graphes, obtenus dans [**AmiCo**], en collaboration avec David Cohen-Steiner, et avec Janne Kool dans [**AmiKool**]. Le premier contient un théorème de transfer qui permet de trouver des bornes supérieures serrées sur le comportement des valeurs propres d'un graphe plongé dans une surface à partir des valeurs propres de la surface elle-même. Cela nous permet d'établir des bornes sur les valeurs propores d'un graphe en fonction de son genre, le nombre de ses sommets et sa valence maximum. Le deuxième établie une inegalité de Li-Yang-Yau pour la gonalité divisorielle d'un graphe métrique, définie à partir de sa théorie des diviseurs.

Dans le chapitre 8 nous présentons les résultats de [**ADGO1, ADGO2, Ami3**]. Les deux premiers articles, écrits en collaboration avec Luc Devroye, Simon Griffiths et Neil Olver, sont consacrés à l'étude du phénomène d'explosion des marches aléatoires dans les arbres. Le troisième définit une générlisation des factorielles de Bhargava au cas des arbres et démontre un théorème d'équirépartition les concernant.

Enfin, le chapitre 9 présente les résultats de [**AEH**] obtenus en collaboration avec Louis Esperet et Jan van den Heuvel. Nous introduisons un nouveau concept de coloration de graphes, et l'étudions dans la classe de graphes de genre borné. Cela nous permet par exemple de déterminer l'asymptotique du nombre chromatique cyclique des graphes plongés dans une surface. La preuve fait appel à des distributions de probabilités sur les couplages d'un graphe, définies à l'aide du polytope de couplage du graphe, et à la méthode de déchargement.

Il est remarquable de voir l'apparition repetée des arbres comme motifs gouvernant la géométrie des structures discrètes, en lien avec d'autres domaines de mathématiques. Ils ont certainement plus de choses à nous dire.

Preface in English

This text presents part of the research I have carried out after my PhD thesis, between 2008 and 2017. The main thematic is the geometry of graphs, and the presented works touch at diverse questions in algebraic and arithmetic geometry, Archimedean and non-Archimedean analysis, probability, mathematical physics, and combinatorics.

A good portion of the results are related or motivated by algebraic geometry, and aim to describe the interesting algebraic and geometric properties of combinatorial objects such as graphs, matroids, or simplicial complexes. These objects, with *a very familiar non-technical nature*, to repeat Grothendieck's words in *Esquisse d'un programme* [**122**], which reveal diverse facets and a rich mathematics already from the point of view of a combinatorist, quite naturally appear in algebraic geometry from different perspective. Moreover, the universality results, such as those of Mnëv [**191, 235**] and Belkale-Brosnan [**32**], suggest that this appearance is not a coincidence and these combinatorial gadgets somehow lie at the heart of algebraic geometry.

We start in Chapter 1 by giving a general introduction to the algebraic geometry of finite and metric graphs, reviewing some of the fundamental theorems, and giving details on a certain number of proofs and tools which will be used later through the text. This chapter contains also some of the results of [**Ami5, Ami6, AmiMan**]. We hope that this allows a more or less independent reading of the other chapters.

Chapter 2 presents the results of [**AmiBa, AmiCa**], written in collaborations with Matt Baker and Lucia Caporaso, respectively, and [**Ami8**]. We introduce metrised complexes, give a brief review of Berkovich analytic curves, and discuss the formalism of slope structures and combinatorial linear series.

Chapter 3 concerns the results of a series of two papers [**ABBR1, ABBR2**] written in collaboration with Matt Baker, Erwan Brugallé and Joe Rabinoff. The objective is to show how graphs and metrised complexes control the algebraic geometry of morphisms of curves. We discuss simultaneous semistable reductions and lifting problems.

Chapter 4 presents the results of the work [**Ami1**]. We use the formalism of slope structures introduced in Chapter 2 to prove an equidistribution theorem concerning Weierstrass points of curves over non-Archimedean fields.

In Chapter 5, we discuss the results of [**ABBF**], obtained in collaboration with Spencer Bloch, José Burgos Gil et Javier Fresán, and the work [**Ami4**]. These concern mathematical problems related to Feynman amplitudes. We study asymptotics of height pairing between degree zero divisors on families of Riemann surfaces, and find pairing between divisors on graphs as the limit. The proof leads to combinatorial problems regarding a graph which controls the exchange properties between spanning forests of a graph.

Chapter 6 is devoted to the results of [**Ami2**] where a Chow ring associated to a product of graphs is studied. This Chow ring controls the combinatorial part of the Chow ring of a product of semistable curves. The results obtained lead to an analytic formula, which generalises the graph pairing studied in Chapter 5.

Chapter 7 contains a discussion of the results of [**AmiCo**] and [**AmiKool**] on the spectral properties of graphs, written jointly with David Cohen-Steiner and Janne Kool, respectively. The first work presents a transfer theorem which allows to obtain upper bounds on the behaviour of Laplacian eigenvalues of graphs embedded in surfaces, in terms of the spectral theory of surfaces. The second work establishes a Li-Yang-Yau for the divisorial gonality of metric graphs, defined in terms of the divisor theory of graphs presented in Chapter 1.

In Chapter 8 we present the results of [**ADGO1, ADGO2, Ami3**]. The first two articles, written in collaboration with Luc Devroye, Simon Griffiths and Neil Olver, study the explosion phenomenon in branching random walks on trees. The last one, presents an extension of the Bhargava factorials to trees and prove an equidistribution theorem in this setting.

Finally, Chapter 9 presents the results of [**AEH**] obtained in collaboration with Louis Esperet and Jan van den Heuvel. We introduce a colouring concept and study it in the class of graphs of bounded genus. This allows to obtain asymptotic theorems on the cyclic chromatic number of graphs on surfaces, and tight bounds on the clique number of squares of graphs. The proof uses hardcore probability distributions associated to the points of the matching polytope, and the discharging method.

It is quite remarkable to see the repeated appearance of trees as motives governing the geometry of graphs, and more general discrete structures, in connection to other branches of mathematics. They certainly have more to tell us.

CHAPTER 1

Algebraic geometry of graphs

This chapter serves as a general self contained introduction to the algebraic geometry of graphs and metric graphs. The materials in this section are throughly used in other chapters, and we have decided to provide a somehow detailed account of the basic foundational results, sometimes with different or more streamlined proofs. The link to the algebraic geometry of curves will be given in other chapters. Other relevant topics can be found in the survey papers [**Ami7, 22**].

1.1. Divisors on graphs

Let $G = (V, E)$ be a finite loopless and connected graph. Multiple edges are allowed.

We denote by $\mathrm{Rat}(G)$ the set of *rational (meromorphic) functions* on G defined by

$$\mathrm{Rat}(G) := \{f : V \to \mathbb{Z}\}.$$

The group of divisors on G denoted by $\mathrm{Div}(G)$ is by definition the free abelian group generated by vertices in V. We write (v) for the generator associated to $v \in V$.

$$\mathrm{Div}(G) := \Big\{\sum_{v\in V} n_v(v) \mid n_v \in \mathbb{Z}\Big\}.$$

Note that $\mathrm{Div}(G)$ and $\mathrm{Rat}(G)$ are isomorphic as abelian groups, however, the formalism is set up in a way to distinguish the role of functions from that of points.

For a divisor $D \in \mathrm{Div}(G)$, the coefficient of (v) in D is denoted by $D(v)$, so we can e.g. write $D = \sum_{v\in V} D(v)\,(v)$.

We now define the order of vanishings of functions at vertices. For any vertex $v \in V$, denote by $\mathrm{ord}_v : \mathrm{Rat}(G) \to \mathbb{Z}$ the *order of vanishing at* v function which on $f \in \mathrm{Rat}(G)$ takes the value

$$\mathrm{ord}_v(f) := \sum_{uv\in E} f(u) - f(v).$$

One notices that this is reminiscent of the definition of the Laplacian in graphs, that we will encounter later in this manuscript. However, we again emphasise the distinction made of the different roles played by points and functions in the theory.

To any $f \in \mathrm{Rat}(G)$ we associate the divisor $\mathrm{div}(f)$ by

$$\mathrm{div}(f) = \sum_{v\in V} \mathrm{ord}_v(f)(v).$$

Elements of this form in $\mathrm{Div}(G)$ are called *principal*, and the subgroup of $\mathrm{Div}(G)$ formed by principal divisors is denoted by $\mathrm{Prin}(G)$.

A divisor D_1 is called *linearly equivalent* to a divisor D_2, and we write $D_1 \sim D_2$, if there exists $f \in \mathrm{Rat}(G)$ such that $D_1 = D_2 + \mathrm{div}(f)$.

The *degree map* $\deg : \mathrm{Div}(G) \to \mathbb{Z}$ is defined by

$$\forall\ D \in \mathrm{Div}(G), \qquad \deg(D) := \sum_{v \in V} D(v).$$

For any integer k, we denote by $\mathrm{Div}^k(G)$ the set of divisors of degree k on G. One easily verifies that $\mathrm{Prin}(G)$ is in fact a subgroup of $\mathrm{Div}^0(G)$. The Picard group of G denoted by $\mathrm{Pic}(G)$ is by definition

$$\mathrm{Pic}(G) := \mathrm{Div}^0(G)/\,\mathrm{Prin}(G).$$

REMARK 1. The group $\mathrm{Pic}(G)$ is the first non-trivial and quite interesting arithmetic invariant associated to a graph. It follows from the matrix-tree theorem that the size of $\mathrm{Pic}(G)$ is the number of spanning trees of the graph, and so we are putting a non-canonical group structure on the set of spanning trees of G. Understanding the behaviour of this group in families of graphs reveals to be an interesting mathematical problem. For example, Cohen-Lenstra heuristics for $\mathrm{Pic}(G)$ in the family of Erdös-Rényi random graphs have been studied in [**73, 245**]; it is in particular conjectured that the probability of having $\mathrm{Pic}(G)$ cyclic tends to $\frac{1}{\zeta(3)}\cdot\frac{1}{\zeta(5)}\cdot\frac{1}{\zeta(7)}\cdot\dots$, when the size of the Erdös-Rényi random graph G tends to infinity.

1.1.1. Abel-Jacobi theorem for finite graphs. Let $G = (V, E)$ be a finite graph without loops. Following the usual convention, see e.g. [**225**], we replace the set of edges of G with the set of oriented edges $\mathbb{E}$, where each edge $e = \{u, v\} \in E$ is replaced with two oriented edges uv and vu in opposite directions in $\mathbb{E}$. By an abuse of notation, it can happen that we denote by e an element of $\mathbb{E}$ and by $\bar{e}$ the inverse of e. Note that $\mathbb{E}$ has size twice the size of E.

For a ring of coefficients A, let $H^1(G, A)$ denote the first cohomology of the graph G seen as a one-dimensional simplicial complex. The group $H^1(G, A)$ is a free A-module of rank g, where $g = |E| - |V| + 1$ is the first betti number of G. It can be identified with the group of A-valued flows in the graph. For $A = \mathbb{Z}$, $H^1(G, \mathbb{Z})$ forms a lattice of full rank in the vector space $H^1(G, \mathbb{R})$. In addition, we have a natural bilinear pairing $C^1(G, A) \times H^1(G, A)$, given by

$$\langle \sigma, \tau \rangle := \frac{1}{2} \sum_{e \in \mathbb{E}} \sigma(e)\tau(e).$$

The pairing induces a symmetric bilinear form on $H^1(G, A)$, which for $A = \mathbb{Z}$ takes integral values. Denote by $H^1(G, \mathbb{Z})^{\#}$ the integral dual of $H^1(G, \mathbb{Z})$ in $H^1(G, \mathbb{R})$ with respect to this form, i.e.,

$$H^1(G, \mathbb{Z})^{\#} := \Big\{ \sigma \in H^1(G, \mathbb{R}) \,\Big|\, \text{for all } \tau \in H^1(G, \mathbb{Z}),\ \langle \sigma, \tau \rangle \in \mathbb{Z} \Big\}.$$

We thus have $H^1(G, \mathbb{Z}) \subseteq H^1(G, \mathbb{Z})^{\#}$, and we can define the Jacobian $J(G)$ of G as the quotient

$$J(G) := H^1(G, \mathbb{Z})^{\#}/H^1(G, \mathbb{Z}).$$

Let v_0 be a fixed vertex of the graph. There exists a natural map $\Phi : V \to J(G)$ defined as follows. For any vertex v, take a path P from v_0 to v in G, that we orient from v_0 to v. Denote by $\mathbb{E}(P)$ the set of oriented edges of P with this orientation. Then $\Phi_0(v)$ is the class in $J(G)$ of the element of the dual $H^1(G, \mathbb{Z})^{\#}$ which sends any element τ of $H^1(G, \mathbb{Z})$ to the integer $\sum_{e \in \mathbb{E}(P)} \tau(e)$. Since changing P to P' results in an element of $H^1(G, \mathbb{Z})$, the map is well-defined.

One can extend the map Φ to a map $\Phi : \mathrm{Div}^0(G) \to J(G)$, by linear combination. The map Φ descends to $\mathrm{Pic}(G)$, and define a well-defined map AJ :$\mathrm{Pic}(G) \to J(G)$, which does not depend on the choice of the base point v_0, and which is called the Abel-Jacobi map. We have the following Abel-Jacobi theorem for finite graphs.

THEOREM 2 (Bacher-de la Harpe-Nagnibeda [**17**]). *The Abel-Jacobi map* AJ *is an isomorphism of finite groups.*

1.1.2. Combinatorial linear systems. A partial order can be defined on divisors by declaring $D \geq D'$ if for any vertex $v \in V$, $D(v) \geq D'(v)$. A divisor D is called *effective* if $D \geq 0$, which means $D(v) \geq 0$ for all $v \in V$. The *complete linear system* $|D|$ associated to D is defined as the set of all the effective divisors E linearly equivalent to D, i.e.,

$$|D| = \Big\{ E \;\Big|\; E \geq 0 \quad \text{and} \quad E \sim D \Big\}.$$

We also define $R(D)$ as the set of all rational functions f on G which verify $\mathrm{div}(f)+D \geq 0$.

We will later propose in Section 2.4.1 a refined notion of a combinatorial linear series which cover the case of non-necessarily complete linear systems, as in classical algebraic geometry.

1.1.3. Chip firing game. Consider the following game played on a connected graph G. Vertices represent people in a group, where edges represent friendship. Each person $v \in V$ has a certain number of pieces $n_v \in \mathbb{Z}$, where $n_v < 0$ means v is in debt. The aim of the group is to achieve a situation where no one is in debt. The only rule of the game is that at each step one person can decide to give one piece to any of its neighbours. The question is then *is there a winning strategy for the group?* Representing the initial configuration with the divisor $\sum_{v \in V} n_v(v)$ one can see that the group has a winning strategy if and only if $|D| \neq \emptyset$.

In the next section, we define a certain type of configurations which allows to answer the above and other similar questions.

1.1.4. Rank of divisors and Riemann-Roch theorem. Ideally, we would like to associate a notion of dimension to the complete linear system $|D|$. The set $|D|$ being discrete, we will see how to associate a combinatorial reformulation of the dimension, which allows to associate a *rank* to each divisor D in the graph.

DEFINITION 3 (Rank of a divisor [**24**]). Let D be a divisor on a graph G. We say

- $r(D) = -1$ if $|D| = \emptyset$.
- $r(D) \geq r$ if for all effective divisors $E \geq 0$ of degree r, we have $|D - E| \neq \emptyset$.

The rank of D is by definition the maximum r with $r(D) \geq r$.

REMARK 4. Let X be a smooth projective curve over an algebraically closed field k. The definition of rank given above makes sense for any divisor D on X. A simple linear algebra argument then shows that the rank of D, as defined above, is equal to $\dim_k(H^0(X, \mathcal{O}(D)))-1$, which is the usual definition of rank of the line bundle $\mathcal{O}(D)$ on X.

EXAMPLE 5.

- *If* $\deg(D) < 0$*, then* $r(D) = -1$.
- *For* D *of non-negative degree, we have* $r(D) \leq \deg(D)$.
- *We have* $r(D) = \max\{\deg(D), -1\}$ *for all* D *if and only if* G *is a tree.*

For a graph G, we define the *arithmetic genus* of G denoted by $g(G)$, or simply g if there is no risk of confusion, the quantity

$$g(G) = |E| - |V| + 1.$$

Remark 6. This is also the first Betti number of the graph G seen as a simplicial complex of dimension one. The term arithmetic has been added to distinguish from the geometric genus (which is the minimum genus of a topological surface on which the graph can be embedded without any crossings, and will appear later in other chapters), and coincides with the arithmetic genus of any semistable curve with dual graph G and with only rational irreducible components.

The *canonical divisor* of G is the divisor $K = \sum_{v\in V}(\mathrm{val}(v) - 2)(v)$. Note that $\deg(K) = 2|E| - 2|V| = 2g - 2$.

With these notations, we can now state Riemann-Roch theorem for graphs.

Theorem 7 (Baker-Norine [**24**]). *Let G be a connected loopless graph of arithmetic genus g. For any divisor D, we have*

$$r(D) - r(K - D) = \deg(D) - g + 1.$$

We will give the proof of the above theorem in the next sections and use this occasion to introduce some tools which will be used and refined in future chapters. Before, let us give some consequences of the theorem. Simple applications of the theorem to appropriate divisors, give

- $r(K) = g - 1$;
- For K' of degree $2g - 2$, we have $r(K') = g - 1$ if and only $K' \sim K$;
- (Riemann's inequality) $r(D) \geq \deg(D) - g$; and
- the equality $r(D) = \deg(D) - g$ holds provided that $\deg(D) \geq 2g - 1$.

Remark 8 (Riemann-Roch formalism in abstract setting). Let X be a set, finite or infinite. Suppose that we have a set $\mathrm{Rat}(X)$, and functions $\mathrm{ord}_x : \mathrm{Rat}(X) \to \mathbb{Z}$. We assume that the functions ord_x verify the following two properties

- (finiteness) For any $f \in \mathrm{Rat}(X)$, for all but finitely many element $x \in X$, we have $\mathrm{ord}_x(f) = 0$.
- (logarithmic product formula) For any $f \in \mathrm{Rat}(X)$, we have $\sum_{x\in X} \mathrm{ord}_x(f) = 0$.

We can generalize all the definitions above to the setting of X equipped with the set $\mathrm{Rat}(X)$ and $\{\mathrm{ord}_x\}$. So the groups $\mathrm{Div}(X)$ and $\mathrm{Prin}(X) \subset \mathrm{Div}^0(X)$ are defined, and we can also define for all $D \in \mathrm{Div}(X)$, the linear system $|D|$ and the integer $r(D) \geq -1$. We say that X satisfies Riemann-Roch if we can associate to X a genus g and a divisor K of degree $2g - 2$ such that for all divisors D, the Riemann-Roch formula holds:

$$r(D) - r(K - D) = \deg(D) - g + 1.$$

Graphs provide examples of such structures satisfying Riemann-Roch theorem. Other examples of X as above include metric graphs and projective smooth curves over algebraically closed fields, or mixed objects like metrized complexes of algebraic curves, that we will discuss later.

1.1.5. Consequences. The following are immediate consequences of the Riemann-Roch theorem.

Theorem 9 (Clifford's theorem). *Let D be a divisor with $r(D), r(K - D) \geq 0$. Then $r(D) \leq \deg(D)/2$.*

Indeed, under the assumptions of the theorem, one easily verifies, $g-1 = r(K) \geq r(D) + r(K-D)$, which combined with Riemann-Roch, gives $r(D) \leq \deg(D)/2$.

A graph G is called *hyperelliptic* if there exists a divisor of degree two and rank one on G. As in the classical version, graph theoretic Clifford's theorem has a second part, which says that if a graph G has a divisor D with equality in the above statement, namely $r(D) = \deg(D)/2$, then either $D = 0$ or $D = K$, or G is hyperelliptic [**82, 169**]. This part of the theorem is more delicate.

Let p be a fixed vertex in G, and $k \in \mathbb{N}$. Consider the map $S^{(k)} : V^k \to \mathrm{J}(G)$, defined by

$$S^{(k)}(v_1, \dots, v_k) = \sum_{i=1}^{k}(v_i) - (p).$$

THEOREM 10 (Jacobi inversion). *The map $S^{(k)}$ is surjective if and only if $k \geq g$.*

Let D be a divisor of degree zero. The divisor $D + k(p)$ has degree k. If $k \geq g$, then $|D + k(p)| \neq \emptyset$. Let $E \in |D + k(p)|$, and write $E = (v_1) + \cdots + (v_k))$. It follows that $S^{(k)}(v_1, \dots, v_k) = D$ which proves the surjectivity for $k \geq g$.

On the other hand, we will see later in the proof of Riemann-Roch theorem the existence of divisors D of degree $g-1$ (linearly equivalent to *moderators*), with $|D| = \emptyset$. The divisor $D - (g-1)(p)$ is not in the image of $S^{(k)}$ for $k \leq g-1$.

1.1.6. Reduced divisors: existence and uniqueness. A useful tool for the study of divisors on graphs is the concept of a reduced divisor (which one takes care of not confusing with reduced notion in algebraic geometry). Reduced divisors provide representatives of each linear equivalence class, with some combinatorial properties which make them manageable to work with. They are intimately related to the concept of critical configurations in the original theory of chip firing [**89, 90**]. Since we do not need critical configurations, we do not precise this here.

For a subset $A \in V$, and a vertex $v \in A$, we denote by $\mathrm{outval}_A(v)$ the number of edges $uv \in E$ with $u \in V \setminus A$. The *border* of A denoted by ∂A is the set of all vertices $v \in A$ with $\mathrm{outval}_A(v) \neq 0$.

Fix a vertex $p \in V$. A divisor $D \in \mathrm{Div}(G)$ is called *p-reduced* if the following two conditions are satisfied

(i) For all $v \in V \setminus \{p\}$, $D(v) \geq 0$, and
(ii) For all $A \subseteq V \setminus \{p\}$, there exists a vertex $v \in A$ such that $D(v) < \mathrm{outval}_A(v)$.

If we denote by 1_A the characteristic function of A which takes value 1 on every vertex $v \in A$ and value 0 outside A, the second condition simply says that $D + \mathrm{div}(1_A)$ violates property (i) in the definition.

We have the following theorem.

THEOREM 11. *For any divisor D there exists a p-reduced divisor D_p linearly equivalent to D. Moreover, D_p is unique.*

Sketch of proof. To see the uniqueness, assume, for the sake of a contradiction, that two distinct divisors $D'_p \sim D_p$ are both p-reduced. Write $D_p = D'_p + \mathrm{div}(f)$ for a non-constant function $f \in \mathrm{Rat}(G)$. Let A be the set of all vertices where f takes its maximum. Changing the role of D_p and $D' + p$ if necessary, we can assume that $p \notin A$.

By property (ii) in the definition of reduced divisors, applied to D'_p and A, there exists $v \in \partial A$ such that $D'_p(v) < \mathrm{outval}_A(v)$. We have

$$\mathrm{ord}_v(f) = \sum_{uv \in E} f(u) - f(v) \leq -\mathrm{outval}_A(v),$$

which implies that

$$D_p(v) = D'_p(v) + \mathrm{ord}_v(f) \leq D'_p(v) - \mathrm{outval}_A(v) < 0,$$

contradicting the condition (i) in the definition of reduced divisors.

To prove the existence, consider $\mathcal{D}$ the set of all divisors $D' \sim D$ which verify condition (i) in the definition of reduced divisors. One can easily show that $\mathcal{D}'$ is not empty (e.g. by using the fact that all the divisors of the form $(v) - (p) \in \mathrm{Div}^0(G)$ have finite order in $\mathrm{Pic}(G)$).

Order the vertices of V as $v_1, \ldots, v_n$ with respect to their distances to p, i.e., in such a way that $\mathrm{dist}_G(v_1, p) \leq \mathrm{dist}_G(v_2, p) \leq \ldots \mathrm{dist}_G(v_n, p)$.

We associate to any divisor $D' \in \mathcal{D}$ the vector of its coefficients in $\mathbb{Z}^n$, defined by

$$\mathrm{Vect}(D') := (D'(v_1), D'(v_2), \ldots, D'(v_n)).$$

The existence now follows by observing that the element D_p in $\mathcal{D}$ which has the maximum vector of coefficients in the lexicographical order in $\mathbb{Z}^n$ is p-reduced. (In the lexicographical order in $\mathbb{Z}^n$, we have $(a_1, \ldots, a_n) <_{lex} (b_1, \ldots, b_n)$ if there exists $0 \leq i < n$ such that $a_j = b_j$ for all $j \leq i$, and $a_{i+1} < b_{i+1}$.) $\square$

The immediate consequence of the existence of p-reduced divisors is an answer to the question of the existence of winning strategies in the chip-firing game. In the formalism above, this states:

Proposition 12. *We have $|D| \neq \emptyset$ if and only if the coefficient of D_p at p is non-negative.*

Indeed, if $D_p(p) \geq 0$, then obviously $D_p \geq 0$. For the other direction, note that D_p maximises its vector of coefficients in the lexicographic order. If there exists an effective divisor $E \sim D$, then $E \in \mathcal{D}$, and necessarily $D_p(p) \geq E(p) \geq 0$.

Here is an intuitive interpretation of the reducedness.

Remark 13 (Dhar's algorithm [**89, 90**]). Let D be a divisor with $D(v) \geq 0$ for $v \neq p$. Suppose a fire starting from p propagates in G by following the edges of the graph in a uniform way, and all edges have the same length. At each vertex $v \neq p$ there are $D(v)$ firefighters and each can stop the propagation in one direction. The fire passes to the vertex v if $D(v)$, the number of firefighters at v, is strictly smaller than the number of edges uv with u already in fire.

One can show that D is p-reduced if and only if the fire eventually passes through all the vertices of the graph.

Remark 14. Proposition 11 allows to provide an explicit bijection between the elements of $\mathrm{Pic}(G)$ and the spanning trees of the graph [**83, 28**]. This is non-canonical and depends on the choice of an order on the edges of the graph. Let $e_1, e_2, \ldots, e_m$ be an enumeration of the edge set. For any divisor D of degree zero, let $D_p \sim D$ be p-reduced. Consider the Dhar's algorithm above, and suppose in addition that the firefighters at each unburned vertex v keep the edges with smallest indices among the edges uv with u already in fire. Keep the edge with smallest index along which the fire passes to v. One can show that the set of these edges form a tree T_D, and that the correspondence D to T_D is a bijection between $\mathrm{Pic}(G)$ and the spanning trees of the graph.

1.1.7. Proof of Riemann-Roch theorem. Let $<_{\mathcal{O}}$ be a total order on the set of vertices of the graph G. We write $\mathrm{val}^{\leftarrow}_{\mathcal{O}}(v)$ for the number of edges $uv \in E$ with $u <_{\mathcal{O}} v$. If we enumerate the vertices as $v_1, v_2, \dots, v_n$ with respect to the order $<_{\mathcal{O}}$, i.e. $i < j$ if $v_i <_{\mathcal{O}} v_j$, then $\mathrm{val}^{\leftarrow}_{\mathcal{O}}(v_j) = \# v_i v_j$ with $i < j$.

To a total order as above, one associates the divisor $D_{\mathcal{O}}$ called a *moderator* as follows:

$$D_{\mathcal{O}} = \sum_{v \in V} (\mathrm{val}^{\leftarrow}_{\mathcal{O}}(v) - 1)(v).$$

The degree of $D_{\mathcal{O}}$ is $g-1$, and for the opposite order $\overline{\mathcal{O}}$ to $\mathcal{O}$ defined by $u <_{\overline{\mathcal{O}}} v$ if and only if $v <_{\mathcal{O}} u$, for any $u, v \in V$, we have $D_{\mathcal{O}} + D_{\overline{\mathcal{O}}} = K$.

The moderators given a characterisation of all the linear equivalence classes of divisors of degree $g-1$ and negative rank on G. To see this, first we show for a total order $\mathcal{O}$ on vertices, we have $|D_{\mathcal{O}}| = \emptyset$.

Otherwise, there would exist $f \in \mathrm{Rat}(G)$ such that $D_{\mathcal{O}} + \mathrm{div}(f) \geq 0$. Since $D_{\mathcal{O}}(v_1) < 0$, f would not be constant. Take A the set of vertices where f takes its maximum, and let v be the smallest vertex in A for the order $<_{\mathcal{O}}$. We would have

$$D_{\mathcal{O}}(v) + \mathrm{ord}_v(f) \leq D_{\mathcal{O}}(v) + \sum_{uv \in E : u <_{\mathcal{O}} v} f(u) - f(v) \leq D_{\mathcal{O}}(v) - \mathrm{val}^{\leftarrow}_{\mathcal{O}}(v) = -1,$$

which would contradict $D_{\mathcal{O}} + \mathrm{div}(f) \geq 0$.

Let now D a divisor and $D_p \sim D$ be p-reduced, for a vertex p in the graph. Dhar's algorithm gives a total order $\mathcal{O}$, $v_1 = p, v_2, \dots, v_n$, on vertices, defined inductively as follows. Start by $v_1 := p$, and inductively apply property (ii) in the definition of reduced divisors to the divisor D_p and the set $A_k = V \setminus \{v_1, \dots, v_k\}$. There thus exists $v_{k+1} \in A_k$ so that $\mathrm{outval}_{A_k}(v_{k+1}) > D_p(v_{k+1})$. By the construction, we have $D_p(v_{k+1}) < \mathrm{val}^{\leftarrow}_{\mathcal{O}}(v_{k+1}) = \mathrm{outval}_{A_k}(v)$, which shows that away from p, we have for all v, $D_p(v) \leq D_{\mathcal{O}}(v)$. This gives the following useful property of moderators.

PROPOSITION 15. *For all divisor D, one and only one of the two following situations happens:*

- *Either, $|D| \neq \emptyset$,*
- *Or, there exists a moderator $D_{\mathcal{O}}$ with $|D_{\mathcal{O}} - D| \neq \emptyset$.*

To see this, following the notations which preceded the proposition, let $\mathcal{O}$ be the order constructed for $D_p \sim D$. We have $|D| = \emptyset$ if and only if $D_p(p) \leq -1 = D_{\mathcal{O}}(p)$. Since for all $v \neq p$, we have $D_p(v) \leq D_{\mathcal{O}}(v)$, we infer that the latter condition is equivalent to $D_p \leq D_{\mathcal{O}}$, or equivalently, $D_{\mathcal{O}} - D_p \geq 0$, which is equivalent to $|D_{\mathcal{O}} - D| \neq \emptyset$.

A an immediate corollary, we get the following formula for the rank. For a divisor D, $\deg^+(D)$ denotes the sum of all the non-negative coefficients of D. Similarly, $\deg^-(D)$ is the sum of all the negative coefficients.

PROPOSITION 16. *For any divisor D, we have*

$$r(D) = \min_{\substack{D' \sim D \\ \textit{total order}\, \mathcal{O}}} \Big[\deg^+\Big(D' - D_{\mathcal{O}}\Big) - 1 \Big].$$

To see this, denote by $r'(D)$ the quantity on the right hand side. Let $D' \sim D$ and let $\mathcal{O}$ a total order on vertices. For E the positive part of $D' - D_{\mathcal{O}}$, we have $D' - D_{\mathcal{O}} - E \leq 0$, which shows that $D' - E \leq D_{\mathcal{O}}$. This implies in part $|D' - E| = |D_{\mathcal{O}}| = \emptyset$, which shows $r(D) = r(D') \leq \deg(E) - 1 = \deg^+(D' - D_{\mathcal{O}}) - 1$. Taking the minimum over all $D' \sim D$ and $\mathcal{O}$, gives one inequality $r(D) \leq r'(D)$.

To see the other inequality, let E be an effective divisor of degree $r(D) + 1$ such that $|D - E| = \emptyset$. There exists then a total order $\mathcal{O}$ on vertices such that $|D_{\mathcal{O}} - D + E| \neq \emptyset$; equivalently, there exists $f \in \mathrm{Rat}(G)$ with $\mathrm{div}(f) - D + D_{\mathcal{O}} + E \geq 0$. For $D' = D - \mathrm{div}(f) \sim D$, we get $D' - D_{\mathcal{O}} \leq E$. Thus, $\deg^+(D' - D_{\mathcal{O}}) \leq \deg(E) = r(D) + 1$, which implies $r'(D) \leq r(D)$.

We can now finish the proof of Riemann-Roch theorem by a magic trick!

Proof of Riemann-Roch theorem. For $D' \sim D$ and a total order $\mathcal{O}$ on vertices with $\overline{\mathcal{O}}$ its opposite order, using $D_{\mathcal{O}} + D_{\overline{\mathcal{O}}}$, it is easy to see that that

$$\deg^+(D' - D_{\mathcal{O}}) = \deg(D) - g + 1 + \deg^+(K - D' - D_{\overline{\mathcal{O}}})$$

Taking the minimum over all $D' \sim D$ and $\mathcal{O}$ on both sides then gives

$$r(D) + 1 = \deg(D) - g + 1 + \min_{\substack{D' \sim D \\ \mathcal{O}}} \deg^+(K - D' - D_{\overline{\mathcal{O}}}).$$

To finish, observe that when D' goes through all divisors linearly equivalent to D, $K - D'$ goes through all $F \sim K - D$. Thus, the minimum on the right hand side is equal to $r(K - D) + 1$, which finishes the proof. $\square$

1.2. Metric graphs

Metric graphs are defined as *metrisation* of finite graphs. They are continuous-type objects which have discrete models. In this section, we define metric graphs, which will be central objects in the remaining chapters, and explain how the theorems discussed in the previous section extends to their setting.

For a metric space X, we denote by $\mathrm{dist}_X(.,.)$ the metric in X. A pointed metric space is a metric space X with a choice of a distinguished point $\star \in X$. For two pointed metric spaces X and Y with distinguished points $\star_X$ and $\star_Y$ in X and Y, respectively, we define the pointed metric space $X \vee Y = X \sqcup Y / \star_X = \star_Y$ with distinguished point $\star = \star_X = \star_Y$, where the metric in $X \vee Y$ is given by

$$\mathrm{dist}_{X \vee Y}(p, q) = \begin{cases} \mathrm{dist}_X(p, q) & \text{if } p, q \in X \\ \mathrm{dist}_X(p, \star_X) + \mathrm{dist}_Y(q, \star_Y) & \text{if } p \in X, q \in Y. \end{cases}$$

For n pointed metric spaces $(X_1, \star_1), \ldots, (X_n, \star_n)$, we define the pointed metric space $X_1 \vee \cdots \vee X_n = \sqcup_i X_i / \star_1 = \cdots = \star_n$ in a similar way.

Let n be a positive integer and $r > 0$ a positive real number. The *star* $S_n(r)$ of radius r with n branches is the metric space $S_n(r) = I_1 \vee \cdots \vee I_n$ where each I_j is a copy of the pointed half open interval $[0, r[$ of length r with distinguished point $\star_j = 0$.

A *metric graph*, unless otherwise stated, is a compact connected metric space Γ which verifies the following two properties:

(1) For every point $p \in \Gamma$ there exists an integer $n_p \geq 1$ and a real number $r_p > 0$ such that p has a neighborhood V_p isometric to the star of radius r_p with n_p branches, i.e., $(V_p, \mathrm{dist}_\Gamma|_{V_p}) \simeq S_{n_p}(r_p)$.

2 The metric in Γ is given by the path metric, in the sense that for any two points x and y of Γ, the distance between x and y is the infimum (in fact minimum) length of any path from x to y.

(In a path-connected metric space X, the length of a (continuous) path $\alpha : [0,1] \to X$ can be defined as $\inf_{n\in\mathbb{N}^*} \sum_{j=0}^{n-1} \mathrm{dist}_X(\alpha(\frac{j}{n}), \alpha(\frac{j+1}{n}))$.)

The integer n_p in the definition above is called the *valence* (or *degree*) of p, and is denoted by $\mathrm{val}(p)$.

Let now $G = (V, E)$ be a finite connected graph and let $\ell : E \to \mathbb{R}_{>0}$ be a (length) function on the edges of G. assigning a positive real number ℓ_e to any edge $e$of G. We define the *metric realisation* of (G, ℓ) which is a metric graph Γ as follows: for each edge $e = uv$ of G take a closed interval I_e of length ℓ_e, and a surjection $\pi_e : \partial I_e \to \{u, v\}$ (which identifies the two extremities of I_e with the vertices of G in e). As a topological space, Γ is defined as

$$\Gamma := (V \sqcup \bigsqcup_e I_e)/\{\, x = \pi_e(x) \qquad \forall e \in E \ \&\ x \in \partial I_e \,\},$$

and it is straightforward to see that Γ is path-connected. In addition, any two points p and q of Γ are connected by a *piecewise isometric path* $\gamma : [0, T_\gamma] \to \Gamma$: i.e. a continuous path γ with $\gamma(0) = p$ and $\gamma(T) = q$, with the property that there is $N \in \mathbb{N}$ and real numbers $t_0 = 0 < t_1 < t_2 < \dots < t_{N-1} < t_N = T_\gamma$ such that for each i, there is an edge e_i with $\gamma([t_i, t_{i+1}]) \subset I_{e_i} \subset \Gamma$, and γ induces an isometry from the interval $[t_i, t_{i+1}]$ to its image in the interval I_e. The distance $\mathrm{dist}_\Gamma(p, q)$ is then defined as the infimum of T_γ over all piecewise isometric path $\gamma : [0, T_\gamma] \to \Gamma$ from p to q, turning Γ into a metric space. By very definition, it is easy to see that Γ is a metric graph. Moreover, using compactness, one can show that any metric graph Γ is a metric realisation of a finite connected graph endowed with a length function ℓ on its edges.

A pair consisting of a graph G and the length function ℓ such that the metric graph Γ is isometric to the metric realisation of the pair (G, ℓ), is called a *finite graph model* of Γ.

Remark 17. Any finite subset V of Γ with the property that $\Gamma \setminus V$ is the disjoint union of topological spaces $I_1, \dots, I_m$ where each I_j is homeomorphic to an open interval, for $j = 1, \dots, m$, forms the vertex set for a finite graph model $G = (V, E)$ of Γ, which in addition comes with a bijection between E and the connected components $I_1, \dots, I_m$ of $\Gamma \setminus V$.

A finite subset V of Γ with the property of the above exercise, that connected components of $\Gamma \setminus V$ are homeomorphic to open intervals, is called a *vertex set* for Γ.

An *essential point* of a metric graph Γ is by definition a point of Γ of valence different from two. It is easy to see that any vertex set of Γ contains all the essential points of Γ.

Remark 18. One can show that in any metric graph which is not isometric to a circle the set of essential points form a vertex set. In particular, any metric graph not isomorphic to a circle has a unique minimal vertex set with respect to the inclusion.

1.2.1. Unit tangent vectors and slopes. Let $I = [0, \ell[$, $\ell > 0$, be a half open interval in $\mathbb{R}$ with the metric induced from the standard Euclidean metric from $\mathbb{R}$ and distinguished point $\star = 0$. Let $\vec{1}$ be the unit vector in $\mathbb{R}$. Define $T^1_\star I := \{\vec{1}\}$. More precisely, one sees I as a manifold with boundary $\partial I = \star$. The tangent space TI gets naturally identified with $I \times \mathbb{R}$, and inherits a Riemannian metric from the standard Euclidean metric on $\mathbb{R}$. The inclusion of the subvariety of codimension one $\{\star\} \subset I$ gives a unit normal vector $\vec{u} \in T_\star I$, which gets identified with the vector $\vec{1} \in \mathbb{R}$. One then defines $T^1_\star I := \{\vec{u}\}$.

Let now $S_n = I_1 \vee \cdots \vee I_n$ be a star with n branches of radius $r > 0$, and $\star$ be the centre of S_n. Define $T^1_\star S_n := \sqcup_{j=1}^n T^1_{\star_j} I_j$.

Finally, let Γ be a metric graph. Each point $p \in \Gamma$ has a neighbourhood U isometric to a star S with $\mathrm{val}(p)$ branches. Define $T^1_p\Gamma := T^1_p V = T^1_\star S$. Note that for each point p, $T^1_p\Gamma$ has precisely $\mathrm{val}(p)$ elements, each corresponding to a branch of Γ incident to p. They are called *unit (outgoing) tangent vectors* to Γ at p. If there is no risk of confusion, we drop "unit" and only refer to them as outgoing tangent vectors at p.

Let now $\vec{u} \in T^1_p\Gamma$ be a unit tangent vector. Let U be a star-shaped neighborhood of p, and let I be the interval in the decomposition of $U = \vee_j I_j$ (as the sum of open half intervals) which corresponds to the vector $\vec{u}$. For $\epsilon > 0$ sufficiently small, we denote by $p + \epsilon\vec{u}$ the unique point at distance ϵ from p on I.

A function $f : \Gamma \to \mathbb{R}$ is called *piecewise smooth* if there exists a simple graph model $G = (V, E)$ of Γ such that the restriction of f to the closed intervals I_e, for $e \in E$, are all of class C^1. A piecewise smooth function is necessarily continuous.

Let $f : \Gamma \to \mathbb{R}$ be a piecewise smooth function on a metric graph Γ. Let $x \in \Gamma$ a point of Γ and $\vec{u} \in T^1_x\Gamma$ a unit tangent vector to Γ at x. The *(outgoing) slope* of f along $\vec{u}$ denoted by $d_{\vec{u}}(f)$, or sometime by $\mathrm{sl}_{\vec{u}}(f)$, is by definition

$$d_{\vec{u}}(f) := \lim_{\epsilon \to 0^+} \frac{f(p + \epsilon\vec{u}) - f(p)}{\epsilon}.$$

A rich class of piecewise smooth functions is provided by *piecewise linear* functions. These are functions $f : \Gamma \to \mathbb{R}$ for which there exists a simple graph model $G = (V, E)$ of Γ such that on any interval I_e in Γ, for any $e \in E$, the function f restricts to an affine linear function. Equivalently, these are piecewise smooth functions whose set of slopes along unit tangent vectors take only a finite number of values. In particular, a piecewise smooth function with slopes in the set of integers is necessarily piecewise linear.

1.2.2. Divisors on metric graphs and Riemann-Roch theorem. We now explain how any metric graph Γ provides a Riemann-Roch formalism, as in the set-up of Remark 8.

Let Γ be a metric graph. Define $\mathrm{Rat}(\Gamma)$ as the set of all piecewise linear functions on Γ with integral slopes, i.e., such that for any $p \in \Gamma$, and $\vec{u} \in T^1_p\Gamma$, $d_{\vec{u}} f \in \mathbb{Z}$. This is a group under point-wise addition of functions.

For any function $f \in \mathrm{Rat}(\Gamma)$, and any point $x \in \Gamma$, define the *order of vanishing* function $\mathrm{ord}_x(.)$ as the sum of outgoing slopes of f along tangent vectors in $T^1_x\Gamma$, i.e.,

$$\mathrm{ord}_x(f) := \sum_{\vec{u} \in T^1_x\Gamma} d_{\vec{u}} f, \qquad \text{for all } f \in \mathrm{Rat}(\Gamma).$$

It is straightforward to see that $\mathrm{Rat}(\Gamma)$ and $\mathrm{ord}_x : \mathrm{Rat}(\Gamma) \to \mathbb{Z}$ verify the properties given in Remark 8, namely, that for any $f \in \mathrm{Rat}(\Gamma)$,

- for all but finitely many $x \in \Gamma$, $\mathrm{ord}_x(f) = 0$; and
- $\sum_{x \in \Gamma} \mathrm{ord}_x(f) = 0$.

This shows that we can formally define a framework of divisors as in the previous section and define rank of divisors. The group of divisors $\mathrm{Div}(\Gamma)$ is the free abelian group on points of Γ. A divisor $D \in \mathrm{Div}(\Gamma)$ is written as

$$D = \sum_{x\in\Gamma} D(x)(x), \text{ with } D(x) \in \mathbb{Z},$$

where $D(x)$ is the coefficient of D at x, and all but finitely many $D(x)$ are zero. The set of points $x \in \Gamma$ with $D(x) \neq 0$ is called the support of D and denoted by $\mathrm{supp}(D)$.

The arithmetic genus of Γ denoted by $g = g(\Gamma)$ is by definition the genus of any loopless graph model $G = (V, E)$ of Γ, so $g = |E| - |V| + 1$. One verifies that it does not depend on the choice of the model. The canonical divisor K of Γ is by definition

$$K := \sum_{x\in\Gamma} (\mathrm{val}(x) - 2)(x),$$

which belongs to $\mathrm{Div}(\Gamma)$ since all but only finitely many points of Γ have valence two.

The next theorem states that Γ with the above data satisfies the Riemann-Roch formalism.

THEOREM 19 (Gathmann-Kerber [**114**], Mikhalkin-Zharkov [**188**]). *Let Γ be a metric graph of genus g. For any divisor D on Γ, we have $r(D) - r(K-D) = \deg(D) - g + 1$.*

The theorem can be proved by a limit argument from the finite graph case [**114**] or by a similar strategy as in the proof given in the finite graph case, by extending the definition of reduced divisors to metric graphs [**188**].

1.2.3. Reduced divisors. Let Γ be a metric graph, and let X be a closed subset of Γ. For any point $x \in X$, the *out-valence* of x from X that we denote by $\mathrm{outval}_X(x)$ is the number of connected components of $U \setminus \{x\}$ which lie $\Gamma \setminus X$ for any star-shaped neighbourhood U of x of radius r, for any sufficiently small $r > 0$. The points which have a strictly positive out-valence are those points which lie on the boundary of X. Equivalently, $\mathrm{outval}_X(x)$ is the number of unit tangent vectors $\vec{u}$ in $T^1_x\Gamma$ which verify the property that for any sufficiently small $\epsilon > 0$, the point $x + \epsilon\vec{u}$ lies outside X.

The definition of reduced divisors can be now extended naturally to metric graphs. Let p be a point of Γ.

A divisor D on Γ is called *p-reduced* if the following two conditions hold:

(i) for any $x \in \Gamma \setminus \{p\}$, $D(x) \geq 0$, and
(ii) for any closed subset $X \subseteq \Gamma$ with $p \notin X$, there exists a point $x \in X$ with $\mathrm{outval}_X(x) > D(x)$.

It is easy to see that the definition extends the definition in the finite graph case in the following sense. Let $G = (V, E)$ be a finite graph model of Γ, and suppose that p belongs to V. A divisor D with support in the set of vertices V of G is p-reduced in Γ if and only if it is p-reduced seen as a divisor in G.

We have the following theorem.

THEOREM 20. *For any divisor D on Γ, there exists a p-reduced divisor D_p linearly equivalent to D. Moreover, D_p is unique.*

The proof strategy is similar to the finite graph case, see e.g. [**Ami5, 135**].

The theorem (and its proof) lead to the following characterisation of divisors of non-negative rank, as in the finite graph case.

PROPOSITION 21. *A divisor D on Γ has a non-negative rank if and only if $D_p(p) \geq 0$ for the p-reduced divisor in the linear equivalence of D (and $p \in \Gamma$).*

1.2.4. The reduced divisor map and variation of reduced divisors. For a non-negative integer d, denote by Σ_d the symmetric group of order d. It acts in a natural way on the d-fold product $\Gamma^d = \Gamma \times \cdots \times \Gamma$. The quotient $\Gamma^{(d)} := \Gamma^d/\Sigma_d$ with the induced quotient topology is called the *d-fold symmetric product* of Γ. It has a natural piecewise linear structure [**Ami5**].

Let now D be a divisor of degree d and non-negative rank. For any point $p \in \Gamma$ the p-reduced divisor D_p can be written as the sum of d points of Γ, $D_p = \sum_{j=1}^{d}(x_j)$. We define the map $\Phi : \Gamma \to \Gamma^{(d)}$ by sending the point p to the point $(x_1, \dots, x_d) \in \Gamma^{(d)}$. We proved in [**Ami5**] the following theorem.

THEOREM 22 ([**Ami5**]). *Let D be a divisor of degree d and of non-negative rank. The map $\Phi : \Gamma \to \Gamma^{(d)}$ defined by reduced divisors is continuous.*

The proof actually provides an explicit description of the behaviour of reduced divisors under the variation of the base-point.

The theorem has some interesting applications which will be discussed later. But before, we will turn to the question of comparing the divisor theory on metric graphs and finite graphs. This will be then the occasion to introduce the useful concept of *specialisation*.

1.2.5. Rank-determining sets. The rank of a divisor D in a metric graph, as defined above, is the maximum integer $r \geq -1$ such that for any effective divisor E of degree r on Γ, there exists $f \in \mathrm{Rat}(\Gamma)$ with $\mathrm{div}(f) + D - E \geq 0$.

From a practical point of view, this definition requires an infinite number of verification. It is natural to wonder if there is a more effective way, in particular finite, of determining the rank of a divisor. This gives rise to the following definition.

A subset $A \subset \Gamma$ is called *rank-determining* if for any divisor D in Γ, the rank of D can be calculated only by taking in the above definition effective divisors of degree r with support in A. More precisely, if the following holds

$$r(D) = \min_{\substack{E \geq 0,\ \mathrm{supp}(E) \subseteq A \\ |D-E| = \emptyset}} \deg(E) - 1.$$

Equivalently, A is rank-determining if the following statements become equivalent in Γ for any integer $r \geq -1$:

- $r(D) \geq r$.
- For any effective divisor E of degree r with support $\mathrm{supp}(E) \subseteq A$, $|D - E| \neq \emptyset$.

REMARK 23. The definition of rank-determining sets can be extended to any set verifying the Riemann-Roch formalism, in the abstract setting of Remark 8.

The main theorem on rank-determining sets is the following.

THEOREM 24 (Luo Ye [**178**]). *Let $A \subset \Gamma$ such that the closure of any connected component of $\Gamma \setminus A$ in Γ is contractible. Then A is rank-determining. In particular, the vertex set of a graph model of Γ without loop is rank-determining.*

Note that the theorem in particular implies that to calculate the rank of a divisor in a metric graph, one needs to only check a finite number of effective divisors E.

We now briefly explain how to obtain the theorem as a direct application of the continuity of the reduced divisor map, Theorem 22, following [**Ami5**]. The proof actually gives the stronger theorem 26, which gives a characterisation of rank-determining sets in metric graphs.

Let D be a divisor on Γ, with $r(D) \geq 0$. The following can be obtained by induction.

PROPOSITION 25. *Let $A \subset \Gamma$. The two following assertions are equivalent.*

(1) A is rank-determining.

(2) For any divisor D, $r(D) \geq 1$ if $D_p(p) \geq 1$ for any $p \in A$.

Let now A be a subset of Γ as in Theorem 24, namely that the closure in Γ of any connected component of $\Gamma \setminus A$ is contractible. By the above proposition, we need to show that for a divisor D we have $r(D) \geq 1$ provided that $D_p(p) \geq 1$ for any $p \in A$. Note that $r(D) \geq 1$ if and only if $D_p(p) \geq 1$ for all $p \in \Gamma$. So it will be enough to show that if $D_p(p) \geq 1$ for all $p \in A$, then the same for all points of Γ.

Let D be a divisor and denote by X the set of all points $x \in \Gamma$ such that $D(x) \geq 1$. Since the reduced divisor map $\Phi : \Gamma \to \Gamma^{(d)}$ is continuous, X is a closed subset of Γ. We prove that if $A \subset X$, then $X = \Gamma$, which proves the theorem. Reasoning by absurd, let U be a connected component of $\Gamma \setminus X$, and denote by T its closure in Γ.

For any point x in the interior of T, we have $D_x(x) = 0$. Since D_x is x-reduced and $D_x(x) = 0$, by the definition of reduced divisors, one easily verifies there is a sufficiently small neighborhood U_x of x such that D_x is y-reduced for any $y \in U_x$. In particular, the map Φ is locally constant in the interior of U, which by continuity shows that it is globally constant on U, and is thus constant on $T = \overline{U}$. Let $D_0 = \Phi(p)$ for a point $p \in T$, so D_0 is x-reduced for any $x \in T$. In addition, $D_x(x) \geq 1$ for all $x \in \partial T$.

Let Y be a connected component of $\Gamma \setminus U$. If A verifies the conditions of the theorem, then T is a metric tree, and for any $y \in \partial Y$, $\mathrm{outval}_Y(y) = 1 \leq D_0(y)$. In particular, D_0 cannot be p-reduced for any point p in the interior of T. This finishes the proof of Theorem 24. □

A subset A of Γ is called *special* if for any open subset $U \subset \Gamma \setminus A$, there is a connected component Y of $\Gamma \setminus U$ such that all the points $y \in \partial Y$ on its boundary have out-valence $\mathrm{outval}_Y(y) = 1$. The following stronger theorem holds.

THEOREM 26 (Luo Ye [**178**]). *A subset A is rank-determining if and only if it is special.*

The non-trivial direction is to show that when A is special, then it is rank-determining. One can follow the proof of Theorem 24, and for the choice of Y in the last step, one takes the connected component Y of $\Gamma \setminus U$ which has the property $\mathrm{outval}_Y(y) = 1$ for all $y \in \partial Y$. For any point p in the interior of U, the divisor D_0 cannot be p-reduced as the closed set Y does not verify the second property in the definition of p-reduced divisors.

REMARK 27. Theorem 26 implies in particular that any metric graph Γ has a rank-determining set of size bounded by $g+1$, where g is genus of Γ. In fact for a spanning tree T of a finite graph model G of Γ, choose a point in the interior of any of the g edges in $E \setminus T$ gives a special set which is rank-determining.

1.2.6. Morphisms of Riemann-Roch structures and specialisation inequality. This is a continuation of Remarks 8 and 23. Let X_1 and X_2 be two structures satisfying the Riemann-Roch formalism. More precisely, X_1 and X_2 are endowed with sets of rational functions $\mathrm{Rat}(X_1)$ and $\mathrm{Rat}(X_2)$, respectively, and the order of vanishing functions ord_{x_1} and ord_{x_2} are defined for any point $x_1 \in X_1$ and $x_2 \in X_2$ such that the two properties in Remark 8, i.e., finiteness and logarithmic product formula, are verified for both X_1 and X_2.

A *morphism* of Riemann-Roch structures from X_1 to X_2 is the data of

- a map of sets $\tau : X_1 \to X_2$, and

- a morphism of groups $\phi : \mathrm{Rat}(X_1) \to \mathrm{Rat}(X_2)$,

such that the following condition holds:

$$\textit{for any function } f \in \mathrm{Rat}(X_1),\ \tau_*(\mathrm{div}(f)) = \mathrm{div}(\phi(f)),$$

where τ_* is the canonical extension by linearity of τ to a morphism of groups from $\mathrm{Div}(X_1)$ to $\mathrm{Div}(X_2)$.

With this preparation, we have the following proposition.

Proposition 28 (Specialisation inequality in the abstract setting). *Notations as above, assume that $\tau(X_1)$ is rank-determining in X_2. Then for any divisor D on X_1, we have*

$$r_{X_2}(\tau_*(D)) \geq r_{X_1}(D)$$

A natural example of a morphism of Riemann-Roch structures is obtained as follows. Let $G = (V, E)$ be a connected finite graph without loops. One can naturally associate a metric graph Γ to G by declaring $\ell_e = 1$ for all edges $e \in E$.

We have the inclusion map $\tau : V \to \Gamma$, which extends to an injective morphism $\tau_* : \mathrm{Div}(G) \hookrightarrow \mathrm{Div}(\Gamma)$. In addition, we have an injective morphism of groups $\phi : \mathrm{Rat}(G) \hookrightarrow \mathrm{Rat}(\Gamma)$ given by interpolation, i.e., which sends a function $f : V \to \mathbb{Z}$ to the function $\phi(f) : \Gamma \to \mathbb{R}$ given on any interval I_e, for $e = uv \in E$, by the linear interpolation of the two values of f on u and v. Since $\ell_e = 1$, one verifies that the function g belongs to $\mathrm{Rat}(\Gamma)$. It is easy to see that $\tau_*(\mathrm{div}(f)) = \mathrm{div}(\phi(f))$. Since V is rank-determining in Γ, from the specialisation inequality, we get for any divisor D with support in $V(G)$ that $r_G(D) \leq r_\Gamma(D)$. Actually, we see that the equality holds in this setting.

1.2.7. Comparison theorem between ranks in graphs and metric graphs. Let G be a finite connected loopless graph and Γ be the metric realisation of G with uniform edge lengths equal to one.

We have the following theorem.

Theorem 29 (Hladký-Kràl'-Norine [**135**]). *For any divisor $D \in \mathrm{Div}(G) \subset \mathrm{Div}(\Gamma)$, the equality $r_G(D) = r_\Gamma(D)$ holds.*

We already explained how the specialisation inequality gives $r_G(D) \leq r_\Gamma(D)$. It remains to prove the inequality $r_\Gamma(D) \leq r_G(D)$. Let $E \geq 0$ a divisor in $\mathrm{Div}(G)$ of degree $r_G(D) + 1$ such that $r_G(D - E) = -1$. Let $D' := D - E$, and fix $p \in V$. The p-reduced divisor $D'_p \sim D'$ in G is at the same time linearly equivalent to D' in Γ and p-reduced in Γ. Observe now that $D'_p(p) < 0$ since $r_G(D') = -1$, which implies $r_\Gamma(D') = -1$. In particular, $|D - E| = \emptyset$ in Γ, and so $r_\Gamma(D) \leq r_G(D)$.

1.2.8. Abel-Jacobi Theorem. In this section, we give the formulation of Abel-Jacobi theorem in metric graphs.

1.2.8.1. *Integration and Jacobian of metric graphs.* Let $G = (V, E)$ be a loopless connected graph. Recall from the previous section that we denote by $\mathbb{E}$ the set of all possible orientations of the edges of G.

Let A be a ring with unite. The space $H^1(G, A)$ can be identified with the space of A-*flows* on the graph. These are the space of applications $\omega : \mathbb{E} \to A$ which verify the following two properties:

- For any edge $e = uv \in E$, $\omega(\vec{uv}) = -\omega(\vec{vu})$.
- For any veretx $v \in V$, $\sum_{uv \in E} \omega(\vec{vu}) = 0$.

We denote the space of A-flows by $\Omega_A(G)$. The space $\Omega_A(G)$ is isomorphic to A^g.

We extend the definition to any metric graph Γ by defining an A-flow in Γ as a flow on any loopless graph model of Γ. The space of A-flows in Γ is denoted by $\Omega_A(\Gamma)$. Note that any $\omega \in \Omega_A(\Gamma)$ gives a map $\omega : T^1\Gamma \to A$, so that elements of $\Omega_A(\Gamma)$ can be regarded as one-forms on Γ (where the notation).

Let the pair $(G = (V, E), \ell)$ be a loopless finite graph model of a metric graph Γ. Let $\omega \in \Omega_{\mathbb{R}}(\Gamma)$ be an $\mathbb{R}$-flow. For any oriented edge $\vec{uv}$ of G, define the integration of ω along $\vec{uv}$ as

$$\int_{\vec{uv}} \omega := \omega(\vec{uv})\ell_{uv},$$

where ℓ_{uv} is the length of the edge uv. Obviously, we have $\int_{\vec{uv}} \omega = -\int_{\vec{vu}} \omega$. The integration extends naturally to define integration of flows on locally isometric paths $\gamma : [0, T] \to \Gamma$: any such path γ can be subdivided by a sequence $t_0 = 0 < \cdots < t_N = T$ such that $\gamma_{[t_i, t_{i+1}]}$ is an isometry. There exists a model $(G = (V, E), \ell)$ of Γ such that all the points $\gamma(t_i)$ belong to the vertex set of the model. We define

$$\int_{\gamma} \omega := \sum_{i=0}^{N-1} \omega(\overrightarrow{\gamma(t_i)\gamma(t_{i+1})})(t_{i+1} - t_i).$$

It is straightforward to check that this definition does not depend on the choice of the model and the points t_i.

In particular any oriented cycle $\vec{\gamma}$ defines a linear form $\Omega_{\mathbb{R}}(\Gamma) \to \mathbb{R}$, which sends ω to $\int_{\vec{\gamma}} \omega$. This defines an inclusion map

$$\Omega_{\mathbb{Z}}(\Gamma) \hookrightarrow \Omega_{\mathbb{R}}(\Gamma)^*.$$

Define H as the image of this map. It is a lattice of full rank g in $\Omega_{\mathbb{R}}(\Gamma)^*$

The *Jacobian* of Γ is the g-dimensional real torus

$$\mathrm{Jac}(\Gamma) := \Omega_{\mathbb{R}}^*/H.$$

EXAMPLE 30. Let $\Gamma = C_\ell$ the cycle of length ℓ. Fixing an orientation $\vec{\gamma}$ for the cycle, gives a canonical identification $\Omega_{\mathbb{R}}(\Gamma) \simeq \mathbb{R}$. Let $\omega_1 \in \Omega_{\mathbb{R}}(\Gamma)$ be the unite flow on Γ, corresponding to $1 \in \mathbb{R}$. The linear form $\int_{\vec{\gamma}}$ sends ω_1 to ℓ. Thus, the subspace H is simply $\ell\mathbb{Z} \subset \mathbb{R}$. It follows that the Jacobian of Γ is given by $\mathrm{Jac}(C_\ell) = \mathbb{R}/\ell\mathbb{Z} \simeq C_\ell$.

1.2.8.2. *The map* $\mathrm{AJ} : \mathrm{Pic}(\Gamma) \to J(\Gamma)$. For any locally isometric path $\gamma : [0, T] \to \Gamma$ in Γ, the *boundary* of γ, denoted by $\partial\gamma$, by

$$\partial\gamma = (\gamma(T)) - (\gamma(0)) \in \mathrm{Div}(\Gamma).$$

Let D be a divisor of degree zero on Γ. We can write $D = \sum_{i=1}^{N} (q_i) - (p_i)$, for integer N and points $p_i, q_i \in \Gamma$. For each i, we choose a locally isometric path γ_i from p_i to q_i, to have a collection of locally isometric paths $\gamma = \{\gamma_i\}$ such that $\partial\gamma = D$.

For any other collection γ' of locally isometric paths with $\partial\gamma' = D$, we have $\partial\gamma + \partial\overline{\gamma}' = 0$, and so we have

$$\int_{\gamma} - \int_{\gamma'} \in H.$$

It follows that there exists a well-defined map

$$\mathrm{AJ} : \mathrm{Div}^0(\Gamma) \longrightarrow \mathrm{Jac}(\Gamma) = \Omega_{\mathbb{R}}(\Gamma)^*/H$$

which send a divisor $D \in \mathrm{Div}^0(\Gamma)$ to $\int_\gamma$ for any collection of paths as above with $\partial\gamma = D$.

Recall that we define $\mathrm{Pic}(\Gamma) := \mathrm{Div}^0(\Gamma)/\mathrm{Prin}(\Gamma)$, where $\mathrm{Prin}(\Gamma)$ consists of all the divisors of the form $\mathrm{div}(f)$ for some $f \in \mathrm{Rat}(\Gamma)$.

We have the following Abel-Jacobi theorem for metric graphs.

THEOREM 31 (Mikhalkin-Zharkov [**188**]). *The map Ψ factorises through $\mathrm{Pic}(\Gamma)$ and induces an isomorphism $\mathrm{AJ} : \mathrm{Pic}(\Gamma) \to J(\Gamma)$.*

1.2.8.3. *Abel-Jacobi map $\mathrm{AJ}_p : \Gamma \to J(\Gamma)$.* Let Γ be a metric graph, and p a fix point of Γ. One defines the Abel-Jacobi map $\mathrm{AJ} : \Gamma \to J(\Gamma)$ by

$$\mathrm{AJ}_p(x) := \mathrm{AJ}((x) - (p_0)).$$

Let $(G = (V, E), \ell)$ be a loopless graph model of Γ. A bridge $e \in E$ is an edge such that $G \setminus \{e\}$ is disconnected. Contracting a bridge $e = uv$ in G leads to a graph G/e defined by identifying the two vertices u and v, removing the edge e and keeping all the other edges of G in G/e. (Thus G/e has one less vertex and one less edge compared to the number of vertices and edges in G.) Denote by $e_1, \dots, e_k$ all the bridges in G. Contracting $e_1, \dots, e_k$ consecutively, leads to a graph $\widetilde{G}$ which does not contain any bridge anymore, and which has edge set $\widetilde{E} = E \setminus \{e_1, \dots, e_k\}$. We denote by $\widetilde{\Gamma}$ the metric realisation of $(\widetilde{G}, \ell_{|\widetilde{E}})$.

PROPOSITION 32. *The Abel-Jacobi map AJ_p retracts all the segments in Γ corresponding to the bridges $e_1, \dots, e_k$, and gives an embedding $\widetilde{\Gamma} \hookrightarrow J(\widetilde{\Gamma})$.*

1.2.9. Torelli theorem. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs, and consider the metric graphs Γ_1 and Γ_2, metric realisations of G_1 and G_2 with uniform edge lengths equal to one, respectively.

We say that G_1 and G_2 are *Whitney equivalent* if there exists a bijection $\phi : E_1 \to E_2$ such that a set of edge X in E_1 contains a cycle in G_1 if any only if $\phi(X)$ contains a cycle in G_2. This is equivalent to saying that the graphical matroid associate to G_1 is isomorphic to the graphical matroid associated to G_2.

Consider the Jacobian tori $J(\Gamma_1)$ and $J(\Gamma_2)$ endowed with the flat metric induced by the pairing $\langle , \rangle_i$ on $H^1(\Gamma_i, \mathbb{R})$, described above, for $i = 1, 2$. We have the following Torelli theorem for graphs.

THEOREM 33 (Gerrizen [**116**], Artamkin [**15**], Caporaso-Viviani [**64**]). *Notations as above, we have $J(\Gamma_1) \simeq J(\Gamma_2)$ if and only if G_1 and G_2 are Whitney equivalent.*

The theorem can be extended to any metric graphs. Consider two metric graphs Γ_1 and Γ_2. Let $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ be the metric graphs obtained from Γ_1 and Γ_2, respectively, by contracting the bridge edges of any simple graph models. Note that we have $J(\Gamma_i) \simeq J(\widetilde{\Gamma}_i)$, for $i = 1, 2$. If Γ_1 and Γ_2 have genus one, then $J(\Gamma_1) \simeq J(\Gamma_2)$ if and only if $\widetilde{\Gamma}_1 \simeq \widetilde{\Gamma}_2$. Otherwise, if they both have genus larger than two, then let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the minimal models of $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ with the corresponding length functions ℓ_1 and ℓ_2, respectively. Then we have $J(\Gamma_1) \simeq J(\Gamma_2)$ if any only if there exists a bijection $\phi : E_1 \to E_2$ under which G_1 and G_2 become Whitney equivalent, and such that in addition, for any edge $e \in E_1$, we have $\ell_1(e) = \ell_2(\phi(e))$ [**116, 64**].

Whitney gives in [**243, 242**] a complete characterisation of the Whitney equivalence.

THEOREM 34. *The Whitney equivalence is the equivalence relation between graphs generated by the following two types of elementary Whitney equivalence relations. Let* H_1 *and* H_2 *be two disjoint graphs, and consider vertices* $v_{1,1}, v_{1,2}$ *in* H_1*, and vertices* $v_{2,1}, v_{2,2}$ *in* H_2*. Then*

- (Change of the cut-vertex) $H_1 \vee_{v_{1,1}=v_{2,1}} H_2$ *and* $H_1 \vee_{v_{1,2}=v_{2,2}} H_2$ *are elementary equivalent, and*
- (Twisting) *the two graphs* $H_1 \sqcup H_2/\{v_{1,1} = v_{2,1}\ ,\ v_{1,2} = v_{2,2}\}$ *and* $H_1 \sqcup H_2/\{v_{1,2} = v_{2,1}\ ,\ v_{1,1} = v_{2,2}\}$ *are elementary equivalent. (I.e., we glue the two graphs by identifying the two pair of vertices in two different ways).*

1.2.10. Combinatorics of the flow Voronoi cells. Let $\mathcal{H}$ be a finite dimensional real vector space endowed with a scalar product, and denote by q the corresponding quadratic form. Let S be a discrete subset of $\mathcal{H}$. For any point λ in S, the *Voronoi cell* Vor_λ of λ with respect to q is defined as

$$\mathrm{Vor}_\lambda \;=\; \Big\{ x \in \mathcal{H} \;\mid\; q(x-\lambda) \le q(x-\mu) \text{ for any other point } \mu \in S \Big\}.$$

The *Voronoi diagram* of S is the decomposition of $\mathcal{H}$ induced by the cells Vor_λ, for $\lambda \in \mathcal{S}$. It is straightforward to see that each cell of the Voronoi diagram is a polytope or a polyhedron (in the case there are cells of infinite volume).

Let now Γ be a metric graph with a finite graph model G and a length function ℓ. Consider the discrete subset H of $\Omega_{\mathbb{R}}(\Gamma)^*$, as defined in the previous section, and let q be the quadratic form induced by $\langle\, ,\rangle$ on $\Omega_{\mathbb{R}}(\Gamma)^*$. By translation-invariance of q, the cells of the Voronoi diagram are all translations of each other. It follows that the Voronoi cell decomposition of $\Omega_{\mathbb{R}}(\Gamma)^*$ is completely understood by the Voronoi cell Vor_0 of the origin, which is a polytope. We call Vor_0 the *flow Voronoi polytope* of Γ.

Consider the set of all faces of the polytope Vor_0 ordered by inclusion. They form a finite poset that we call the *flow face poset* of Γ and denote by $\mathcal{FP}$. Affirming a conjecture of Caporaso and Viviani [**64**], and answring a question raised by Bacher-de la Harpe-Nagnibeda [**17**], we give in [**Ami6**] a complete description of this poset in terms of the combinatorics of the graph G, that we now explain.

An orientation D of a connected graph G is called *strongly connected* if any pair of vertices u and v in D are connected both by an oriented path from u to v and by an oriented path from v to u. An orientation of a (non necessarily connected) graph G is called strongly connected if the orientation induced on each of the connected components is strongly connected.

Let G be a given graph (with possibly multiple edges and loops). Define the following poset $\mathcal{SC}$, that we call the poset of strongly connected orientations of subgraphs of G: the elements of $\mathcal{SC}(G)$ are all the pairs (H, D_H) where H is a subgraph of G and D_H is a strongly connected orientation of the edges of H. (Vertices and the edges of G are labeled, so parallel edges are distinguished in dealing with subgraphs.) A partial order $\preceq$ is defined on $\mathcal{SC}$ as follows: given two elements (H, D_H) and $(H', D_{H'})$ in $\mathcal{SC}$, we have $(H, D_H) \preceq (H', D_{H'})$ if and only if $H' \subseteq H$ and the orientation $D_{H'}$ is the orientation induced by D_H on H'. It is quite straightforward to see that $(\mathcal{SC}, \preceq)$ is a graded poset, and the grading is given by g minus the genus of the underlying subgraph. Note that $(\emptyset, \emptyset)$ is the maximum element of $(\mathcal{SC}, \preceq)$.

We have the following theorem.

Theorem 35 ([**Ami6**]). *The two posets $\mathcal{FP}$ and $\mathcal{SC}$ are isomorphic.*

In [**AmiEst1**], we formulate and prove an analogue description of the *cut Voronoi cell*, which is the Voronoi cell of the cut lattice [**17**]. The results are then used to study limits of line bundles and linear series on degenerating families of curves, and to provide compactifications of those spaces, generalising results of [**104**] to more general semistable curves.

1.3. Extension to sublattices of the root lattice A_n

In [**AmiMan**], we discuss the existence of Riemann-Roch formalism for other sublattices of the root lattice A_n. The lattice $A_n \subset \mathbb{R}^{n+1}$ is the set of all points of $\mathbb{Z}^{n+1}$ whose coordinates sum up to zero. Let now L be a sub-lattice of A_n of full-rank. By considering the quotient $\mathbb{Z}^{n+1}/L$, it defines a *a linear equivalence* relation on the set of points of $\mathbb{Z}^{n+1}$ $D \sim D'$ if and only if $D - D' \in L$. One can then talk about the positivity of points and define rank of an integral point in the same way the rank of divisors are defined in graphs. In the case, the lattice L is the image of $\mathbb{Z}^{n+1}$ by a graph Laplacian, this coincides with the the theory of divisors and their rank exposed in the first section. The natural question is then: *which lattices satisfy a Riemann–Roch theorem?*

In [**AmiMan**] we study this question. We show that Riemann-Roch theory associated to a full rank sub-lattice L of A_n is related to the study of the Voronoi diagram of the lattice L in the hyperplane H_0 generated byA_n under a certain *simplicial distance function* (these are (non-symmetric) distance functions whose balls are simplices). The whole theory is then captured by the corresponding critical points of this simplicial distance function.

To such a sublattice, we associate two geometric invariants, the *min-* and the *max-genus*, denoted respectively by $g_{\min}$ and $g_{\max}$. The invariant $g_{\max}$ is a generalisation of the Frobenius number and appears also in Lorenzini's work [**174**] in defining zeta functions for lattices. We call a lattice *uniform* if both its genera are equal.

We define a characteristic property for a given sub-lattice of A_n that we call *reflection invariance*. It is a sort of symmetry for the critical points of the simplicial distance function. Our main theorem is then

Theorem 36 (Riemann-Roch [**AmiMan**]). *Let L be a uniform reflection invariant sub-lattice of A_n. Then there exists a point $K \in \mathbb{Z}^{n+1}$, called* canonical*, such that for every point $D \in \mathbb{Z}^{n+1}$, we have*

$$r(D) - r(K - D) = \deg(D) - g + 1,$$

where $g = g_{min} = g_{max}$.

It is possible to prove directly that Laplacian lattices of undirected connected graphs are uniform and reflection invariant, obtaining in this way an alternative geometric proof of Riemann-Roch theorem for graphs, see [**AmiMan**] for more details. In this way, one can give a purely combinatorial justification to the canonical divisor of a graph.

CHAPTER 2

Metrised complexes and limit linear series

This chapter presents an overview of the results obtained in our papers [**AmiBa, AmiCa, Ami8**]. We use the occasion to recall results underlying the structure of Berkovich analytic curves, which is the elegant and ideally situated framework for our purpose. The section on Berkovich curves will be also important for us in the next chapters.

Limit linear series were introduced by Eisenbud and Harris in the eighties [**98**], and allowed them to obtain a number of interesting results concerning the geometry of curves and their moduli spaces, e.g. [**99, 100, 101**]. Their theory has undergone numerous developments and improvements in the last thirty years. However, the theory of limit linear series only applies so far, for the most part, to a rather restricted class of reducible curves, namely those of *compact type*, i.e., nodal curves whose dual graph is a tree[1]

Baker studied in [**20**] the well-known specialisation framework for degenerating divisors on curves from the perspective of divisor theory on graphs, which degenerates complete linear series on a regular semistable family of curves to a complete linear series on the dual graph of the special fibre. This happens to be more or less orthogonal to the Eisenbud-Harris theory, in that it works best for special fibres which are maximally degenerate, meaning that the dual graph has arithmetic genus (Betti number) equal to the genus of the generic fibre. In particular, specialising to the dual graph provides no information whatsoever when the special fibre is of compact type. Intriguingly, both the Eisenbud-Harris theory and Baker's work [**20**] lead to simple proofs of the celebrated Brill-Noether theorem of Griffiths-Harris [**98, 80**].

In [**AmiCa**], we partially remedy this loss of information by looking at vertex-weighted graphs, where the weight attached to a vertex is the genus of the corresponding irreducible component of the special fibre.

In our paper [**AmiBa**], we introduce a theoretical framework suitable for generalising both the Eisenbud-Harris theory of limit linear series and the work of Baker on specialisation of divisors from curves to graphs by introducing *metrised complex of algebraic curves*; metrised complexes have a mixture of both combinatorial and algebraic points; the combinatorial points are the points of a metric graph and the algebraic points are points in curves (which can even live in different characteristics). To a curve X defined over a non-Archimedean field K, together with a semistable model $\mathfrak{X}$ for X over the valuation ring R of K, we can naturally associate a corresponding metrised complex $\mathfrak{C}\mathfrak{X}$ of curves over the residue field κ of K. We show that metrised complexes have an algebraic geometry analogous to that of usual algebraic curves in the sense that they are examples of Riemann-Roch structures, as we defined in the previous section. Furthermore, they come both with a forgetful map to the underlying

[1]Note however very recent developments in the work of Osserman where a generalisation to curves of pseudo-compact type curves is proposed [**201, 202**], and in our series of papers [**AmiEst1, AmiEst2, AmiEst3**] where a different approach based on toric tilings is proposed.

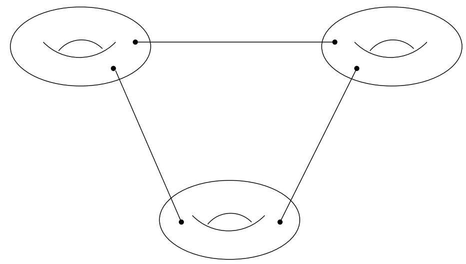

FIGURE 1. The geometric realization of a metrised complex of genus four.

combinatorial data and a specialisation map from the *generic fibre*, in the case they are associated to semistable models, which both provide morphisms of Riemann-Roch structures. This allows to apply the formalism of the previous chapter and to get a specialisation theorem for rank of divisors, from which we recover both the original specialisation lemma of Baker [**20**] and the extension obtained in [**AmiCa**].

As an application, we establish a concrete link between specialisation of divisors from curves to metrised complexes and the Eisenbud-Harris theory of limit linear series. This allows us to formulate a generalisation of the notion of limit linear series to curves which are not necessarily of compact type and prove, among other things, that any degeneration of a $\mathfrak{g}^r_d$ in a regular family of semistable curves is a limit $\mathfrak{g}^r_d$ on the special fibre. Furthermore, the consideration of this work allows to propose a purely combinatorial definition of limit linear series, which have applications we will discuss in the next chapter.

Finally, we note that metrised complexes over a field κ could be alternatively viewed as a sort of (extended) *tropicalisation of logarithmic curves*, in logarithmic geometry. Since we do not use any logarithmic geometry in what follows, we simply refer the interested reader to the two following recent papers [**58, 109**].

2.1. Definition and Riemann-Roch theorem

A *metrised complex* $\mathfrak{C}$ *of curves* consists of the following data:

- A connected finite graph G with vertex set V and edge set E.
- A metric graph Γ having G as a model, with a length function $\ell : E \to \mathbb{R}_{>0}$.
- For each vertex v of G, a complete, nonsingular, irreducible curve C_v over an algebraically closed field κ_v.
- For each vertex v of G, a bijection $e \mapsto x^e_v$ between the edges of G incident to v (with loop edges counted twice) and a subset $\mathcal{A}_v = \{x^e_v\}_{e \ni v}$ of $C_v(\kappa_v)$.

The *geometric realisation* $|\mathfrak{C}|$ of $\mathfrak{C}$ is defined to be the union of the intervals I_e of length ℓ_e associated to the edges e of G, as in the previous chapter, and the collection of curves C_v, with each endpoint v of an edge e (the interval I_e) identified with the corresponding marked point x^e_v. See Figure 1. When we think of $|\mathfrak{C}|$ as a set, we identify it with the disjoint union of $\Gamma \backslash V$ and $\bigcup_{v \in V} C_v(\kappa_v)$. Thus, when we write $x \in |\mathfrak{C}|$, we mean that x is either a non-vertex point of Γ (a *graphical point* of $\mathfrak{C}$) or a point of $C_v(\kappa_v)$ for some $v \in V$ (a *geometric point* of $\mathfrak{C}$).

The *genus* of a metrised complex of curves $\mathfrak{C}$, denoted $g(\mathfrak{C})$, is by definition $g(\mathfrak{C}) = g(\Gamma) + \sum_{v \in V} g_v$, where g_v is the genus of C_v and $g(\Gamma)$ is the genus of Γ.

Given a metrised complex $\mathfrak{C}$ of κ-curves, with all the κ_v equal to an algebraically closed field κ, there is an associated semistable curve X_0 over κ obtained by gluing the curves C_v along the points x_v^e (one intersection for each edge e of G) and forgetting the metric structure on Γ. Conversely, given a semistable curve X_0 over κ together with a positive real number for each node (which can be called the *thickness* of the node), one obtains an associated metrised complex of κ-curves by letting G be the dual graph of X_0, Γ the metric graph associated to G and the length function given by the thickness, C_v the normalisation of the irreducible component X_v of X_0 corresponding to v, and $\mathcal{A}_v$ the preimage in C_v of the set of nodes of X_0 belonging to X_v.

A *divisor* on a metrised complex of curves $\mathfrak{C}$ is an element $\mathcal{D}$ of the free abelian group on $|\mathfrak{C}|$. Thus a divisor on $\mathfrak{C}$ can be written uniquely as $\mathcal{D} = \sum_{x\in|\mathfrak{C}|} a_x(x)$ where $a_x \in \mathbb{Z}$, all but finitely many of the a_x are zero, and the sum is over all points of $\Gamma\backslash V$ as well as $C_v(\kappa_v)$ for $v \in V$. The *degree* of $\mathcal{D}$ is defined to be $\sum a_x$.

To a divisor on $\mathfrak{C}$, we can naturally associate a divisor D_Γ on Γ, called the Γ-*part* of $\mathcal{D}$, as well as, for each $v \in V$, a divisor D_v on C_v (called the C_v-*part* of $\mathcal{D}$). The divisor D_v is simply the restriction of $\mathcal{D}$ to C_v, i.e. $D_v = \sum_{x\in C_v(\kappa)} \mathcal{D}(x)(x)$, and D_Γ is defined as

$$D_\Gamma = \sum_{x\in\Gamma\backslash V} \mathcal{D}(x)(x) + \sum_{v\in V} \deg(D_v)(v),$$

where $\mathcal{D}(x)$ denotes the coefficient of x in $\mathcal{D}$.

In particular, the degree of $\mathcal{D}$ equals the degree of D_Γ. One could equivalently define a divisor on $\mathfrak{C}$ to be an element of the form $\mathcal{D} = D_\Gamma \oplus \sum_v D_v$ of $\mathrm{Div}(\Gamma) \oplus (\oplus_v \mathrm{Div}(C_v))$ such that $\deg(D_v) = D_\Gamma(v)$ for all v in V.

A *nonzero rational function* $\mathfrak{f}$ on a metrised complex of curves $\mathfrak{C}$ is the data of a rational function f_Γ on Γ and nonzero rational functions f_v on C_v for each $v \in V$. (We do *not* impose any compatibility conditions on the rational functions f_Γ and f_v.) We call f_Γ the Γ-*part* of $\mathfrak{f}$ and f_v the C_v-*part* of $\mathfrak{f}$.

We define *order of vanishing* functions $\mathrm{ord}_x(.)$ at points of $\mathfrak{C}$ as follows. For a rational function $\mathfrak{f}$ on $\mathfrak{C}$ and a point $x \in \mathfrak{C}$, $\mathrm{ord}_x(\mathfrak{f})$ is defined as follows:

- If $x \in \Gamma\backslash V$, then $\mathrm{ord}_x(\mathfrak{f}) = \mathrm{ord}_x(f_\Gamma)$, as defined in the previous chapter (namely, this is the sum of the slopes of f_Γ in all unit tangent directions emanating from x).
- If $x \in C_v(\kappa_v)\backslash\mathcal{A}_v$, then $\mathrm{ord}_x(\mathfrak{f}) = \mathrm{ord}_x(f_v)$.
- If $x = x_v^e \in \mathcal{A}_v$, then $\mathrm{ord}_x(\mathfrak{f})$ is the sum of $\mathrm{ord}_x(f_v)$ and the slope f_Γ in the unite (outgoing) tangent vector at v in the direction of the edge e.

With these definitions, one easily verifies for a metrised complex $\mathfrak{C}$ and $\mathrm{Rat}(\mathfrak{C})$ the two properties of finiteness and logarithmic product formula of Remark 8 in the previous chapter. It follows that we have Riemann-Roch formalism in this setting, and so we can define linear equivalence and rank of divisors on any metrised curve complex.

We already defined the genus of $\mathfrak{C}$. A *canonical divisor* on $\mathfrak{C}$, denoted $\mathcal{K}$, is defined to be any divisor linearly equivalent to $\sum_{v\in V}(K_v+A_v)$, where K_v is a canonical divisor on C_v and A_v is the sum of the $\deg_G(v)$ points in $\mathcal{A}_v$. The Γ-part of $\mathcal{K}$ is $K^\# = \sum_v (\deg_G(v) + 2g_v - 2)(v)$, and the C_v-part of $\mathcal{K}$ is $K_v + A_v$.

With these definitions, the following theorem states that $\mathfrak{C}$ satisfies Riemann-Roch.

THEOREM 37 (Riemann-Roch for metrised curve complexes [**AmiBa**]). *Let $\mathfrak{C}$ be a metrised complex of algebraic curves, and $\mathcal{K}$ a divisor in the canonical class of $\mathfrak{C}$. For any divisor $\mathcal{D} \in \mathrm{Div}(\mathfrak{C})$, we have*

$$r_{\mathfrak{C}}(\mathcal{D}) - r_{\mathfrak{C}}(\mathcal{K} - \mathcal{D}) = \deg(\mathcal{D}) - g(\mathfrak{C}) + 1.$$

We remark that the above theorem generalises both the classical theory for algebraic curves and the corresponding theory for metric graphs, exposed in the previous chapter. The former corresponds to the case where G consists of a single vertex v and no edges and $C = C_v$ is an arbitrary smooth curve. The latter corresponds to the case where the curves C_v have genus zero for all $v \in V$. Since any two points on a curve of genus zero are linearly equivalent, it is easy to see that the divisor theories and rank functions on $\mathfrak{C}$ and Γ are essentially equivalent. In the presence of higher genus curves among the C_v, the divisor theories on Γ and $\mathfrak{C}$ can be very different. In addition, different choices of $\mathcal{A}_v$ can drastically change both the linear equivalence relation and the rank function.

2.2. Background on Berkovich curves

In this section we provide a brief overview of the parts of Berkovich's theory of non-Archimedean analytic spaces which are important for our purposes. More details can be found in [**27, 94, 232**].

2.2.1. Non-Archimedean analytic spaces. Let K be a complete and algebraically closed non-trivially valued non-Archimedean field with valuation ring R and residue field κ. Let X/K be an algebraic variety, i.e., a reduced, separated scheme of finite type over K. The (Berkovich) *analytification* X^{an} of X, as a topological space, can be defined as follows. Points of X^{an} can be identified with pairs $x = (\xi, |\,|_\xi)$ consisting of a (scheme-theoretic) point $\xi \in X$ and an extension $|\,|_\xi$ of the absolute value on K to the residue field $K(\xi)$ of ξ. The topology on X^{an} is the weakest one for which $U^{\mathrm{an}} \subset X^{\mathrm{an}}$ is open for every open affine subset $U \subset X$ and the function $x \mapsto |f(x)|$ is continuous for every $f \in \mathcal{O}_X(U)$. (Here, by a standard abuse of notation, $f(x)$ denotes the image of f in the residue field of ξ, where $x = (\xi, |\,|_\xi)$, and $|f(x)|$ is shorthand for $|f(x)|_\xi$.) There is a canonical inclusion from the set $X(K)$ of closed points of X into X^{an} with dense image. The induced topology on $X(K)$ is the canonical analytic topology with respect to which $X(K)$ is totally disconnected and not locally compact. By contrast, the larger space X^{an} is locally compact, Hausdorff, and locally path-connected. Furthermore, X^{an} is compact iff X is proper and path-connected iff X is connected. There is a category of (Berkovich) K-analytic spaces which includes as a special case the analytifications of algebraic varieties over K. The association $X \mapsto X^{\mathrm{an}}$ is a functor from algebraic varieties over K to K-analytic spaces. Any open subset X^{an} can be given a K-analytic structure in a natural way.

2.2.2. Open discs and open annuli. Points x in the Berkovich affine line $(\mathbf{A}^1)^{\mathrm{an}}$ over K can be identified with multiplicative seminorms $|\,|_x$ on the one-variable polynomial ring $K[T]$ extending the given absolute value on K. An *open ball* in $(\mathbf{A}^1)^{\mathrm{an}}$ is an open subset of the form $\{x \in (\mathbf{A}^1)^{\mathrm{an}} \mid |T|_x < R\}$ for some $R > 0$ (with the induced analytic structure). Similarly, a *punctured open ball* is an open subset of the form $\{x \in (\mathbf{A}^1)^{\mathrm{an}} \mid 0 < |T|_x < R\}$ and an *open annulus* is an open subset of the form $\{x \in (\mathbf{A}^1)^{\mathrm{an}} \mid r < |T|_x < R\}$ for some $0 < r < R$. A *generalised open annulus* is either a punctured open ball (considered as an open annulus with $r = 0$) or an open annulus. Each generalised open annulus $\mathbf{A}$ contains a canonical *skeleton* $\Sigma = \Sigma(\mathbf{A})$, which is the set of all multiplicative norms $|\,|_x$ on $K[T]$ of the form $|\sum a_i T^i|_x = \max\{|a_i|\rho^i\}$ with $r < \rho < R$. As topological spaces, Σ is homeomorphic to an open interval and there is a canonical retraction map from $\mathbf{A}$ onto Σ. There is also a *canonical metric* on Σ: if $\mathbf{A}$ is an open annulus, we identify Σ with the open interval $(\log r, \log R)$, and if $\mathbf{A}$ is an punctured open ball then we identify Σ with the open interval $(-\infty, \log R)$.

For the rest of this section, let X/K be a smooth, proper, connected algebraic curve.

2.2.3. Types of points on curves. Points of X^{an} are traditionally classified into four types, according to algebraic properties of their completed residue fields. The *completed residue field* $\mathcal{H}(x)$ of $x = (\xi, |\,|_\xi) \in X^{\mathrm{an}}$ is the completion of $K(\xi)$ with respect to $|\,|_\xi$. We define $s(x)$ to be the transcendence degree of the residue field of $\mathcal{H}(x)$ over the residue field of K, and $t(x)$ to be the rank of the finitely generated abelian group $|\mathcal{H}(x)^\times|/|K^\times|$. By *Abhyankar's inequality*, $s(x) + t(x) \le 1$. We say that x is of *type 1* if $\mathcal{H}(x) \cong K$, of *type 2* if $s(x) = 1$, of *type 3* if $t(x) = 1$, and of *type 4* if $s(x) = t(x) = 0$ but $\mathcal{H}(x) \ncong K$. The points of type 1 in X^{an} are precisely the points of $X(K)$.

2.2.4. Semistable vertex sets and skeleta. A *semistable vertex set* for X^{an} is a finite set V of type 2 points of X^{an} such that the complement of V in X^{an} is isomorphic (as a K-analytic space) to the disjoint union of a finite number of open annuli and an infinite number of open balls. (Such a disjoint union is called the *semistable decomposition* of X^{an} associated to V.) It follows from the *semistable reduction theorem* that semistable vertex sets always exist, and more generally that any finite set of type 2 points of X^{an} is contained in a semistable vertex set. The *skeleton* $\Gamma = \Sigma(X^{\mathrm{an}}, V)$ of X^{an} with respect to a semistable vertex set V is the union (inside X^{an}) of V and the skeletons of each of the open annuli in the semistable decomposition $\Delta(V)$ associated to V. Using the canonical metric on the skeletons of these open annuli, Γ can be naturally viewed as a (finite) *metric graph* contained in X^{an}.

One can define in a similar way semistable vertex sets and skeleta for an affine curve X'. In this case, one must also allow a finite number of punctured open balls in the semistable decomposition and the skeleton is a topologically finite but not necessarily finite length metric graph – it will contain a finite number of infinite rays corresponding to the points of $X \setminus X'$, where X is the projective completion of X'.

The skeleton Γ of a semistable vertex set V comes equipped with a natural model G whose vertices are the points of V and whose edges correspond bijectively to the open annuli in the semistable decomposition associated to V. A semistable vertex set V is called *strongly semistable* if the graph G has no loop edges. Every semistable vertex set is contained in a strongly semistable vertex set.

There is a canonical retraction map $\tau = \tau_V : X^{\mathrm{an}} \twoheadrightarrow \Gamma$ which sends a point $x \in X^{\mathrm{an}}$ to itself (if $x \in V$), to the retraction of x to $\Sigma(\mathbf{A})$ (if x belongs to an open annulus $\mathbf{A}$ in $\Delta(V)$), or to the unique point of $\bar{\mathbf{B}}$ belonging to $\Sigma(X^{\mathrm{an}}, V)$ (if x belongs to an open ball $\mathbf{B}$ in $\Delta(V)$). By a theorem of Berkovich, the map τ is in fact a strong deformation retraction. In particular, X^{an} and Γ have the same homotopy type (and the homotopy type of Γ is independent of the choice of V).

If V is a semistable vertex set for an affine curve X', there is a similarly defined retraction map $\tau : (X')^{\mathrm{an}} \twoheadrightarrow \Sigma((X')^{\mathrm{an}}, V)$.

2.2.5. The genus formula. For $x \in X^{\mathrm{an}}$ of type 2, recall that the residue field $\widetilde{\mathcal{H}(x)}$ of $\mathcal{H}(x)$ (which could be called the *double residue field" of* x) has transcendence degree one over the residue field κ of K. (Recall that K is algebraically closed by hypothesis, and therefore κ is algebraically closed as well.) Let C_x be the unique smooth projective curve over κ with function field $\widetilde{\mathcal{H}(x)}$, and let $g_x = \omega(x)$ be the genus of C_x. The metric graph Γ enhanced with this genus function is referred to as an *augmented metric graph.*

If V is any semistable vertex set for X^{an}, then $g_x = 0$ for all $x \notin V$ and the *genus formula* asserts that

$$g(X) = \sum_{x \in V} g_x + g(\Sigma(X^{\mathrm{an}}, V)).$$

2.2.6. Tangent vectors. There is a canonical metric on $\mathbf{H}(X^{\mathrm{an}}) := X^{\mathrm{an}} \setminus X(K)$ which restricts to the metric on $\Gamma = \Sigma(X^{\mathrm{an}}, V)$ for any semistable vertex set V for X^{an}. Note, however, that the metric topology on $\mathbf{H}(X^{\mathrm{an}})$ is much finer than the subspace topology inherited from X^{an}. A *geodesic segment* starting at $x \in \mathbf{H}(X^{\mathrm{an}})$ is an isometric embedding $\alpha : [0, a] \to \mathbf{H}(X^{\mathrm{an}})$ for some $a > 0$ such that $\alpha(0) = x$. We say that two geodesic segments starting at x are *equivalent* if they agree on a neighborhood of 0. A *tangent direction* at a point $x \in \mathbf{H}(X^{\mathrm{an}})$ is an equivalence class of geodesic segments starting at x, which we represent by a *unit tangent direction* with respect to the metric to the canonical metric. We denote by $T_x = T_x(X^{\mathrm{an}})$ the set of all tangent directions at x (and identify it with the unit tangent directions T_x^1).

Every point $x \in X^{\mathrm{an}}$ has a neighborhood base consisting of simply connected open sets. For any simply connected neighborhood U of $x \in X^{\mathrm{an}}$, there is a natural bijection between T_x and the connected components of $U \setminus \{x\}$. Since $U \setminus \{x\}$ is connected for any type 1 point $x \in X(K) \subset X^{\mathrm{an}}$, it is natural to say that there is only one tangent direction at x when x is of type 1. For x of type 4 we also have $|T_x| = 1$, and for x of type 3 we have $|T_x| = 2$.

If x is of type 2, on the other hand, then $|T_x|$ is infinite. In fact, there is a canonical bijection between T_x and $C_x(\kappa)$, the set of closed points on the κ-curve C_x corresponding to x. Equivalently, there is a canonical bijection between T_x and the set of discrete valuations on $\widetilde{\mathcal{H}(x)} = \kappa(C_x)$ which are trivial on κ. For $\vec{\nu} \in T_x$, we denote by $\mathrm{ord}_\nu : \kappa(C_x)^\times \to \mathbb{Z}$ the corresponding discrete valuation, so that for every nonzero rational function $\widetilde{f} \in \kappa(C_x)$ we have $\mathrm{ord}_\nu(\widetilde{f}) = \mathrm{ord}_{p_\nu}(\widetilde{f})$, where p_ν is the closed point of C_x corresponding to $\vec{\nu}$.

2.2.7. The metrised complex associated to a semistable vertex set. Given a semistable vertex set V for X^{an}, there is a canonical corresponding metrised complex $\mathfrak{C}V$ of κ-curves. Indeed, we have already defined a metric graph $\Gamma = \Sigma(X^{\mathrm{an}}, V)$ corresponding to V, and Γ comes equipped with a natural model G whose vertices are the points of V and whose edges correspond bijectively to the open annuli in the semistable decomposition associated to V. For each $v \in V$, we have also defined a smooth projective curve C_v over κ. It remains to specify, for each $v \in V$, a bijection ψ_v from the edges of G incident to v to a subset $\mathcal{A}_v$ of $C_v(\kappa)$. Given such an edge e, we define $\psi(e)$ to be the point of $C_v(\kappa)$ corresponding to the tangent direction at v defined by e.

REMARK 38. Passing to a larger semistable vertex set is compatible with linear equivalence of divisors and does not change the rank of divisors. More precisely, given two semistable vertex sets $V_1 \subseteq V_2$, there is a natural (retraction) map $\tau_{2,1} : \mathfrak{C}V_2 \to \mathfrak{C}V_1$ which induces a surjective map $\tau_{2,1} : \mathrm{Div}(\mathfrak{C}V_2) \to \mathrm{Div}(\mathfrak{C}V_1)$. In addition, $\tau_{2,1}$ sends principal divisors on $\mathfrak{C}V_2$ to principal divisors on $\mathfrak{C}V_1$ and so induces a map $\tau_{2,1} : \mathrm{Pic}(\mathfrak{C}V_2) \to \mathrm{Pic}(\mathfrak{C}V_1)$. It is easy to see that $\tau_{2,1}$ is an isomorphism and that for any divisor $\mathcal{D}$ on $\mathfrak{C}V_2$, one has $r_{\mathfrak{C}V_2}(\mathcal{D}) = r_{\mathfrak{C}V_1}(\tau_{2,1}(\mathcal{D}))$. As a consequence, there is a canonical group $\mathrm{Pic}\,\mathfrak{C}X^{\mathrm{an}}$ associated to X^{an}, defined as $\mathrm{Pic}(\mathfrak{C}V)$ for any semistable vertex set V, together with a canonical rank function $r : \mathrm{Pic}\,\mathfrak{C}X^{\mathrm{an}} \to \mathbb{Z}$.

2.2.8. Semistable vertex sets and semistable models. Semistable vertex sets for X^{an} correspond bijectively to semistable R-models $\mathfrak{X}$ for X. (Since we are assuming that X is proper, it does not matter if we work with algebraic or formal models for X; for simplicity, we will work with algebraic models.) In order to explain this in more detail, recall that a connected reduced algebraic curve over κ is called *semistable* if all of its singularities are ordinary double points, and is called *strongly semistable* if in addition its irreducible components are all smooth. A *(strongly) semistable model* for X is a flat and integral proper relative curve $\mathfrak{X}$ over R whose generic fiber is isomorphic to X and whose special fiber $\bar{\mathfrak{X}}$ is a (strongly) semistable curve.

Given a semistable model $\mathfrak{X}$ for X, there is a canonical associated reduction map red : $X(K) \to \bar{\mathfrak{X}}(\kappa)$ which is defined using the natural bijection between $X(K)$ and $\mathfrak{X}(R)$. This extends naturally to a map $\mathrm{red} : X^{\mathrm{an}} \to \bar{\mathfrak{X}}$. Let $\bar{\mathfrak{X}}^{\mathrm{gen}}$ be the set of generic points of irreducible components of $\bar{\mathfrak{X}}$. Then $V(\mathfrak{X}) := \mathrm{red}^{-1}(\bar{\mathfrak{X}}^{\mathrm{gen}})$ is a finite set of type 2 points mapping bijectively onto $\bar{\mathfrak{X}}^{\mathrm{gen}}$, and moreover $V(\mathfrak{X})$ is a semistable vertex set for X^{an}. The association $\mathfrak{X} \mapsto V(\mathfrak{X})$ gives a *bijection* between semistable formal models of X and semistable vertex sets for X^{an}. The model $\mathfrak{X}$ is strongly semistable if and only if $V(\mathfrak{X})$ is a strongly semistable vertex set.

2.2.9. The metrised complex associated to a semistable model. Since there is a natural bijection between (strongly) semistable vertex sets for X^{an} and semistable models for X, it follows that we can canonically associate to any semistable model $\mathfrak{X}$ a corresponding metrised complex of κ-curves $\mathfrak{C}\mathfrak{X}$. However, one can give a more direct description of $\mathfrak{C}\mathfrak{X}$ as follows.

Let G be the dual graph of $\bar{\mathfrak{X}}$, so that vertices of G correspond to irreducible components of $\bar{\mathfrak{X}}$ and edges of G correspond to intersections between irreducible components. If x^e is the ordinary double point of $\bar{\mathfrak{X}}$ corresponding to an edge e of G, the *formal fiber* $\mathrm{red}^{-1}(x^e)$ is isomorphic to an open annulus $\mathbf{A}$. We define the *length of the edge* e to be the length of the skeleton of $\mathbf{A}$, i.e., the *modulus* $\log(R) - \log(r)$ of $\mathbf{A} \cong \{x \in (\mathbf{A}^1)^{\mathrm{an}} \mid r < |T|_x < R\}$. (The modulus of an open annulus is well-defined independent of the choice of such an analytic isomorphism.) In this way, we have defined a metric graph $\Gamma = \Gamma_{\mathfrak{X}}$ associated to $\mathfrak{X}$ together with a model G. The irreducible components C_v of $\bar{\mathfrak{X}}$ correspond bijectively to the vertices v of G, and we let $A_v \subset C_v$ be the finite set of double points of $\bar{\mathfrak{X}}$ contained in C_v, so that there is a natural bijection between A_v and the edges of G incident to v. In this way we have defined a metrised complex $\mathfrak{C}\mathfrak{X}$ canonically associated to $\mathfrak{X}$. The metrised complex $\mathfrak{C}\mathfrak{X}$ is the same as the metrised complex $\mathfrak{C}V$ which we previously associated to $V = V(\mathfrak{X})$.

One can show that essentially every metrised complex of curves comes from this construction [**ABBR1**].

2.2.10. Reduction of rational functions and Poincaré-Lelong. Let $x \in X^{\mathrm{an}}$ be a point of type 2. Given a nonzero rational function f on X, choose $c \in K^\times$ such that $|f(x)| = |c|$. Define $f_x \in \kappa(C_x)^\times$ to be the image of $c^{-1}f$ in $\widetilde{\mathcal{H}(x)} \cong \kappa(C_x)$. Although f_x is only well-defined up to multiplication by an element of $\kappa^\times$, its divisor $\mathrm{div}(f_x)$ is canonical and the resulting map $\mathrm{Prin}(X) \to \mathrm{Prin}(C_x)$ is a homomorphism. Note also that if ord is a discrete valuation on $\kappa(C_x)$ which is trivial on κ then $\mathrm{ord}(f_x)$ is intrinsic to f.

If H is a K-linear subspace of $K(X)$, the collection of all possible reductions of nonzero elements of H, together with $\{0\}$, forms a κ-vector space H_x. The following elementary lemma says that $\dim H = \dim H_x$:

LEMMA 39 ([**AmiBa**]). *Let X be a smooth proper curve over K, and $x \in X^{\mathrm{an}}$ a point of type 2. The κ-vector space H_x defined by the reduction to $\widetilde{\mathcal{H}(x)}$ of an $(r+1)$-dimensional K-subspace $H \subset K(X)$ has dimension $r+1$.*

We say that a function $F : X^{\mathrm{an}} \to \mathbb{R}$ is *piecewise linear* if for any geodesic segment $\alpha : [a,b] \hookrightarrow \mathbf{H}(X^{\mathrm{an}})$, the pullback map $F \circ \alpha : [a,b] \to \mathbb{R}$ is piecewise linear. The *outgoing slope* of a piecewise linear function F at a point $x \in \mathbf{H}(X^{\mathrm{an}})$ along a (unit) tangent direction $\vec{v} \in T_x$ is defined to be

$$d_{\vec{v}}F(x) = \lim_{\epsilon \to 0}(F \circ \alpha)'(\epsilon),$$

where $\alpha : [0,a] \hookrightarrow X^{\mathrm{an}}$ is a geodesic segment starting at x which represents $\vec{v}$. A piecewise linear function F is called *harmonic* at a point $x \in \mathbf{H}(X^{\mathrm{an}})$ if the outgoing slope $d_{\vec{v}}F(x)$ is zero for all but finitely many $\vec{v} \in T_x$, and $\sum_{\vec{v} \in T_x} d_{\vec{v}}F(x) = 0$.

The following theorem will be essential all through the next few chapters. It is called the slope formula in [**27**] and is an immediate consequence of the non-Archimedean Poincaré-Lelong formula worked out in detail in Thuillier's thesis [**234**] (see also [**49**] and [**248**] where an implicit form of the theorem appears).

THEOREM 40 (Non-Archimedean Poincaré-Lelong - Slope formula [**27, 234, 49, 248**]). *Let f be a nonzero rational function on X, let X' be an open affine subset of X on which f has no zeros or poles, and let $F = -\log|f| : (X')^{\mathrm{an}} \to \mathbb{R}$. Let V be a semistable vertex set of X' and let $\Sigma = \Sigma(X', V)$ be its skeleton. Then:*

(1) *$F = F \circ \tau_\Sigma$ where $\tau_\Sigma : (X')^{\mathrm{an}} \to \Sigma$ is the retraction.*
(2) *F is piecewise linear with integer slopes, and F is linear on each edge of Σ.*
(3) *If x is a type-2 point of X^{an} and $\vec{v} \in T_x$, then $d_{\vec{v}}F(x) = \mathrm{ord}_v(f_x)$.*
(4) *F is harmonic at all $x \in \mathbf{H}(X^{\mathrm{an}})$.*
(5) *Let $x \in X \setminus X'$, let e be the ray in Σ whose closure in X^{an} contains x, let $y \in V$ be the other endpoint of e, and let $\vec{v} \in T_y$ be the tangent direction represented by e. Then $d_{\vec{v}}F(y) = \mathrm{ord}_x(f)$.*

2.2.11. Higher-dimensional metrised complexes. We restrict our attention to metrised complexes of curves. Nevertheless, it is clear that many of the above basic definitions and constructions should have higher-dimensional analogues. We sketch here a possible approach which we feel merits further investigation.

Let K be a complete and algebraically closed non-trivially valued non-Archimedean field, R the valuation ring of K, and κ its (algebraically closed) residue field. Let $\mathfrak{X}$ be a proper and flat strongly semistable R-scheme with smooth and irreducible generic fiber X/K. (By *strongly semistable*, we mean that the special fiber $\bar{\mathfrak{X}}$ has smooth irreducible components with simple normal crossings.) The *metrised complex of κ-schemes associated to* $\mathfrak{X}$ consists of the following data:

- The simplicial complex $G_{\mathfrak{X}}$ (the "dual complex") associated to $\bar{\mathfrak{X}}$.

- The skeleton $\Gamma_{\mathfrak{X}}$ of $\mathfrak{X}$ in the sense of [**35**], which is the geometric realization of $G_{\mathfrak{X}}$ together with its canonical integer affine structure. (Berkovich proves in *loc. cit.* that there is a canonical embedding of $\Gamma_{\mathfrak{X}}$ in X^{an} as well as a strong deformation retraction $\tau : X^{\mathrm{an}} \to \Gamma_{\mathfrak{X}}$.)
- For each vertex (0-dimensional cell) v of $G_{\mathfrak{X}}$, the corresponding κ-variety X_v.
- For each face F of $G_{\mathfrak{X}}$ corresponding to a subset $S \subseteq V$, the corresponding non-empty scheme $X_F = \cap_{v \in S} X_v$ together with its compatible family of embeddings in $X_{F'}$ for $F' \subseteq F$. (In the terminology of some algebraic geometers, $\{X_F\}_{F \in G_{\mathfrak{X}}}$ is the "cubical variety" associated to $\bar{\mathfrak{X}}$.)

When κ has characteristic zero, it is known that every smooth proper algebraic variety X/K has a strongly semistable R-model. Berkovich has introduced in [**34, 35**] a procedure for generalising the skeleton $\Gamma_{\mathfrak{X}}$ and the corresponding deformation retraction τ to *pluristable* R-models, which allows one to utilise de Jong's theory of alterations instead of resolution of singularities. It would be interesting to study line bundles and intersection theory on metrised complexes of κ-schemes associated to semistable and/or pluristable models. This would combine the algebraic geometry of simplicial schemes with that of tropical varieties.

2.3. Specialisation

Let X be a smooth proper curve over K and let $\mathfrak{X}$ be a semistable model of X. As discussed in the previous section, there is a canonical embedding of $\Gamma_{\mathfrak{X}}$ in the Berkovich analytification X^{an} of X, as well as a canonical retraction map $\tau : X^{\mathrm{an}} \to \Gamma_{\mathfrak{X}}$. In addition, there is a canonical reduction map $\mathrm{red} : X^{\mathrm{an}} \to \bar{\mathfrak{X}}$ sending $X(K)$ to surjectively onto the closed points of $\bar{\mathfrak{X}}$. The retraction map τ induces by linearity a specialisation map $\tau_* : \mathrm{Div}(X) \to \mathrm{Div}(\Gamma_{\mathfrak{X}})$. We can promote this to a map $\tau_*^{\mathfrak{C}\mathfrak{X}}$ whose target is the larger group $\mathrm{Div}(\mathfrak{C}\mathfrak{X})$ as follows.

First, we define the map $\tau^{\mathfrak{C}\mathfrak{X}} : X(K) \to |\mathfrak{C}\mathcal{X}|$. Let P be a point of $X(K)$. If P satisfies $\tau(P) = v \in V$, then $\mathrm{red}(P)$ is a nonsingular closed point of C_v, and we let $\bar{P} \in C_v(\kappa) \setminus \mathcal{A}_v$ denote the reduction of P. The map $\tau^{\mathfrak{C}\mathfrak{X}}$ is then defined by

$$\tau^{\mathfrak{C}\mathfrak{X}}(P) := \begin{cases} \tau(P) & \tau(P) \notin V \\ \bar{P} & \tau(P) \in V. \end{cases}$$

We also described a way to define the restriction of any rational function f on X to a rational function on $\mathfrak{C}\mathfrak{X}$. For f and $\mathfrak{f}$ as above, an application of Theorem 40 allows to obtain the following important formula [**AmiBa**]

$$\tau_*^{\mathfrak{C}\mathfrak{X}}(\mathrm{div}(f)) = \mathrm{div}(\mathfrak{f}),$$

which simply implies that the map τ_X and the restriction of rational functions provide a morphism of Riemann-Roch structures, as defined in Chapter 1 in Remark 8 and Proposition 28. From this observation, we get the following general specialisation inequality.

THEOREM 41 (Specialisation Theorem [**AmiBa**]). *Let K be a complete and algebraically closed non-Archimedean field with residue field κ, X/K a smooth, proper, connected algebraic curve, and $\mathfrak{X}$ a strongly semistable model of X over the valuation ring R of K. Denote by $\mathfrak{C}\mathfrak{X}$ the metrised complex associated to $\mathfrak{X}$. Then for any divisor $D \in \mathrm{Div}(X)$, we have*

$$r_X(D) \leq r_{\mathfrak{C}\mathfrak{X}}(\tau_*^{\mathfrak{C}\mathfrak{X}}(D)).$$

Since the morphism of Riemann-Roch structures from $\mathfrak{C}\mathfrak{X}$ to $\Gamma_{\mathfrak{X}}$ is surjective, by Proposition 28 of the last chapter, we get the inequality $r_{\mathfrak{C}\mathfrak{X}}(\tau_*^{\mathfrak{C}\mathfrak{X}}(D)) \leq r_{\Gamma_{\mathfrak{X}}}(\tau_*(D))$, from which, combined with the above theorem, we immediately obtain the specialisation inequality of Baker [**20**].

The above theorem is strong enough to imply as well, through a more tricky combinatorial argument though, the specialisation inequality of our paper [**AmiCa**]. We recall the theorem. Let $\Gamma = \Gamma_{\mathfrak{X}}$ and define a metric graph $\Gamma^{\#}$ by attaching g_v loops of arbitrary positive length at each point $v \in \Gamma$. The genus of $\Gamma^{\#}$ thus coincides with $g(X)$.

Any divisor D in Γ defines a divisor on $\Gamma^{\#}$ with the same support and coefficients (by viewing Γ as a subgraph of $\Gamma^{\#}$). By an abuse of notation, we will also denote this divisor by D. However, we distinguish the rank of D in Γ and in $\Gamma^{\#}$. We have

Theorem 42 (Weighted specialisation inequality [**AmiCa**]). *For every divisor $D \in \mathrm{Div}(X)$, we have $r_X(D) \leq r_{\Gamma^{\#}}(\tau_*(D))$.*

We refer to [**AmiBa**] for other stronger forms of the specialisation inequality.

Taken together, Theorems 41 and 37 have some interesting consequences. For example, they allow to show that for any canonical divisor K_X on X, its specialisation $\tau_*^{\mathfrak{C}\mathfrak{X}}(K_X)$ belongs to the canonical divisor class on $\mathfrak{C}\mathfrak{X}$. For discretely valued R, this can be shown by adjunction formula for arithmetic surfaces [**20**]. However, it does not seem obvious to prove the general (not necessarily discretely valued) case using arithmetic intersection theory, since there does not seem to be a satisfactory theory of relative dualising sheaves and adjunction in the non-Noetherian setting.

2.4. Limit linear series

In this section we discuss a framework for the generalisation of Eisenbud-Harris.

A proper nodal curve X_0 over κ is of *compact type* if its dual graph G is a tree. For such curves, Eisenbud and Harris [**98**] define a notion of (crude) *limit* $\mathfrak{g}^r_d$. A *crude limit* $\mathfrak{g}^r_d$ L on X_0 is the data of a (not necessarily complete) degree d and rank r linear series L_v on X_v for each vertex $v \in V$ such that if two components X_u and X_v of X_0 meet at a node p, then for any $0 \leq i \leq r$,

$$a_i^{L_v}(p) + a_{r-i}^{L_u}(p) \;\geq\; d\,,$$

where $a_i^L(p)$ denotes the i^{th} term in the vanishing sequence of a linear series L at p. A crude limit series is *refined* if all the inequalities in the above definition are equalities.

We can canonically associate to a proper strongly curve X_0 a metrised complex $\mathfrak{C}X_0$ of κ-curves, called the *regularization* of X_0, by assigning a length of 1 to each edge of G, and we write X_v for the irreducible component of X_0 corresponding to a vertex $v \in V$. (This is the metrised complex associated to any regular smoothing $\mathfrak{X}$ of X_0 over any discrete valuation ring R with residue field κ.) We prove the following theorem.

Theorem 43 ([**AmiBa**]). *Let $\mathfrak{C}X_0$ be the metrised complex of curves associated to a proper strongly semistable curve X_0/κ of compact type. Then there is a bijective correspondence between the following:*

- *Crude limit $\mathfrak{g}^r_d$'s on X_0.*
- *Equivalence classes of pairs $(\mathcal{H}, \mathcal{D})$, where $\mathcal{H} = \{H_v\}$, H_v is an $(r+1)$-dimensional subspace of $\kappa(X_v)$ for each $v \in V$, and $\mathcal{D}$ is a divisor of degree d supported on the vertices of $\mathfrak{C}X_0$ with $r_{\mathfrak{C}X_0,\mathcal{H}}(\mathcal{D}) = r$. Here we say that $(\mathcal{H}, \mathcal{D}) \sim (\mathcal{H}', \mathcal{D}')$ if there is*

a rational function $\mathfrak{f}$ *on* $\mathfrak{C}X_0$ *such that* $D' = D + \mathrm{div}(\mathfrak{f})$ *and* $H_v = H'_v \cdot f_v$ *for all* $v \in V$*, where* f_v *denotes the* C_v*-part of* $\mathfrak{f}$.

In this theorem, $r_{\mathfrak{C}X_0,\mathcal{H}}(.)$ refers to the *restricted rank* of a divisor, where the rational functions on curves are only allowed to be taken from the fixed subspace H_v, see [**AmiBa**] for a precise definition.

Theorem 43, combined with our Riemann-Roch theorem for metrised complexes of curves, provides an arguably more conceptual proof of the fact (originally established in [**98**]) that limit linear series satisfy analogues of the classical theorems of Riemann and Clifford.

Theorem 43 proposes the following generalised definition of limit linear series on any semistable curve. Let X_0 be a proper strongly semistable (but not necessarily compact type) curve over κ with regularization $\mathfrak{C}X_0$. A *limit* $\mathfrak{g}^r_d$ on X_0 is an equivalence class of pairs $(\mathcal{H} = \{H_v\}, \mathcal{D})$ as above, where H_v is an $(r+1)$-dimensional subspace of $\kappa(X_v)$ for each $v \in V$, and $\mathcal{D}$ is a degree d divisor on $\mathfrak{C}X_0$ with $r_{\mathfrak{C}X_0,\mathcal{H}}(\mathcal{D}) = r$.

We prove that if R is a discrete valuation ring with residue field κ and $\mathfrak{X}/R$ is a regular arithmetic surface whose generic fiber X is smooth and whose special fiber X_0 is strongly semistable, then for any divisor D on X with $r_X(D) = r$ and $\deg(D) = d$, our specialisation machine gives rise in a natural way to a limit $\mathfrak{g}^r_d$ on X_0 [**AmiBa**].

In the next section, we refine this to a purely combinatorial definition of limit linear series.

2.4.1. Combinatorial linear series on metric graphs. In this section, we define combinatorial linear series on metric graphs by the help of an auxiliary structure that we call *slope structure*. Slope structures provide a parametrised version of rank functions of order r, verifying a finiteness condition.

2.4.1.1. *Rank functions on hypercubes and permutation arrays.* Let r be a non-negative integer, and set $[r] := \{0, \dots, 1\}$. For a positive integer d, the *hypercube* $\square^d_r$ of dimension d and width r is the product $[r]^d$. We denote the elements of $\square^d_r$ by vectors $\underline{a} = (a_1, \dots, a_d)$, for $0 \le a_1, \dots, a_d \le r$. There is a partial order $\le$ on $\square^d_r$ where for two elements $\underline{a}, \underline{b} \in \square^d_r$, we have $\underline{a} \le \underline{b}$ if for all $i = 1, \dots, d$, we have $a_i \le b_i$.

The smallest (resp. largest) element of $\square^d_r$ is $\underline{0} := (0, \dots, 0)$ (resp. $\underline{r} = (r, \dots, r)$). For each integer $1 \le i \le d$, we denote by $\underline{e}_i$ the vector whose coordinates are all zero expect the i-th coordinate which is equal to one. For $0 \le t \le r$, the vector $t\underline{e}_i$ lies in the hypercube $\square^d_r$. There is a *lattice* structure on $\square^d_r$ induced by the two operations $\vee$ and $\wedge$ defined by taking the maximum and the minimum coordinate-wise, respectively: for $\underline{a}$ and $\underline{b}$ in $\square^d_r$, we have

$$\underline{a} \vee \underline{b} := (\max(a_1, b_1), \dots, \max(a_d, b_d)) \qquad \underline{a} \wedge \underline{b} = (\min(a_1, b_1), \dots, \min(a_d, b_d)).$$

A function $f : \square^d_r \to \mathbb{Z}$ is called *supermodular* if for any two elements $\underline{a}$ and $\underline{b}$, one has

$$f(\underline{a}) + f(\underline{b}) \le f(\underline{a} \vee \underline{b}) + f(\underline{a} \wedge \underline{b}).$$

We will be interested in a special kind of supermodular functions on $\square^d_r$.

Definition 44 (Rank function). A function $\rho : \square^d_r \to \mathbb{Z}$ is called a *rank function* if it is supermodular and in addition, satisfies the following conditions:

(1) The values of ρ are in the set $[r] \cup \{-1\}$.
(2) ρ is decreasing with respect to the partial order of $\square^d_r$, i.e., if $\underline{a} \le \underline{b}$, then $\rho(\underline{b}) \le \rho(\underline{a})$.
(3) for any $1 \le i \le d$, and all $0 \le t \le r$, we have $\rho(t\,\underline{e}_m) = r - t$.

For a rank function ρ on $\square_r^d$, for an $1 \leq i \leq d$, and for all $\underline{a} \in \square_r^d$ such that $\underline{a} + \underline{e}_i \in \square_r^d$, one easily verifies from the supermodularity that

$$\rho(\underline{a}) - 1 \leq \rho(\underline{a} + \underline{e}_i) \leq \rho(\underline{a}).$$

The leads to the following definition.

Definition 45 (Jumps of a rank function). Let ρ be a rank function on $\square_r^d$. A point $\underline{a}$ is a *jump* for ρ if $\rho(\underline{a}) \geq 0$ and for any $1 \leq i \leq d$ such that $\underline{a} + \underline{e}_i$ belongs to $\square_r^d$, we have $\rho(\underline{a} + \underline{e}_i) = \rho(\underline{a}) - 1$. The set of jumps of ρ is denoted by J_ρ.

Example 46 (Rank functions induced by complete flags). Let r be a non-negative integer, and let H be a vector space of dimension $r+1$ over some field κ. A complete flag of H consists of a chain of vector subspaces

$$F^0 = H \supsetneq F^1 \supsetneq \cdots \supsetneq F^{r-1} \supsetneq F^r \supsetneq (0),$$

where for each $i \in [r]$, F^i is a vector subspace of codimension i in H. Let d be a positive integer, and let $F_1^\bullet, \dots, F_d^\bullet$ be a collection of d complete flags of H. Define the function $\rho : \square_r^d \to \mathbb{Z}$ by

$$\rho(a_1, \dots, a_d) := \dim_\kappa\big(F_1^{a_1} \cap \cdots \cap F_d^{a_d} \big) - 1.$$

This defines a rank function on $\square_r^d$.

Remark 47 (Rank functions and permutation arrays). As we discovered later [**Ami8**], rank functions on hypercubes are equivalent to the data of permutation arrays introduced by Eriksson-Linusson [**102, 103**] as a combinatorial set-up to study the intersection patterns of the complete flags.

Let κ be a field. A rank function ρ on $\square_r^d$ is called *realisable* if it is the rank function associated to a collection of d complete flags $F_\bullet^1, \dots, F_\bullet^d$ on some vector space of dimension $r+1$ over κ. By the above remark, the realisability of a rank function $\square_r^d$ is equivalent to the realisability of the corresponding permutation array in the terminology of [**103**]. Billey and Vakil [**41**] provided several examples which show the existence of non-realisable permutation arrays. We show in [**Ami8**] that the examples in [**41**] are matroidal obstructions in the following sense. Let ρ be a rank function on $\square_r^d$. Let $\underline{a} \in \square_r^d$, and define $E_{\underline{a}}$ as the set of all indices $1 \leq i \leq d$ such that $\underline{a} + \underline{e}_i \in \square_r^d$. Define a function $\rho_{\underline{a}} : 2^{E_{\underline{a}}} \to \mathbb{N}$ as follows. For any subset $X \subset E_{\underline{a}}$, define

$$\rho_{\underline{a}}(X) := \rho(\underline{a}) - \rho(\underline{a} + \sum_{i \in X} \underline{e}_i)$$

Then one can show that the pair $(E_{\underline{a}}, \rho_{\underline{a}})$ defines a matroid $M_{\underline{a}}$ on the set of elements $E_{\underline{a}}$. Thus, non-realisability of $M_{\underline{a}}$ is an obstruction for the realisability of ρ.

Example 48 (Geometric rank functions over a field κ). We now describe a geometric situation which naturally leads to rank functions (and thus permutation arrays).

Let κ be an algebraically closed field, and let C be a smooth proper curve C over κ. Let r be non-negative integer, and let x be a κ-points on C. Let $\kappa(C)$ be the function field of C, and let $H \subset \kappa(C)$ be a vector subspace of rational functions of dimension $r+1$ over κ. The point x leads to a complete flag $F_x^\bullet$ of H by looking at the orders of vanishing of functions in H at the point x_i, as follows. Define the set $S_x := \{\mathrm{ord}_x(f) \,|\, f \in H - \{0\}\}$. The set S_x is finite of order $r+1$. Denoting by $s^x_0 < \cdots < s^x_r$ the elements of S, enumerated in an increasing order, we define the flag $F^\bullet_x$ by setting, for $j = 0, \dots, r$,

$$F^j_x = \{f \in H - \{0\} \,|\, \mathrm{ord}_x(f) \geq s_j\} \cup \{0\}.$$

Each space F^j_x has codimension j in H. For a collection $A = \{x_1, \dots, x_d\}$ of d distinct κ-points on C, one gets an associated rank function ρ on the hypercube $\square^d_r$, by Example 46.

2.4.1.2. *Slope structures on graphs.* Let first $G = (V, E)$ be a simple graph. We denote by $\mathbb{E}$ the set of all the orientations of edges of G, so for an edge $\{u, v\}$ in E, we have two orientations $uv, vu \in \mathbb{E}$. For a vertex $v \in V$, we denote by $\mathbb{E}_v \subset \mathbb{E}$ the set of all the oriented edges $vu \in \mathbb{E}$, for edges $\{v, u\} \in E$. For an oriented edge $e = uv \in \mathbb{E}$, we denote by $\bar{e} = vu$ the oriented edge of $\mathbb{E}$ with reverse orientation.

A *slope structure* $\mathfrak{S} = \Big\{S^v; S^e\Big\}_{v \in V, e \in A}$ of *order* r on G, or simply an *r-slope structure*, is the data of

- For any oriented edge $e = uv \in \mathbb{E}$ of G, a collection S^e of $r+1$ integers $s^e_0 < s^e_1 < \cdots < s^e_r$, subject to the requirement that $s^{uv}_i + s^{vu}_{r-i} = 0$ for any edge $\{u, v\} \in E$.
- For any vertex v of G, a rank function ρ_v on $\square^{\mathrm{val}(v)}_r$. If J_{ρ_v} denotes the set of jumps of ρ_v, we denote by $S^v \subseteq \prod_{e \in \mathbb{E}_v} S^e$, the set of all points $s_{\underline{a}}$ for $\underline{a} \in J_{\rho_v}$: Here, for a point $\underline{a} = (a_e)_{e \in \mathbb{E}_v}$ of the hypercube, the point $s_{\underline{a}} \in \prod_{e \in \mathbb{E}_v} S^e$ denotes the point in the product which has coordinate at $e \in \mathbb{E}_v$ equal to $s^e_{a_e}$.

2.4.1.3. *Slope structures on metric graphs.* Let now Γ be a metric graph. By an *r-slope structure* on Γ we mean an r-slope structure $\mathfrak{S}$ on a simple graph model $G = (V, E)$ of Γ, that we enrich by extending to any point of Γ as follows. For a point $x \in \Gamma$, denote by $\mathrm{T}_x(\Gamma)$ the set of all the $\mathrm{val}(x)$ (out-going) tangent vectors to Γ at x. For any point x and $\nu \in \mathrm{T}_x(\Gamma)$, there exists a unique oriented edge uv of G which is parallel to ν. Define $S^\nu = S^{uv}$. Also for any point $x \in \Gamma \backslash V$ in the interior of an edge $\{u, v\}$, define ρ_x to be the standard rank function on $\square^2_r$. In particular, $S^x \subseteq S^{uv} \times S^{vu}$ can be identified with the set of all pairs (s^{uv}_i, s^{vu}_j) with $i + j \leq r$. We call the collection $\Big\{S^x; S^\nu\Big\}_{x \in \Gamma, \nu \in \mathrm{T}_x(\Gamma)}$ a *slope structure of order* r on Γ that we denote by $\mathfrak{S}_\Gamma$, or simply $\mathfrak{S}$, if there is no risk of confusion. Note that a slope structure on a metric graph can arise from choices of slope structures on different graph models of Γ.

2.4.1.4. *Rational functions on a metric graph compatible with a slope structure.* Let Γ be a metric graph and let $\mathfrak{S} = \Big\{S^x; S^\nu\Big\}_{x \in \Gamma, \nu \in \mathrm{T}_x(\Gamma)}$ be a slope structure of order r on Γ. A continuous piecewise affine function $f : \Gamma \to \mathbb{R}$ is said to be *compatible* with $\mathfrak{S}$ if the two conditions (i) and (ii) below are verified:

(i) for any point $x \in \Gamma$ and any tangent direction $\nu \in \mathrm{T}_x(\Gamma)$, the outgoing slope of f along ν lies in S^ν.

Denote by $\delta_x(f)$ the vector in $\prod_{\nu \in \mathrm{T}_x(\Gamma)} S^\nu$ which consists of outgoing slopes of f along $\nu \in \mathrm{T}_x(\Gamma)$. Then

(ii) for any point x in Γ, the vector $\delta_x(f)$ belongs to S^x.

For any rational function f on Γ, the corresponding *principal divisor* is denoted by

$$\mathrm{div}(f) = \sum_x \mathrm{div}_x(f)(x), \qquad \text{where } \mathrm{div}_x(f) := \sum_{\nu \in \mathrm{T}_x(\Gamma)} \mathrm{slope}_\nu(f).$$

2.4.1.5. *Linear series* $\mathfrak{g}^r_d$ *on* Γ. We define a $\mathfrak{g}^r_d$ on Γ as the linear equivalence class of the data of a pair $(D, \mathfrak{S})$ where D is a divisor of degree d on Γ and $\mathfrak{S}$ is an r-slope structure on Γ subject to the following (rank) property:

$(*)$ For any effective divisor E on Γ of degree r, there exists a rational function $f \in \mathrm{Rat}(\mathfrak{S})$ such that

(1) For any point $x \in \Gamma$, $\rho_x(\delta_x(f)) \geq E(x)$; and in addition,
(2) $\mathrm{div}(f) + D - E \geq 0$.

Let $(D, \mathfrak{S})$ define a $\mathfrak{g}^r_d$ on Γ. We denote by $\mathrm{Rat}(D; \mathfrak{S})$ the space of all $f \in \mathrm{Rat}(\mathfrak{S})$ with the property that $\mathrm{div}(f) + D \geq 0$, and define the linear system $|(D, \mathfrak{S})|$ associated to $(D, \mathfrak{S})$ as the space of all effective divisors E on Γ of the form $\mathrm{div}(f) + D$ for some $f \in \mathrm{Rat}(D; \mathfrak{S})$. The set $|(D, \mathfrak{S})|$ is independent of the choice of the pair $(D, \mathfrak{S})$ in its linear equivalence class.

2.4.1.6. *Specialisation.* Let X be a smooth proper curve ove an algebraically closed non-archimedean field K. Let $\mathcal{D}$ be divisor of degree d on X, and $(\mathcal{O}(\mathcal{D}), H)$ be a $\mathfrak{g}^r_d$ on X. We identify H with a subspace of $K(X)$ of dimension $r+1$. Let Γ be a skeleton of X^{an}. By the slope formula, the reduction $F = -\log(|f|)$ of any function $f \in H$ to Γ is a piecewise affine function on Γ with integer slopes. Let first x be a type II point of Γ, and ν a tangent direction in $\mathrm{T}_x(\Gamma)$. Denote by x^ν the point of $\mathfrak{C}_x(\kappa)$ which corresponds to ν. The dimension of $\widetilde{H} \subset \kappa(\mathfrak{C}_x)$ is $(r+1)$. The orders of vanishing of $\widetilde{f} \in \widetilde{H}$ at x^ν define a sequence of integers $s^\nu_0 < s^\nu_1 < \cdots < s^\nu_r$. Denote by $S^\nu = \{s^\nu_i\}$. In addition, the collection of points $x^\nu \in \mathfrak{C}_x(\kappa)$ for $\nu \in \mathrm{T}_x(\Gamma)$ define a rank function ρ_x associated to the corresponding filtrations on $\widetilde{H}$. The set of jumps of ρ_x is denoted byS^x. Then we have

THEOREM 49 (Specialisation of linear series). *Notations as above, there exists a semistable vertex V for X such that $\Sigma(X, V) = \Gamma$, and such that the slopes of rational functions f in H along tangent directions in Γ define a well-defined $\mathfrak{g}^r_d$ $(D, \mathfrak{S})$ on Γ, with $D = \tau_*(\mathcal{D})$.*

2.4.2. Applications. The connection of the framework of limit linear series on metrised complexes to the refinement of Chabauty-Coleman method, to bound for the number of rational points of curves over number fields, obtained by Katz and Zureick-Brown [**147**] is discussed in [**AmiBa**]. We note that this has been recently refined by Katz, Rabinoff and Zureick-Brown [**148**] to a uniform bound for curves of bounded Mordel-Weil rank, as suggested by Stoll [**230**]. The argument of [**148**], leading to the uniform bound, can be recast in the language of slope structures discussed above: namely that all the slopes appearing in the slope structures defined by the canonical linear series have absolute value bounded by an absolute constant.

We will see an application of the framework discussed above in Chapter 4 in the study of Weierstrass points on curves over non-Archimedean fields.

CHAPTER 3

Lifting harmonic morphisms

Let K be an algebraically closed, complete non-Archimedean field. In [**ABBR1, ABBR2**] we study the extent to which finite morphisms of algebraic K-curves are controlled by their skeleta, which are metric graph embedded in the Berkovich analytification of X, and their associated metrised complexes.

We prove that a finite morphism of K-curves gives rise to a finite harmonic morphism of a suitable choice of skeleta. We use this to give analytic proofs of stronger skeletonised versions of some foundational results of Liu-Lorenzini, Coleman, and Liu on simultaneous semistable reduction of curves. We then consider the inverse problem of lifting finite harmonic morphisms of metrised complexes to morphisms of curves over K. We prove that every tamely ramified finite harmonic morphism of Λ-metrised complexes of k-curves lifts to a finite morphism of K-curves. If in addition the ramification points are marked, we obtain a complete classification of all such lifts along with their automorphisms. This generalises and provides new analytic proofs of earlier results of Saïdi and Wewers.

In this section, we briefly discuss these results, as well as some of their applications.

3.1. Morphisms of metric graphs

We recall some standard definitions regarding the morphisms between metric graphs, and the corresponding tropical curves, see [**ABBR1**] and the references there for a more detailed discussion of the following definitions with several examples.

Let Γ and Γ' be two metric graphs, and fix vertex sets $V = V(\Gamma)$ and $V' = V(\Gamma')$ for Γ and Γ', respectively. Denote by E and E' the edge sets $E(\Gamma)$ and $E(\Gamma')$, respectively. Let $\phi : \Gamma \to \Gamma'$ be a continuous map.

- The map ϕ is called a (V, V')-*morphism of metric graphs* if we have $\phi(V) \subset V'$, $\phi^{-1}(E') \subset E$, and the restriction of ϕ to any edge e in E is a dilation by some factor $d_e(\phi) \in \mathbb{Z}_{\geq 0}$.
- The map ϕ is called a *morphism of metric graphs* if there exists a vertex set $V = V(\Gamma)$ of Γ and a vertex set $V' = V(\Gamma')$ of Γ' such that ϕ is a (V, V')-morphism of metric graphs.
- The map ϕ is said to be *finite* if $d_e(\phi) > 0$ for any edge $e \in E(\Gamma)$.

The integer $d_e(\phi) \in \mathbb{Z}_{\geq 0}$ in the definition above is called the *degree* of ϕ along e. Let $p \in V(\Gamma)$, let $w \in T_p(\Gamma)$, and let $e \in E(\Gamma)$ be the edge of Γ in the direction of w. The directional derivative of ϕ in the direction w is by definition the quantity $d_w(\phi) := d_e(\phi)$. If we set $p' = \phi(p)$, then ϕ induces a map

$$d\phi(p) \ : \ \big\{w \in T_p(\Gamma) \ : \ d_w(\phi) \neq 0\big\} \to T_p(\Gamma')$$

in the obvious way.

Let $\phi : \Gamma \to \Gamma'$ be a morphism of metric graphs, let $p \in \Gamma$, and let $p' = \phi(p)$. The morphism ϕ is harmonic at p provided that, for every tangent direction $w' \in T_{p'}(\Gamma')$, the number

$$d_p(\phi) := \sum_{\substack{w \in T_p(\Gamma) \\ w \mapsto w'}} d_w(\phi)$$

is independent of w'. The number $d_p(\phi)$ is called the *degree* of ϕ at p.

We say that ϕ is *harmonic* if it is surjective and harmonic at all $p \in \Gamma$; in this case the number $\deg(\phi) = \sum_{p \mapsto p'} d_p(\phi)$ is independent of $p' \in \Gamma'$, and is called the degree of ϕ.

There is an equivalence relation between metric graphs, whose equivalence classes ares called *tropical curves*. An *elementary tropical modification* of a metric graph Γ_0 is a metric graph $\Gamma = [0, \ell] \cup \Gamma_0$ obtained from Γ_0 by attaching a segment $[0, \ell]$ of (an arbitrary) length $\ell > 0$ to Γ_0 in such a way that $0 \in [0, \ell]$ gets identified with a point $p \in \Gamma_0$. A metric graph Γ obtained from a metric graph Γ_0 by a finite sequence of elementary tropical modifications is called a *tropical modification* of Γ_0. Tropical modifications generate an equivalence relation $\sim$ on the set of metric graphs. A tropical curve is an equivalence class of metric graphs with respect to $\sim$.

There exists a unique rational tropical curve, which is denoted by $\mathbb{TP}^1$: it is the class of all finite metric trees (which are all equivalent under tropical modifications).

A *tropical morphism* of tropical curves $\phi : C \to C'$ is a harmonic morphism of metric graphs between some metric graph representatives of C and C', considered up to tropical equivalence.

A tropical curve C is said to have a (non-metric) graph G as its *combinatorial type* if C admits a metric graph representative which admits G as a graph model.

A tropical curve C is called *d-gonal* if there exists a tropical morphism $C \to \mathbb{TP}^1$ of degree d. A metric graph Γ has *geometric gonality* d, if the tropical curve associated to Γ is d-gonal, and d is the smallest integer satisfying this condition. The geometric gonality of a metric graph is denoted by $\gamma_{gm}(\Gamma)$.

A related notion is the *divisorial gonality* denoted by $\gamma_{div}(\Gamma)$ which is defined by

$$\gamma_{div}(\Gamma) := \min\{d : \text{ there exists a } D \in \mathrm{Div}(\Gamma), \text{ with } \deg(D) = d \text{ and } r(D) = 1\}.$$

It is easy to see that the fibres of any finite harmonic morphisms from a metric graph Γ to a finite tree are linearly equivalent, and define a linear equivalence class of divisors on Γ of rank at least one. It thus follows that $\gamma_{gm}(\Gamma) \geq \gamma_{\mathrm{div}}(\Gamma)$, for any metric graph Γ.

3.2. Simultaneous semistable reduction theorem

The following theorem shows that finite morphisms of curves induce morphisms of skeleta, and so tropical curves.

THEOREM 50 ([**ABBR1**]). *Let $\phi : X' \to X$ be a finite morphism of smooth, proper, connected K-curves and let $D \subset X(K)$ be a finite set. There exists a skeleton $\Gamma \subset X^{\mathrm{an}}$ such that $\Gamma' := \phi^{-1}(\Gamma)$ is a skeleton of X'^{an}. For any such Γ the map $\phi : \Gamma' \to \Gamma$ is a finite harmonic morphism of metric graphs.*

Due to the close relationship between semistable models and skeleta, briefly discussed in the previous chapter, Theorem 50 should be interpreted as a very strong simultaneous semistable reduction theorem. In fact, it is possible to derive from this theorem, the simultaneous semistable reduction theorems of Liu–Lorenzini [**173**], Coleman [**79**], and Liu [**172**], see [**ABBR1**] for details. Our version of these results hold over any non-Archimedean field K_0, not assumed to be discretely valued. As an example, the following weak form of Liu's theorem is a consequence of Theorem 50:

Corollary 51. *Let X, X' be smooth, proper, geometrically connected curves over a non-Archimedean field K_0 and let $\phi : X' \to X$ be a finite morphism. Then there exists a finite, separable extension K_1 of K_0 and semistable models $\mathfrak{X}, \mathfrak{X}'$ of the curves X_{K_1}, X'_{K_1}, respectively, such that ϕ_{K_1} extends to a finite morphism $\mathfrak{X}' \to \mathfrak{X}$.*

The proof of Theorem 50 is a consequence of a relative version of the Poincaré-Lelong formula, which gives the harmonicity of the appliction ${X'}^{\mathrm{an}} \to X^{\mathrm{an}}$.

3.3. Lifting harmonic morphisms

Our second goal is to lift finite harmonic morphisms of metric graphs to finite morphisms of curves. More precisely, let (X, D) be a punctured K-curve, let Γ be a skeleton, and let $\bar{\phi} : \Gamma' \to \Gamma$ be a finite harmonic morphism of metric graphs. It is natural to ask whether there exists a curve X' and a finite morphism $\phi : X' \to X$ such that $\phi^{-1}(\Gamma)$ is a skeleton of X' and $\phi^{-1}(\Gamma) \cong \Gamma'$ as metric graphs over Γ. In general the answer is "no": there are subtle obstructions of Hurwitz type, c.f. [**ABBR1**]. One solution to remedy this is to enrich Γ, Γ' with the extra structure of metrised complexes of curves. There is a notion of a finite harmonic morphism of metrised complexes of curves, which consists of a finite harmonic morphism $\phi : \Gamma' \to \Gamma$ of underlying metric graphs, and for every vertex $p' \in \Gamma'$ with $p = \phi(p')$, a finite morphism $\phi_{p'} : C_{p'} \to C_p$, satisfying various compatibility conditions.

Now let $\Gamma = \Sigma(X, V \cup D)$ for a triangulated punctured curve $(X, V \cup D)$, and let $\mathfrak{C}$ be the corresponding metrised complex of curves. Let $\mathfrak{C}' \to \mathfrak{C}$ be a *tame covering* of metrised complexes of curves (see below or [**ABBR1**] for the precise definition). The main lifting theorem can be stated as follows.

Theorem 52 (Lifting theorem [**ABBR1**]). *Notations as above, there exists a curve X' and a finite morphism $\phi : X' \to X$, branched only over D, such that $\phi^{-1}(\Gamma)$ is a skeleton of $(X', \phi^{-1}(D))$ with corresponding metrised complex isomorphic to $\mathfrak{C}'$ as metrised complexes of curves over $\mathfrak{C}$. In addition, there are only finitely many such lifts X' up to X-isomorphism.*

Moreover, we are able to give explicit descriptions of the set of lifts and the automorphism group of each lift (as a cover of X) in terms of the morphism $\Sigma' \to \Sigma$. We omit the precise statement which can be found in [**ABBR1**, Section 7].

This lifting theorem extends several theorems in the literature, making them more precise. Saïdi [**216**] proves a version of the lifting theorem for semistable formal curves (without punctures) over a complete discrete valuation ring. Wewers [**241**] works with marked curves over a complete Noetherian local ring using deformation-theoretic arguments, proving that every tamely ramified admissible cover of a marked semistable curve over the residue field of a complete Noetherian local ring lifts. Wewers classifies the possible lifts in terms of certain non-canonical deformation data depending on compatible choices of formal coordinates. The use of metrised complexes of curves in the formulation of the lifting theorems allows us to obtain canonical gluing data, resulting in an action of the automorphism group of the morphism of metrised complexes on the set of gluing data. This results in a classification of lifts up to isomorphism as covers of the target curve, and allows to determine the automorphism group of such a lift as well.

The question of lifting (and classification of all possible liftings) in the wildly ramified case is more subtle and cannot be guaranteed in general [**ABBR1**]. See however the very recent paper of Brezner and Temkin [**58**] where interesting results in this direction are obtained.

3.3.1. Local lifting theorem. The proof of Theorem 52 is based on a local lifting result, that we now discuss. Let X be a smooth, proper, connected curve over K and let $x \in X^{\mathrm{an}}$ be a type-2 point, c.f. Chapter 2. Consider a semistable vertex set $V \subset X^{\mathrm{an}}$ of X such that V contains x, and so that the simple graph model of the metric graph $\Gamma = \Sigma(X, V)$ given by V does not have loop edges. Let $\tau : X^{\mathrm{an}} \to \Gamma$ be the retraction. Let $e_1, \dots, e_r$ be the edges of Γ adjacent to x and let $\Gamma_0 = \{x\} \cup e_1^\circ \cup \cdots \cup e_r^\circ$ be the *open star* of X in the metric graph Γ. Here for a (closed) edge e we let e° denote the corresponding open edge, i.e. the edge without its endpoints. Then $\tau^{-1}(\Gamma_0)$ is an open neighbourhood of x in X^{an}. Such open sets (for some choice of V) are called *simple neighborhood* of x.

Definition 53. A *star-shaped curve* is a pointed K-analytic space (Y, y) which is isomorphic to (U, x) where x is a type-2 point in the analytification of a smooth, proper, connected curve over K and U is a simple neighborhood of x. The point y is the *central vertex* of Y.

Let now (Y, y) be a star-shaped curve, so that $Y \setminus \{y\}$ is a disjoint union of open balls and finitely many open annuli $A_1, \dots, A_r$. A *compatible divisor in* Y is a finite set $D \subset Y(K)$ whose points are contained in distinct open ball connected components of $Y \setminus \{y\}$, so the connected components of $Y \setminus (\{y\} \cup D)$ are open balls, the open annuli $A_1, \dots, A_r$, and (finitely many) open balls $B_1, \dots, B_s$ punctured at a point of D. The data (Y, y, D) of a star-shaped curve along with a compatible divisor is called a *punctured star-shaped curve*. The *skeleton of* (Y, y, D) is the set

$$\Sigma(Y, \{y\} \cup D) = \{y\} \cup D \cup \bigcup_{i=1}^{r} \Sigma(A_i) \cup \bigcup_{j=1}^{s} \Sigma(B_j).$$

Fix a compatible divisor D in Y, and let $\Gamma_0 = \Sigma(Y, \{y\} \cup D)$. As for skeleta of algebraic curves, we get a canonical continuous retraction map $\tau : Y \to \Gamma_0$. Each connected components of $\Gamma_0 \setminus \{y\}$ is called and edge of Γ_0; an edge which contains a point of D is infinite, all the other edges are finite.

Let (Y, y) be a star-shaped curve. The smooth, proper, connected k-curve C_y with function field $\widetilde{\mathscr{H}}(y)$ is called the residue curve of Y. The tangent vectors in $T_y Y$ are in bijection with the points of C_y

Let (Y, y, D) be a punctured star-shaped curve with skeleton $\Gamma_0 = \Sigma(Y, \{y\} \cup D)$ and let C_y be the residue curve of Y. A *tame covering* of (Y, y, D) consists of a punctured star-shaped curve (Y', y', D') and a finite morphism $\phi : Y' \to Y$ satisfying the following properties:

(1) $\phi^{-1}(y) = \{y'\}$,
(2) $D' = \phi^{-1}(D)$, and
(3) if $C_{y'}$ denotes the residue curve of Y' and $\phi_{y'} : C_{y'} \to C_y$ is the morphism induced by ϕ, then $\phi_{y'}$ is tamely ramified and is branched only over the points of C_y corresponding to tangent directions at y represented by edges in Σ_0.

With these notations, the local lifting theorem can be stated as follows.

THEOREM 54 ([**ABBR1**]). *Notations as above, let C' be a smooth, proper, connected k-curve, and let $\bar{\phi} : C' \to C_y$ be a finite, tamely ramified morphism branched only over the points of C_y corresponding to tangent directions at y represented by edges in Γ_0. Then there exists a lifting of C' to a punctured star-shaped curve over (Y, y, D), and this lifting is unique up to unique isomorphism.*

Proof of Theorem 54 eventually reduces to classical results about the tamely ramified étale fundamental group.

3.4. Applications

The main question we like to answer is the following:

Which tropical morphisms between (augmented) tropical curves can be lifted to a morphism of algebraic curves?

Recall from Chapter 2 that an augmented metric graph is a metric graph enriched with a genus function on vertices of a graph model.

The results of the previous section show that in order to answer this question, it will be enough to lift a morphism of augmented tropical curves to a finite harmonic morphisms of metrised complexes.

Essentially by definition, the answer to the above question reduces to an existence problem for ramified coverings $\phi_{p'} : C'_{p'} \to C_p$ of a given degree with some prescribed ramification profiles. This is intimately linked with the question of non-vanishing of Hurwitz numbers. In particular one can easily construct tropical morphisms between augmented tropical curves which cannot be promoted to a finite harmonic morphism of metrised complexes (and hence cannot be lifted to a finite morphism of smooth proper curves over K), see [**ABBR2**] for such an example.

Understanding when Hurwitz numbers vanish remains mysterious in general, and so at present there is no satisfying combinatorial answer to the above question, if we require that the genus of the objects in question be preserved by our lifts. However, if we drop the latter condition, i.e., if we use different augmentations, we can show the answer to the question is *all.*

THEOREM 55 ([**ABBR2**]). *Any finite harmonic morphism $\phi : \Gamma' \to \Gamma$ of metric graphs is liftable if* $\mathrm{char}(k) = 0$.

3.4.1. Tame group actions.

Let $\mathfrak{C}$ be a metrised complex with underlying metric graph Γ, and let H be a finite subgroup of $\mathrm{Aut}(\mathfrak{C})$. We say the action of H on $\mathfrak{C}$ is *tame* if for any vertex p of Γ, the stabiliser group H_p acts freely on a dense open subset of C_p, and for any point x of C_p, the stabiliser subgroup H_x of H is cyclic of the form $\mathbf{Z}/d\mathbf{Z}$ for some integer d, with $(d, p) = 1$ if $\mathrm{char}(k) = p > 0$. It follows from (the strong form of) our lifting Theorem 52 that we can lift $\mathfrak{C}$ together with a tame action of H if and only if the quotient $\mathfrak{C}/H$ exists in the category of metrised complexes. We give a characterisation of when such a quotient exists in [**ABBR2**], of which the following result is a special case.

THEOREM 56 ([**ABBR2**]). *Suppose that the action of H is tame and has no isolated fixed points on the underlying metric graph Γ of $\mathfrak{C}$. Then there exists a smooth, proper, and geometrically connected algebraic K-curve X lifting $\mathfrak{C}$ which is equipped with an action of H compatible with the action of Γ on $\mathfrak{C}$.*

In presence of isolated fixed points, there are additional hypothesis on the action of H to be liftable to a K-curve. As a concrete example, we prove the following characterisation of all augmented tropical curves arising from a hyperelliptic K-curve.

THEOREM 57 (Characterisation of liftable hyperelliptic metric graphs [**ABBR2**]). *Let Γ be an augmented metric graph of genus $g \geq 2$ having no infinite vertices or degree one vertices of genus 0. Then there is a smooth proper hyperelliptic curve X over K of genus g having Γ as its minimal skeleton if and only if (a) there exists an involution s on Γ such that s fixes all the points $p \in \Gamma$ with $g(p) > 0$ and the quotient Γ/s is a metric tree, and (b) for every $p \in \Gamma$ the number of bridge edges adjacent to p is at most $2g(p) + 2$.*

3.4.2. Linear equivalence of divisors. When the target curve has genus zero, a variant of the lifting question in which the genus of the source curve may be prescribed, at the cost of losing control over the degree of the morphism, can be used to show that linear equivalence of divisors on a tropical curve C coincides with the equivalence relation generated by declaring that the fibres of every tropical morphism from C to the tropical projective line are equivalent.

3.4.3. Smoothing of combinatorial $\mathfrak{g}^1_d$s on metric graphs. We prove in [**Ami8**]:

THEOREM 58 ([**Ami8**]). *Notations as in Section 2.4.1, let $(D, \mathfrak{S})$ be a $\mathfrak{g}^1_d$ on Γ. There is metric graph $\widetilde{\Gamma}$ which is a tropical modification of Γ, and a finite harmonic morphism $\phi : \widetilde{\Gamma} \to T$ to a tree T, such that $(D, \mathfrak{S})$ is the induced $\mathfrak{g}^1_d$ on Γ from ϕ.*

Combined with our lifting result we get the following smoothing theorem.

THEOREM 59 (Smoothing theorem for $\mathfrak{g}^1_d$). *A $\mathfrak{g}^1_d$ $(D, \mathfrak{S})$on Γ is smoothable if for a model $G = (V, E)$ of Γ which defines $(D, \mathfrak{S})$, all the rank functions ρ_v for $v \in V$ are geometric over a field κ (as defined in Example 48).*

The theorem should be compared to (and probably could lead to a generalisation of) the recent work of Bainbridge-Chen-Gendron-Grushevsky-Möller [**19**], where a smoothing theorem for $\mathfrak{g}^1_{2g-2}$ given by sections of the canonical sheaves is proved.

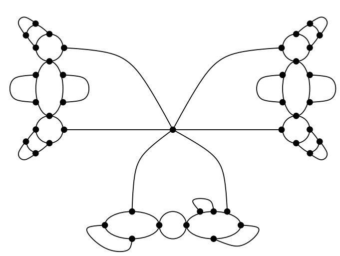

FIGURE 1. The graph G_{27}

3.4.4. Geometric gonality of tropical curves. We prove the following theorem which shows that the geometric gonality inequality can be strict. More precisely,

THEOREM 60 ([**ABBR2**]). *Let G_{27} be the graph of genus 27 depicted in Figure 1. There exists a tropical curve C of combinatorial type G which has geometric gonality four. However for any curve of genus 27 X over K with a skeleton of combinatorial type G, the curve X has gonality at least five. This means that G is not the dual graph of any 4-gonal curve X/K.*

CHAPTER 4

Variation of Okounkov bodies and equidistribution of Weierstrass points

A theorem of Mumford and Neeman [**195, 197**] states that for a compact Riemann surface S of positive genus, and for an ample line bundle L on S, the discrete measures μ_n supported on Weierstrass points of $L^{\otimes n}$ converge weakly to the Bergman measure on S when n goes to infinity. In our paper [**Ami1**], we prove a non-Archimedean analogue of Mumford-Neeman theorem. In this chapter we describe these results.

To simplify the presentation, we assume that all the fields appearing in this chapter are of characteristic zero. For the discussion of the positive characteristic case, we refer to [**Ami1**].

4.1. Weierstrass divisor

Let K be an algebraically closed field of characteristic zero, X a smooth proper curve of genus g over K, and let L be an ample line bundle on X.

A point p of X is called a *Weierstrass point* if there exists a holomorphic differential ω in $H^0(X,\omega_X)$ which has an order of vanishing at least g at p.

One generalises the above definition to any line bundle L on X by looking at the orders of vanishings of global sections of L at points of X. If the dimension of $H^0(X,L)$ is $r+1$, then apart from a finite number of points, at every point the global section of L have orders of vanishing forming the *standard sequence* $0,1,\dots,r$. The Weierstrass points of L are precisely those points which have orders of vanishing different from the standard sequence. If p is a Weierstrass point and the orders of vanishing of elements of $H^0(X,L)$ are given by the sequence of non-negative integer numbers $s_0,\dots,s_r$, then the *multiplicity* w_p of p is given by the quantity $\sum_{i=0}^{r}(s_i-i)$. The *Weierstrass divisor* of L is the divisor $W=\sum_p w_p(p)$.

Alternatively, one can define the Weierstrass divisor as follows. Consider a basis $f_0,\dots,f_r$ be a basis for $H^0(X,L)$, and set $\mathcal{F}=\{f_0,\dots,f_r\}$.

We define the *Wronskian* $\mathrm{Wr}_{\mathcal{F}}$ of the family $\mathcal{F}$, which is a global section of $L^{\otimes(r+1)}\Omega^{\otimes\frac{r(r+1)}{2}}$. Intrinsically, it is defined as follows: consider the jet bundle $J_r=\pi_{1*}\Big(\pi_2^*(L)/I^{r+1}\Big)$, where I is the ideal of the diagonal in $X\times X$, and π_1 and π_2 are the projections into the first and the second factor. There is a natural induced filtration on J_r by powers of I, and the quotients are identified with $L\otimes\Omega_X^{\otimes i}$. The set of global sections $f_0,\dots,f_r$ define sections $\pi_2^*(f_i)$ of J_r, whose determinant will be a global section of $L^{\otimes(r+1)}\otimes\Omega^{\otimes r(r+1)/2}$

The zero divisor of $\mathrm{Wr}_{\mathcal{F}}$ is independent of all the choices, and coincides with the Weierstrass divisor of L defined by multiplicities [**162, 163**]. It follows that the Weierstrass divisor has degree $(r+1)(\deg(L)+r(g-1))$.

4.2. Arakelov-Bergman measure and Mumford-Neeman theorem

Suppose that $K = \mathbb{C}$, the field of complex numbers. The analytification X^{an} of the curve X is a Riemann surface of genus g, that we assume to be non-zero. It comes with a canonical volume form, called Bergman or Arakelov measure, that we denote by μ_{Ar}. It is defined as follows. The space of holomorphic one-forms $H^0(X^{\mathrm{an}}, \omega_{X^{an}})$ has a Hermitian inner product defined by

$$\langle \omega_1, \omega_2 \rangle = \frac{2}{i} \int_{X^{\mathrm{an}}} \omega_1 \wedge \overline{\omega}_2.$$

The volume form μ_{Ar} is then defined as

$$\mu_{\mathrm{Ar}} := \frac{i}{2g} \sum_{j=1}^{g} \omega_j \wedge \overline{\omega}_j,$$

for a choice of orthonormal basis $\omega_1, \dots, \omega_g$ of $H^0(X^{\mathrm{an}}, \omega_{X^{\mathrm{an}}})$. It is independent of the choice of the orthonormal basis.

The theorem of Mumford and Neeman states that for any ample line bundle L on X the (multiset of) Weierstrass points of $L^{\otimes n}$, seen in the analytified variety X^{an}, become equidistributed according to the measure μ_{Ar}.

Theorem 61 (Mumford-Neeman [**195, 197**]). *Notations as above, the sequence of discrete measures*

$$\mu_n = \frac{1}{(r_n+1)(n\deg(L) + r_n(g-1))} \sum_{p \in X^{\mathrm{an}}} c_p(L^{\otimes n}) \delta_p$$

converges weakly to μ_{Ar} when n tends to infinity.

In the above statement, $c_p(L^{\otimes n})$ is the Weierstrass multiplicity of $L^{\otimes n}$ at point p, and δ_p is the Dirac measure supported at p.

Note that the theorem in particular implies that Weierstrass points of powers of L are dense in X^{an}, which was proved previously by Olsen [**199**].

4.3. Non-Archimedean Arakelov measure

In this section, we describe the non-Archimedean Arakelov measure as defined by Zhang [**248**] (see also the independent work of Bloch, Gillet and Soulé in [**42**] for a partial result in defining the measure).

Let Γ be a metric graph with a loopless graph model $G = (V, E)$, and with a length function ℓ. Each edge e in E has a length $\ell_e > 0$.

The metric graph Γ has a natural Lebesgue measure denoted by $d\theta$. The Laplacian of Γ is the (measure valued) operator Δ on Γ which to a piecewise smooth function f associates the measure

$$\Delta(f) := -f'' d\theta - \sum_{p \in \Gamma} \sigma_p \delta_p,$$

where δ_p is the Dirac measure at p, and σ_p is the sum of the directional derivatives of f along (outgoing) unit tangent vectors in $T_p^1\Gamma$.

Consider a measure μ on Γ of total mass one. Fix a point $x \in \Gamma$ and consider the following Laplace equation

$$\Delta_y \, g(x,y) = \delta_x - \mu \,, \tag{1}$$

with uniformizing condition $\int_\Gamma g(x,y)\,d\mu(y) = 0$. It has a unique solution denoted by $g_\mu(x,y)$. In addition, $g_\mu(x,y)$ has the following explicit representation in terms of the three-parameters-Green-function $g_z(x,y)$ on Γ. Recall first that $g_z(x,y)$ is the unique solution to the equation

$$\Delta_y g_z(x,y) = \delta_x - \delta_z,$$

under the condition $g_z(x,z) = 0$. We have $g_\mu(x,y) = \int_\Gamma g_z(x,y)d\mu(z)$. We refer to [**25**] for more details.

4.3.1. Canonical admissible measure μ_{ad}. Let D be a divisor and let μ be a measure on Γ. We use the conventional notation that $g_\mu(D,y) = \sum_{p\in\Gamma} D(p) g_\mu(p,y)$.

The following theorem is proved by Zhang [**248**], and is a generalisation of a theorem of Chinburg-Rumley [**71**] for $D = 0$ to any divisor of degree $\deg(D) \neq -2$.

THEOREM 62 (Zhang [**248**]). *Let D be a divisor of degree different from -2 on a metric graph Γ of positive genus. There is a measure μ_D and a constant c_D such that for any point x of Γ, one has*

$$c_D + g_{\mu_D}(D,x) + g_{\mu_D}(x,x) = 0.$$

In addition, the pair (μ_D, c_D) is unique.

Let now K be a complete algebraically closed non-Archimedean field with a non-trivial valuation. Let X be a smooth proper connected curve on K, and denote by Γ a skeleton of X^{an} associated to a semistable vertex set V, that is a metric graph augmented with the genus function, as was defined in the Chapter 2. If the genus $g(X) \geq 1$, the canonical divisor K_Γ of Γ has non-negative degree $2g-2$, and the measure associated to it by Zhang's theorem is called the *canonical admissible measure*. It coincides with the limit of measures defined by Bloch, Gillet and Soulé [**42**]. We denote it by μ_{ad}.

Zhang provides in addition the following explicit formula for μ_{ad}.

THEOREM 63 (Zhang [**248**]). *Notations as above, we have*

$$\mu_{\mathrm{ad}} = \frac{1}{g}\sum_{x\in\Gamma} g_x\,\delta_x + \frac{1}{g}\sum_{e\in E}\frac{1}{\ell_e+\rho_e}d\theta_e,$$

where g is the genus of X, $g_.$ is the genus function, and for any edge e in E (the set of edges of the graph model of Γ), $d\theta_e$ denotes the Lebesgues measure on the interval $I_e \subset \Gamma$ associated to the edge e, and ρ_e denotes the effective resistance between the two end points of e in $\Gamma \setminus e$.

Here the metric graph $\Gamma\setminus e$ is defined by removing the interior of the edge e from Γ. The effective resistance ρ_e in $\Gamma\setminus e$ between the two points u and v of the edge $e = \{u,v\}$ is formally defined as $\rho_e = g^e_u(v,v)$, where $g^e_z(z,y)$ is the Green function on $\Gamma\setminus e$, as described above.

4.3.2. Description of the admissible measure in terms of spanning trees. It follows from the probabilistic theory of electrical networks [**181, 21**] that the coefficients of $d\theta_e$ admit the following probabilistic interpretation in terms of the spanning trees of the simple graph model G.

Denote the set of spanning trees of G by $\mathcal{ST}$. The length function ℓ allows to define a probability measure on the set of spanning trees of G as follows. We define the weight of a spanning tree T to be equal to $\lambda(T) := \prod_{e\notin T}\ell_e$. The *partition function* is then defined by $\Phi = \sum_{T\in\mathcal{ST}}\lambda(T)$, and coincides with the evaluation at ℓ of the first Symanzik polynomial studied later in Chapter 5. The probability measure on spanning trees is then defined by $\sum_{T\in\mathcal{ST}}\frac{\lambda(T)}{\Phi}\delta_T$. With this description we have

$$\mu_{\mathrm{ad}} = \frac{1}{g}\delta_{K_g} + \frac{1}{g}\sum_{e\in E}\Pr(e\notin T)d\theta_e,$$

where T denotes a random spanning tree chosen with respect to the above probability measure.

For an edge $e=\{u,v\}$, the probability $\Pr(e\notin T)$ is also the probability that a certain Markovian random walk in G, defined by the length function, starting at u arrives at v for the first time without passing through the edge e [**181, 21**].

It is now clear from this discussion that μ_{ad} does not have support on the interior of any bridge edge of G.

As a consequence, it follows that the measure $\iota_*(\mu_{\mathrm{ad}})$ is a well-defined measure on X^{an} independent of the choice of a skeleton Γ of X^{an}, where $\iota:\Gamma\hookrightarrow X^{\mathrm{an}}$ denotes the natural inclusion of Γ in X^{an}. By an abuse of the notation, we denote that measure by μ_{ad}.

4.4. Statement of the equidistribution theorem

Let K be a complete algebraically closed non-Archimedean field (all fields are supposed to be of characteristic zero), and let X be a smooth proper curve of positive genus g over K. Let L be an ample line bundle on X, and denote by $\mathcal{W}_n=\sum_{x\in X(K)}w_{n,x}(x)$ the Weierstrass divisor of $L^{\otimes n}$, that we view as a divisor with support in the set of Zariski points $X(K)\subset X^{\mathrm{an}}$. Consider the discrete measure

$$\frac{1}{\deg(\mathcal{W}_n)}\delta_{\mathcal{W}_n} = \frac{1}{\deg(\mathcal{W}_n)}\sum_{x\in X(\mathbb{K})\subset X^{\mathrm{an}}} w_{n,x}\delta_x$$

supported on a finite set of points of type I in X^{an}. We have

Theorem 64 ([**Ami1**]). *Notations as above, the measures* $\frac{1}{\deg(\mathcal{W}_n)}\delta_{\mathcal{W}_n}$ *converge weakly to the canonical admissible measure* μ_{ad} *on* X^{an}.

The analytic curve X^{an} comes with a retraction $\tau: X^{\mathrm{an}}\to\Gamma$ to a skeleton Γ of X^{an}. In order to prove Theorem 64, it will be enough to fix a skeleton Γ of X^{an} and prove the following equidistribution theorem. Denote by $W_n=\tau_*(\mathcal{W}_n)$ the reduction of $\mathcal{W}_n$ on Γ. Define the discrete measure $\mu_n := \frac{1}{\deg(W_n)}\delta_{W_n}$.

Theorem 65 ([**Ami1**]). *Notations as above, the measures* μ_n *converge weakly to the canonical admissible measure* μ_{Zh} *on* Γ.

Remark 66. Although in the presence of bridge edges, the asymptotic distribution does not have any support in the bridge edges of Γ, we still believe that Weierstrass points should be dense in the Berkovich space, i.e., we should have the analogue of Olsen's theorem in the non-Archimedean setting [**199**].

We now reformulate the above theorems for curves defined over discrete valuation fields. This in particular applies to the relative situation, where we have a family of smooth proper algebraic curves over a one dimensional base $B/\mathbb{C}$, and we look at the behaviour of the Weierstrasss points when we approach a point in the boundary $\overline{B}\setminus B$, for $\overline{B}$ a compactification of B.

Let K be a discretely valued field with an algebraically closed residue field of characteristic zero. Let X be a smooth proper curve of genus $g>0$ over K. Let $\mathfrak{X}$ be a regular semistable model of X over R', the valuation ring of a finite extension K' of K, and denote by G the dual graph of the special fiber $\mathfrak{X}_s$ of $\mathfrak{X}$. Let Γ be the metric realisation of the edge-weighted graph (G, ℓ) with the length $\ell(e)$ of an edge e is defined to be $\frac{1}{[K':K]}$. (So each edge e of G is replaced by a segment of length $\ell(e)$ in the metric graph Γ). Let $\overline{K}$ be an algebraic closure of K, and denote by $\tau : X(\overline{K}) \to \Gamma$ the specialisation map [**71**]. Let now L be a line bundle of positive degree on X, and denote by $\mathcal{W}_n = \sum_{x \in X(\bar{K})} w_{n,x}(x)$ the Weierstrass divisor of $L^{\otimes n}$, and let $\mu_n = \frac{1}{\deg(\mathcal{W}_n)} \delta_{\tau_*(\mathcal{W}_n)}$. We have

THEOREM 67. *The measures μ_n converge to the Zhang measure on Γ.*

A special case of the above theorem, which seems to be also new, is the following.

COROLLARY 68. *Let X be a smooth proper curve of genus $g > 0$ over a discrete valuation field K of equicharacteristic zero. Let $\mathfrak{X}$ be a regular semistable model of X over R', the valuation ring of a finite extension K' of K. Let L be a line bundle of positive degree on X and denote by $\mathcal{W}_n$ (by an abuse of the notation) the (multi)set of Weierstrass points of $L^{\otimes n}$. Let $\mathfrak{C}_0$ be an irreducible component of the special fiber $\mathfrak{X}_s$ of genus g_0, and denote by $\mathcal{W}_{0,n}$ the (multi)set of Weierstrass points whose specialisation lie on $\mathfrak{C}_0$. Then $|\mathcal{W}_{0,n}|/|\mathcal{W}_n|$ tends to g_0/g.*

The proof of the above theorem uses two main ingredients: first, limit linear series as described in Chapter 2 which allows to give an explicit description of the reduction of Weierstrass divisors on skeleta, and second, the theory of Okounkov bodies [**149, 165**]. Furthermore, the appearance of Okounkov bodies in the proof seems to reflect a deeper phenomenon that merits further investigation. We think in particular that it is possible to associate polytopes to (divisorial) points of higher dimensional Berkovich spaces whose variations provide interesting measures on the analytic space.

We briefly discuss these ingredients.

4.5. Reduction of the Weierstrass divisor

A key role in the proof of our equidistribution theorem is played by slope structures on metric graphs, described in the previous chapter. Recall that a slope structure $\mathfrak{S}$ of rank r is the data of strictly increasing integers numbers $s_0^\nu < \cdots < s_r^\nu$ associated to any unit tangent directions ν at any point x, which satisfy certain natural conditions. The following theorem provides a description of the reduction of Weierstrass divisors on skeleta.

Let $\mathcal{D}$ be a divisor of degree d on X and rank r, and denote by $L = \mathcal{O}(\mathcal{D})$ the corresponding line bundle. Denote by $\mathcal{W}$ the Weierstrass divisor of L on X. Let Γ be a skeleton of X^{an}, $D = \tau_*(\mathcal{D})$ the reduction of $\mathcal{D}$ on Γ, and let $\mathfrak{S}$ be the slope structure of rank r induced on Γ by the specialisation theorem of the previous chapter. Write $D = \sum_{x \in \Gamma} d_x(x)$.

THEOREM 69 ([**Ami1**]). *We have $\tau_*(\mathcal{W}) = \sum_{x \in \Gamma} c_x(x)$, where the coefficient c_x of a point x has the following expression:*

$$c_x = (r+1)d_x + \frac{r(r+1)}{2}(2g_x - 2 + \mathrm{val}(x)) - \sum_{\nu \in \mathrm{T}_x^1\Gamma} \sum_{i=0}^{r} s_i^\nu.$$

In the above theorem g_x and $\mathrm{val}(x)$ denote the genus and the valence of x in Γ, respectively, $\mathrm{T}_x^1(\Gamma)$ denotes the set of unit tangent directions to Γ at x, and s_i^ν are the integers underlying the definition of the slope structure $\mathfrak{S}$ on Γ. The proof of the theorem is obtained by applications of appropriate local-global versions of the Poincaré-Leong formula.

The above result refines a previous result of Esteves-Medeiros [**104**], and answers a question raised by Eisenbud-Harris [**99**] in the eighties on the reduction of Weierstrass points on a (semistable) degenerating family of curves over complex numbers. Indeed, any family of smooth proper curves $\pi : \mathcal{C} \to \Delta^*$ over the punctured disc gives rise to a smooth proper curve over the discrete valued field $\mathbb{C}((t))$, and the metric graph Γ is the metric realisation of the weighted dual graph of a semistable extension of the family $\widetilde{\pi} : \widetilde{\mathcal{C}} \to \Delta$ (possibly after an étale extension of the base). In particular, denoting by $\mathcal{C}_0$ the fiber over zero of $\widetilde{\pi}$, we thus get a recipe in terms of the combinatorial data of slope structure for the number of Weierstrass points in the family which are specialised to each irreducible component of $\mathcal{C}_0$ and to each node of $\mathcal{C}_0$.

4.6. Local Okounkov bodies

A crucial tool in the proof of the equidistribution theorem above is a well-known equidistribution phenomena for Okounkov bodies [**149, 165, 52**]. To any point x of the skeleton Γ which is a point of type II in X^{an} under the embedding $\iota : \Gamma \hookrightarrow X^{\mathrm{an}}$ is associated a curve $\mathfrak{C}_x$ over κ, whose points are in bijection with the unit tangent vectors to X^{an} at x (think about semistable models and irreducible components of the special fibre). Fixing such a branch $\nu \in T^1_x\Gamma \subset T^1_x(X^{\mathrm{an}})$, and considering (the normalisation of) the slopes $s^{\nu}_{n,i}$ of the slope structure $\mathfrak{S}_n$ induced by the complete linear series of the n-th power $L^{\otimes n}$ allows to define, when n varies in $\mathbb{N}$, an interval Λ^{ν} of length $d = \deg(L)$ in $\mathbb{R}$, that we can name $[s^{\nu}_{\min}, s^{\nu}_{\max}]$. The local equidistribution theorem then asserts that the normalised slopes are equidistributed in Λ^{ν} according to the Lebesgue measure.

4.6.1. Proof of the main theorem. The local equidistribution theorem and the fact that $s_{\max} - s_{\min} = d$ allow to replace, in the limit at infinity, the sum of the slopes in Theorem 69 by an expression involving only the minimum slopes, and some constants. The theorem is then proved by using results on reduced divisors which allows to give precise bounds on the sum of the minimum slopes $s^{\nu}_{\min}$ around each vertex, making an appearance of the Laplacian of the metric graph.

CHAPTER 5

Trees, forests, Feynman amplitudes and limit of height pairing

Physicists see quantum field theory as a limit of string theory when a certain parameter α' (the square of the string length) goes to zero, see for example [**233**]. Both the papers [**ABBF, Ami4**] grew out of an attempt to understand from a mathematical perspective this idea. In this chapter, we discuss these results.

Throughout we work in space-time $\mathbb{R}^D$ with a given Minkowski bilinear form $\langle \cdot\,, \cdot\rangle$.

5.1. Stringy amplitudes

String amplitudes are integrals over the moduli space $\mathcal{M}_{g,n}$ of genus $g \geq 1$ curves with n marked points. They are associated to a fixed collection of external momenta $\underline{\mathbf{p}} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$, which are vectors in $\mathbb{R}^D$ satisfying the conservation law $\sum_{i=1}^n \mathbf{p}_i = 0$. Up to some factors carrying information about the physical process being studied, the string amplitude can be written as (see e.g. [**239**, p.182])

$$A_{\alpha'}(g, \underline{\mathbf{p}}) = \int_{\mathcal{M}_{g,n}} \exp(-i\,\alpha' \mathcal{F})\; d\nu_{g,n}. \tag{2}$$

In this expression, $d\nu_{g,n}$ is a volume form on $\mathcal{M}_{g,n}$, independent of the momenta, α' is a positive real number, which one thinks of as the square of the string length, and $\mathcal{F}\colon \mathcal{M}_{g,n} \to \mathbb{R}$ is the continuous function defined at the point $[C, \sigma_1, \dots, \sigma_n]$ of $\mathcal{M}_{g,n}$ by

$$\mathcal{F}([C, \sigma_1, \dots, \sigma_n]) = \sum_{1\leq i,j\leq n} \langle \mathbf{p}_i, \mathbf{p}_j\rangle\, \mathfrak{g}'_{\mathrm{Ar},C}(\sigma_i, \sigma_j),$$

where $\mathfrak{g}'_{\mathrm{Ar},C}$ denotes a regularised version of the canonical Green function on C so that it takes finite values on the diagonal. We now make the expression (2) more explicit.

5.1.1. Green functions on Riemann surfaces. The regularised Green function in the above expression can be describes as follows. Let C be a smooth projective complex curve, together with a smooth positive $(1,1)$-form μ. For example, if C has genus $g \geq 1$, a very natural choice for μ is the Arakelov form of the previous chapter

$$\mu_{\mathrm{Ar}} = \frac{i}{2g} \sum_{j=1}^{g} \omega_i \wedge \overline{\omega_i},$$

where $\omega_1, \dots, \omega_g$ is any orthonormal basis of the holomorphic differentials $H^0(C, \Omega^1_C)$ for the Hermitian product $(\omega, \omega') = \frac{i}{2}\int_C \omega \wedge \overline{\omega'}$.

To μ one associates a Green function $\mathfrak{g}_\mu$ as follows, see e.g. [**164**]. For a fixed point x of C, consider the differential equation

$$\partial\overline{\partial}\varphi = \pi i(\delta_x - \mu), \tag{3}$$

where δ_x is the Dirac delta distribution. It admits a unique solution

$$\mathfrak{g}_\mu(x,\cdot)\colon C \setminus \{x\} \longrightarrow \mathbb{R}$$

satisfying the following conditions:

- If we choose local coordinates in an analytic chart U, then, for fixed $x \in U$ there exists a smooth function α such that $\mathfrak{g}_\mu(x,y) = -\log|y-x| + \alpha(y)$ for any $y \in U \setminus \{x\}$.
- (Normalisation) $\int_C \mathfrak{g}_\mu(x,y)\mu(y) = 0$.

Letting x vary, we can view $\mathfrak{g}_\mu$ as a function on $C \times C \setminus \Delta$. The chosen normalisation implies that $\mathfrak{g}_\mu$ is symmetric. In addition, if μ' is another positive $(1,1)$-form on C, then it can be shown (see e.g. [**164**, Chap, II, Prop. 1.3]) that there exists a smooth function f on C such that

$$\mathfrak{g}_{\mu'}(x,y) = \mathfrak{g}_\mu(x,y) + f(x) + f(y). \tag{4}$$

The *regularised Green function* $\mathfrak{g}'_\mu\colon C \times C \to \mathbb{R}$ agrees with $\mathfrak{g}_\mu$ outside the diagonal, and is defined on Δ by

$$\mathfrak{g}'_\mu(x,x) = \lim_{x' \to x} \big(\mathfrak{g}_\mu(x',x) + \log d_\mu(x',x)\big),$$

where x' is a holomorphic coordinate in a small neighbourhood of x and d_μ denotes the distance function associated to the metric μ. The function $\mathfrak{g}_{\mathrm{Ar},C}$ which appears in the action function in the integral 2 is the regularized Green function associated to the Arakelov measure μ_{Ar} on C.

5.1.2. Height pairing on Riemann surfaces. The action function $\mathcal{F}$ can be described in terms of the height pairing between degree zero divisors on C. We briefly recall the definition and refer to [**124, 51**] for further details on the height pairing.

Let C be a smooth projective curve over the field of complex numbers and $\Sigma \subset C$ a finite set of points in C, which we also think of as a reduced effective divisor. We denote by $(\bigoplus_\Sigma \mathbb{R})^0$ the set of degree zero divisors on C with real coefficients and with support in Σ.

The inclusion $j\colon C \setminus \Sigma \hookrightarrow C$ yields an exact sequence of mixed Hodge structures:

$$0 \to H^1(C,\mathbb{Z}(1)) \xrightarrow{j^*} H^1(C\setminus\Sigma,\mathbb{Z}(1)) \longrightarrow (\bigoplus_\Sigma \mathbb{Z})^0 \to 0, \tag{5}$$

which once tensored with $\mathbb{R}$ becomes canonically split.

The Hodge filtration on $H^1(C\setminus\Sigma,\mathbb{C})$ comes from the exact sequence of sheaves

$$0 \to \Omega^1_C \to \Omega^1_C(\log\Sigma) \xrightarrow{\mathrm{Res}_\Sigma} \bigoplus_\Sigma \underline{\mathbb{C}} \to 0, \tag{6}$$

were $\mathrm{Res}_\Sigma = \sum_{p\in\Sigma} \mathrm{Res}_p$, and Res_Σ is defined, for a local section ω, by

$$\mathrm{Res}_p(\omega) = \frac{1}{2\pi i}\int_{\gamma_{p,\varepsilon}} \omega,$$

for a small suitably oriented circle $\gamma_{p,\varepsilon}$ around the point p.

From the exact sequence (6) we deduce that $F^0H^1(C\setminus\Sigma,\mathbb{C}(1)) = H^0(C,\Omega^1_C(\log\Sigma))$, from which we get a canonical map

$$\begin{aligned}(\bigoplus_\Sigma \mathbb{R})^0 &\longrightarrow H^0(C,\Omega^1_C(\log\Sigma)) \cap H^1(C\setminus\Sigma,\mathbb{R}(1))\\ \mathfrak{D} &\longmapsto \omega_{\mathfrak{D},\mathbb{R}}.\end{aligned}$$

More concretely, from (6), we see that the condition $\mathrm{Res}_\Sigma \omega_{\mathfrak{D}} = \mathfrak{D}$ determines a logarithmic differential $\omega_{\mathfrak{D}}$ only up to addition of elements in $H^0(C, \Omega^1_C)$. To fix $\omega_{\mathfrak{D}}$ uniquely, we require that $\int_\gamma \omega_{\mathfrak{D},\mathbb{R}} \in \mathbb{R}(1)$ for every real valued cycle γ in $C \setminus \Sigma$.

Let $\mathfrak{A}$ be a degree zero $\mathbb{R}$-divisor on C with support Σ and let $\omega_{\mathfrak{A},\mathbb{R}}$ the form just defined. Given another degree zero $\mathbb{R}$-divisor $\mathfrak{B}$ with disjoint support, we can find a real-valued 1-chain $\gamma_{\mathfrak{B}}$ on $C \setminus \Sigma$ such that $\mathfrak{B} = \partial \gamma_{\mathfrak{B}}$. The *archimedean height pairing* between $\mathfrak{A}$ and $\mathfrak{B}$ is the real number

$$\langle \mathfrak{A}, \mathfrak{B} \rangle := \mathrm{Re}\left(\int_{\gamma_{\mathfrak{B}}} \omega_{\mathfrak{A},\mathbb{R}} \right). \tag{7}$$

The above definition is independent of the choice of $\gamma_{\mathfrak{B}}$. Though not apparent from (7), the Archimedean height pairing is symmetric.

EXAMPLE 70. When the divisor $\mathfrak{A}$ is of the form $\mathrm{div}(f)$ for a rational function f on C, the differential $\omega_{\mathfrak{A},\mathbb{R}}$ is nothing else than $-\frac{df}{f}$, hence $\langle \mathfrak{A}, \mathrm{div}(f) \rangle = -\log |f(\mathfrak{B})|$.

Tensoring with $\mathbb{R}^D$ and using the Minkowski metric, the archimedean height pairing extends to a pairing between degree zero $\mathbb{R}^D$-valued divisors with disjoint support, which is the case of interest to us. It can be expressed in terms of Green functions. For two divisors $\mathfrak{A} = \sum \mathbf{p}_{i,1}\sigma_{i,1}$ and $\mathfrak{B} = \sum \mathbf{p}_{j,2}\sigma_{j,2}$ with coefficients in $\mathbb{R}^D$ of degree zero and with disjoint support on C, on can show that

$$\langle \mathfrak{A}, \mathfrak{B} \rangle = \sum_{i,j} \langle \mathbf{p}_{i,1}, \mathbf{p}_{j,2} \rangle \mathfrak{g}_\mu(\sigma_{i,1}, \sigma_{j,2}) \tag{8}$$

for any positive $(1,1)$-form μ on C.

Using the above expression, and replacing the Green function by its regularisation in (8), we can extend the definition of the height pairing to arbitrary $\mathbb{R}^D$-valued divisors and, in particular, define

$$\langle \mathfrak{A}, \mathfrak{A} \rangle'_\mu := \sum_{1 \le i,j \le n} \langle \mathbf{p}_i, \mathbf{p}_j \rangle \mathfrak{g}'_\mu(\sigma_i, \sigma_j).$$

Without further assumptions, the real number $\langle \mathfrak{A}, \mathfrak{A} \rangle'_\mu$ depends on the choice of μ. However, if we assume that the external momenta $\mathbf{p}_i \in \mathbb{R}^D$ satisfy the conservation law $\sum_{i=1}^n \mathbf{p}_i = 0$ and the *on shell condition* $\langle \mathbf{p}_i, \mathbf{p}_i \rangle = 0$ for all i, then $\langle \mathfrak{A}, \mathfrak{A} \rangle'_\mu$ is independent of the choice of μ, and so we can drop μ. It follows that under the same assumption on the extrenal momenta, we have the following more familiar expression for the action function $\mathcal{F}$:

$$\mathcal{F}([C, \sigma_1, \ldots, \sigma_n]) = \langle \mathfrak{A}_C, \mathfrak{A}_C \rangle',$$

for the $\mathbb{R}^D$-valued divisor $\mathcal{A}_C = \sum_j \mathbf{p}_j \sigma_j$ on C.

5.1.3. Polyakov measure. The volume form $d\nu_{g,n}$ exists in dimension $D = 26$ [**50**] and is called the Polyakov measure. It has been made explicit by the works of Belavin and Knizhnik [**30**] and of Beilinson and Manin [**29**] in the following way.

First note that we have a fibration $\pi_n : \mathcal{M}_{g,n} \to \mathcal{M}_g$, where the fibers over a point $[C]$ of $\mathcal{M}_g$ are a configuration of n points over C. Endowing the Riemann surface with the Arakelov measure, and the base space with the Polyakov measure $d\nu_g$, and the fibration π_n with the product measure, we are led to consider only the case $n = 0$.

Let $\pi : \mathcal{C} \to \mathcal{M}_g$ be the universal curve over the moduli space of curves, and consider the sheave of relative 1-forms $\omega_{\mathcal{C}/\mathcal{M}_g}$. For each $j \in \mathbb{N}$, define $\lambda_j := \det(R\pi_* \omega^{\otimes j}_{\mathcal{C}/\mathcal{M}_g})$ in the determinantal formalism of Knudson and Mumford [**155**]. By Mumford's isomorphism theorem [**196**], we have a canonical isomorphism $\lambda_{j+1} \simeq \lambda_1^{6j^2+6j+1}$. This gives rise to the Mumford form μ_g, a global section of $\lambda_2 \otimes \lambda_1^{\otimes(-13)}$ on $\mathcal{M}_g$, which can be locally written in the form

$$\mu_g = f \frac{\eta_1 \wedge \cdots \wedge \eta_{3g-3}}{(\omega_1 \wedge \cdots \wedge \omega_g)^{13}},$$

for choices of a basis $\eta_1, \ldots, \eta_{3g-3}$ of globals sections of $\omega_\cdot^{\otimes 2}$ and a basis $\omega_1, \ldots, \omega_g$ of the global sections of $\omega_\cdot^{\otimes 2}$, and a holomorphic function f which depends on the choices of local bases.

The Polyakov measure $d\nu_g$ is then the squared norm of μ_g, which can be written locally in the form

$$d\nu_g = |f|^2 i^g \frac{\eta_1 \wedge \bar{\eta_1} \wedge \cdots \wedge \eta_{3g-3} \wedge \bar{\eta}_{3g-3}}{\det\Big(\int \omega_j \wedge \bar{\omega_k}\Big)^{13}},$$

where $\Big(\int \omega_j \wedge \bar{\omega_k}\Big)$ denotes a matrix-valued form on $\mathcal{M}_g$ which at point $X \in \mathcal{M}_g$ takes the value $\Big(\int_x \omega_j \wedge \bar{\omega_k}\Big)_{j,k=1}^{g}$. Fixing a symplectic basis $a_1, \ldots, a_g, b_1, \ldots, b_g$ on the topological surface of genus g, i.e., with $(a_i, b_j) = \delta_{i,j}$, $(a_i, a_j) = (b_i, b_j) = 0$, for $i, j = 1, \ldots, g$, defines the Riemann basis $\omega_1, \ldots, \omega_g$ of $H^0(\omega_\cdot)$ with the condition $\int_{a_i} \omega_j = \delta_{i,j}$. The period matrix is given by $A = \Big(\int_{b_j} \omega_k\Big)$, and the quantity $\det(\operatorname{Im} A)$ calculates the volume of the Jacobian $J(X)$ of X with the flat metric induced by the Arakelov pairing on $H^0(\omega_X)$. By the work of Beilinson and Manin [**29**], $d\nu_g$ can be further described in the form

$$d\nu_g(X) = F(X) \frac{\eta_1 \wedge \bar{\eta_1} \wedge \cdots \wedge \eta_{3g-3} \wedge \bar{\eta}_{3g-3}}{\mathrm{Vol}\Big(J(X)\Big)^{13}},$$

for a function F which we do not explicit here.

5.2. Symanzik polynomials and Feynman amplitudes

Correlation functions in quantum field theory are calculated using Feynman amplitudes, which are certain finite dimensional integrals associated to graphs. A (massless) Feynman graph $(G, \mathbf{p})$ consists of a finite graph $G = (V, E)$, with vertex and edge sets V and E, respectively, together with a collection of external momenta $\underline{\mathbf{p}} = (\mathbf{p}_v)_{v \in V}$, $\mathbf{p}_v \in \mathbb{R}^D$, such that $\sum_{v \in V} \mathbf{p}_v = 0$.

To the Feynman graph $(G, \underline{\mathbf{p}})$ one associates two polynomials in the variables $\underline{Y} = (Y_e)_{e \in E}$. Denote by $\mathcal{ST}$ the set of all the spanning trees of the graph G. (Recall that a spanning tree of a connected graph is a maximal subgraph which does not contain any cycle. It has precisely $|V| - 1$ edges.) The first Symanzik ψ_G, which depends only on the graph G, is given by the following sum over the spanning trees of G:

$$\psi_G(\underline{Y}) := \sum_{T \in \mathcal{ST}} \prod_{e \notin T} Y_e.$$

A spanning 2-forest in a connected graph G is a maximal subgraph of G without any cycle and with precisely two connected components. Such a subgraph has precisely $|V| - 2$ edges.

Denote by $\mathcal{SF}_2$ the set of all the spanning 2-forests of G. The second Symanzik polynomial ϕ_G, which depends on the external momenta as well, is defined by

$$\phi_G(\underline{\mathbf{p}}, \underline{Y}) := \sum_{F \in \mathcal{SF}_2} q(F) \prod_{e \notin F} Y_e.$$

Here F runs through the set of spanning 2-forests of G, and for F_1 and F_2 the two connected components of F, $q(F)$ is the real number $-\langle \mathbf{p}_{F_1}, \mathbf{p}_{F_2} \rangle$, where $\mathbf{p}_{F_1}$ and $\mathbf{p}_{F_2}$ denote the total momentum entering the two connected components F_1 and F_2 of F, i.e.,

$$\mathbf{p}_{F_1} := \sum_{v \in V(F_1)} \mathbf{p}_v \qquad \mathbf{p}_{F_2} := \sum_{u \in V(F_2)} \mathbf{p}_u.$$

One of the various representations of the Feynman amplitude associated to $(G, \underline{\mathbf{p}})$ is, up to some elementary factors which we omit, a path integral on the space of metrics (i.e., edge lengths) on G with the action given by ϕ_G/ψ_G. It is given by

$$I_G(\underline{\mathbf{p}}) = C \int_{[0,\infty]^E} \exp(-i\, \phi_G/\psi_G)\; d\pi_G,$$

for a constant C, and the volume form $d\pi_G = \psi_G^{-D/2} \prod_E dY_e$ on $\mathbb{R}_+^E$, c.f. [**140**, Equation (6-89)].

We would now like to put the above integral in a form somehow similar to expression of the String theory amplitude, as we did in the previous section.

5.2.1. The Feynman measure. In Chapter 1 we explained how to associate to any metric graph Γ the Jacobian variety, which is a real torus of dimension g, the genus of Γ, endowed with a flat metric. Let G a graph, and Y_e, for $e \in E(G)$, a sequence of positive real numbers associated to the edges of G. Let $\Gamma_{\underline{Y}}$ be the associated metric graph, and $J(\Gamma_{\underline{Y}})$ the associated Jacobian. We have the following well-known theorem, which is the dual form of the matrix tree theorem, see e.g. [**14**].

THEOREM 71. *The volume of $J(\Gamma_{\underline{Y}})$ with respect to the Lebesgue measure induced from the flat metric on $J(\Gamma_{\underline{Y}})$ is equal to the value of the first Symanzik polynomial $\psi_G(\underline{Y})$.*

It follows that in dimension D=26 the Feynman measure $d\pi_G$ in the strata associated to the graph G in the moduli space of metric graphs can be rewritten in the form

$$d\pi_G = \frac{\prod_e dY_e}{\mathrm{Vol}\Big(J(\Gamma_{\underline{Y}})\Big)^{13}},$$

which is reminiscent of the form of Polyakov measure described in the previous section.

5.2.2. Height pairing on a metric graph. Let Γ be a metric graph, and consider two divisors of degree zero D_1 and D_2 on Γ. Recall that we define the height pairing, or intersection pairing, between D_1 and D_2 as follows. Consider the Laplacian operator Δ on Γ. There is a continuous piecewise linear function f_1, resp. f_2, on Γ (without necessarily integral slopes) which verify the Laplace equation

$$\Delta(f_1) = D_1\,, \text{ resp., } \Delta(f_2) = D_2.$$

We define

$$\langle D_1, D_2 \rangle := -f_1(D_2) = -f_2(D_1) = \int_\Gamma f_1' f_2' dt,$$

where dt denotes the Lebesgue measure on Γ. (Recall that we use the convention $f(D) = \sum_{x\in\Gamma} f(x)D(x)$ for a function f and a divisor D on Γ.)

The definition can be extended to divisors with coefficients in $\mathbb{R}^D$ using the Minkowski bilinear form on $\mathbb{R}^D$.

Let now G be a Feynman graph, $\underline{Y} \in \mathbb{R}_+^E$, and $\underline{\mathbf{p}}$ a collection of external momenta associated to the vertices of G. Consider the divisor $D_{\underline{\mathbf{p}}} = \sum_{v\in V(G)} \mathbf{p}_v(v)$ on Γ with coefficients in $\mathbb{R}^D$. Let $\Gamma_{\underline{Y}}$ be the metric graph associated to G with edge lengths $\underline{\mathbf{p}}$. We have the following theorem.

THEOREM 72. *Notations as above, we have*

$$\langle D_{\underline{\mathbf{p}}}, D_{\underline{\mathbf{p}}}\rangle = \frac{\phi_G(\underline{\mathbf{p}}, \underline{Y})}{\psi_G(\underline{\mathbf{p}})},$$

for the height pairing in the metric graph $\Gamma_{\underline{Y}}$.

This is reminiscent of the form of the action in the string amplitudes.

5.3. Formulation of the problem

Given the form of the integrals detailed above, despite the fact that both amplitudes diverge in general, one may still ask if $I_G(\underline{\mathbf{p}})$ is related to the asymptotic of $A_{\alpha'}(g, \underline{\mathbf{p}})$ when α' goes to zero, as physics suggests. The graph G appears as the dual graph of a stable curve C_0 with n marked points lying on the boundary of the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ (recall that the irreducible components X_v of C_0 are indexed by the vertices of G, whereas the singular points correspond to the edges). Given the form of the amplitudes, the question can be then split into two different problems, namely

(i) the convergence of the integrands, or equivalently, the convergence of the archimedean heigh pairing to the height pairing in the corresponding metric graph, and
(ii) the convergence of the measure $\nu_{g,n}$, in an appropriate sense, and along the boundary of $\overline{\mathcal{M}}_{g,n}$, to a linear combination of the measures π_G for the (marked) dual graphs G associated to the strata.

Our main result in [**ABBF**] answers question (i) in the affirmative when the external momenta satisfy the on shell condition: the integrand in string theory converges indeed to the integrand appearing in the Feynman amplitude. The question (ii) is more subtle. The volume forms π_G on cones $\mathbb{R}_+^{E(G)}$ can be glued together to define a volume form on the tropical moduli space $\mathcal{M}_{g,n}^{\mathrm{trop}}$. We refer to [**2, 63, 67**] for tropical moduli spaces. Nevertheless, it is not clear how the moduli space $\mathcal{M}_{g,n}$ endowed with the string theory volume form can *converge* to the tropical moduli space $\mathcal{M}_{g,n}^{\mathrm{trop}}$ with the Feynman measure. We mention that a related question in the context of degenerations of varieties endowed with volume forms is considered in the recent work of Boucksom and Jonsson, see [**53**]. However, our context is somehow different: Mumford form has poles of order two at infinity [**31, 246**], there is no dependence of the volume form on $\mathcal{M}_{g,n}$ on α', and it is not clear how to give a rigorous sense to the convergence of measures. This and other related questions are currently under investigation.

In the next section, we discuss the answer to question (i).

5.4. The limit of height pairing

Let C_0 be a stable curve of genus g lying on the boundary of $\overline{\mathcal{M}}_{g,n}$, and let $G = (V, E)$ be the dual graph of C_0. Consider the versal analytic deformation $\pi\colon \mathcal{C}' \to S'$ of the marked curve C_0, which we think of as a smooth neighborhood of C_0 in the analytic stack $\overline{\mathcal{M}}_{g,n}$. Here S' is a polydisc of dimension $3g-3+n$, the total space $\mathcal{C}'$ is regular and $\mathcal{C}'_0$, the fibre of π at 0, is isomorphic to C_0. For each edge $e \in E$, let $D_e \subset S'$ denote the divisor parametrising those deformations in which the point associated to e remains singular. Then $D = \bigcup_{e\in E} D_e$ is a normal crossings divisor whose complement $U' = S' \setminus D$ can be identified with $(\Delta^*)^E \times \Delta^{3g-3-|E|+n}$. Over U', the fibres $\mathcal{C}'_s$ are smooth curves of genus g. Moreover, the versal family comes together with n disjoint sections $\sigma_i\colon S' \to \mathcal{C}'$ which do not meet the double points of C_0. We denote by $\underline{\mathbf{p}}^G = (\mathbf{p}^G_v)_{v\in V}$ the *restriction* of $\underline{\mathbf{p}}$ to G. By this we mean that, for each vertex $v \in V$, the external momentum $\mathbf{p}^G_v$ is obtained by summing those $\mathbf{p}_i$ associated to the sections σ_i which meet C_0 on the irreducible component X_v.

We define an *admissible segment* in [**ABBF**] as a continuous maps $\underline{t}\colon [0,\varepsilon] \to S'$ from an interval of length $\varepsilon > 0$ such that $\underline{t}((0,\varepsilon]) \in U'$ and, letting t_e denote the coordinate corresponding to $e \in E$ in the factor $(\Delta^*)^E$ of U', the limit $\lim_{\alpha'\to 0} |t_e(\alpha')|^{\alpha'}$ exists and belongs to $(0,1)$. To any admissible segment we attach a collection $\underline{Y} = (Y_e)_{e\in E}$ of positive real numbers (the edge lengths) as follows:

$$Y_e = -\lim_{\alpha'\to 0} \log |t_e(\alpha')|^{\alpha'}.$$

The main theorem of [**ABBF**] can be stated as follows.

THEOREM 73 ([**ABBF**]). *Let C_0 be a stable curve of genus $g \geq 1$ with n marked points $\sigma_1, \ldots, \sigma_n$ and dual graph $G = (V, E)$, and let $\underline{\mathbf{p}} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ be a collection of external momenta satisfying the conservation law $\sum_{i=1}^n \mathbf{p}_i = 0$ and the "on shell" condition $\langle \mathbf{p}_i, \mathbf{p}_i\rangle = 0$ for all i. Then, for any admissible segment $\underline{t}\colon I \to \overline{\mathcal{M}}_{g,n}$ such that $\underline{t}(0) = [C_0, \sigma_1, \ldots, \sigma_n]$, we have*

$$\lim_{\alpha'\to 0} \alpha' \mathcal{F}(\underline{t}(\alpha')) = \frac{\phi_G(\underline{\mathbf{p}}^G, \underline{Y})}{\psi_G(\underline{Y})},$$

where $\underline{Y} = (Y_e)_{e\in E}$ denotes the edge lengths determined by $\underline{t}$.

We derive the above theorem from results describing the asymptotic behaviour of the archimedean height pairing. We start with the case of disjoint divisors. We consider the versal analytic deformation $\pi\colon \mathcal{C} \to S$ of C_0 (without the marked points), which we think of as a smooth neighborhood of C_0 in the analytic stack $\overline{\mathcal{M}}_g$. Now S is a polydisc of dimension $3g-3$. Again the total space $\mathcal{C}$ is regular and we repeat the construction above letting $D_e \subset S$ denote the divisor parametrising those deformations in which the point associated to e remains singular. Then $D = \bigcup_{e\in E} D_e$ is a normal crossing divisor whose complement $U = S \setminus D$ can be identified with $(\Delta^*)^E \times \Delta^{3g-3-|E|}$. To accommodate external momenta, we assume that we are given two collections of sections of π, which we denote by $\sigma_1 = (\sigma_{\ell,1})_{\ell=1,\ldots,n}$ and $\sigma_2 = (\sigma_{\ell,2})_{\ell=1,\ldots,n}$. Since $\mathcal{C}$ is regular, the points $\sigma_{l,i}(0)$ lie on the smooth locus of C_0. We label the markings with two vectors $\underline{\mathbf{p}}_1 = (\mathbf{p}_{l,1})_{l=1}^n$ and $\underline{\mathbf{p}}_2 = (\mathbf{p}_{l,2})_{l=1}^n$ with $\mathbf{p}_{l,i} \in \mathbb{R}^D$ subject to the conservation of momentum, thus obtaining a pair of relative degree zero $\mathbb{R}^D$-valued divisors

$$\mathfrak{A}_s = \sum_{l=1}^{n} \mathbf{p}_{l,1}\sigma_{l,1}, \qquad \mathfrak{B}_s = \sum_{l=1}^{n} \mathbf{p}_{l,2}\sigma_{l,2}.$$

Assume first that σ_1 and σ_2 are disjoint on each fiber of π. Archimedean height pairing on each fiber of π thus gives a real-valued function

$$s \mapsto \langle \mathfrak{A}_s, \mathfrak{B}_s \rangle. \tag{9}$$

For each $e \in E$, denote by s_e the coordinate in the factor corresponding to e in $U = (\Delta^*)^E \times \Delta^{3g-3-|E|}$, write $y_e = \frac{-1}{2\pi} \log |s_e|$ and put $\underline{y} = (y_e)_{e\in E}$. After shrinking U if necessary, the asymptotic of the height pairing is given by the following result:

THEOREM 74 ([**ABBF**]). *Assume, as above, that σ_1 and σ_2 are disjoint on each fibre. Then there exists a bounded function $h\colon U \to \mathbb{R}$ such that*

$$\langle \mathfrak{A}_s, \mathfrak{B}_s \rangle = 2\pi \frac{\phi_G(\underline{\mathbf{p}}_1^G, \underline{\mathbf{p}}_2^G, \underline{y})}{\psi_G(\underline{y})} + h(s). \tag{10}$$

Theorem 74 only deals with disjoint sections. In order to derive Theorem 73 we need to allow the supports of the divisors to intersect, which requires a regularisation of the height pairing, as was previously discussed. Pointe-wise, the regularisation depends on the choice of a metric on the tangent space of the given curve, so to regularise the height pairing globally we choose a smooth $(1,1)$-form μ on $\pi^{-1}(U)$ such that the restriction to each fibre $\mathcal{C}_s$ is positive. Using μ we define a regularised height pairing $\langle \mathfrak{A}_s, \mathfrak{A}_s \rangle'_\mu$. In the case where $g \geq 1$ and μ is the Arakelov metric μ_{Ar} we recover the function $\mathcal{F}$ from the string amplitude:

$$\mathcal{F}([\mathcal{C}_s, \sigma_1(s), \dots, \sigma_n(s)]) = \langle \mathfrak{A}_s, \mathfrak{A}_s \rangle'_{\mu_{\mathrm{Ar}}}.$$

The asymptotic of the regularised height pairing is described in the following result.

THEOREM 75 ([**ABBF**]). *Let $\underline{\mathbf{p}} = (\mathbf{p}_i)_{i=1,\dots,n}$ be external momenta satisfying the conservation law and $(\sigma_i)_{i=1,\dots,n}$ a collection of sections $\sigma_i\colon S \to \mathcal{C}$. Put $\mathfrak{A}_s = \sum \mathbf{p}_i \sigma_i(s)$, and let μ be a smooth $(1,1)$-form on $\pi^{-1}(U)$ whose restriction to each curve $\mathcal{C}_s$ is positive. Assume that one of the following conditions hold:*

(1) μ extends to a continuous $(1,1)$-form on $\mathcal{C}$, or
(2) the $\mathbf{p}_i$ satisfy the "on shell" condition $\langle \mathbf{p}_i, \mathbf{p}_i \rangle = 0$.

Then there exists a bounded function $h\colon U \to \mathbb{R}$ such that

$$\langle \mathfrak{A}_s, \mathfrak{A}_s \rangle'_\mu = 2\pi \frac{\phi_G(\underline{\mathbf{p}}^G, \underline{y})}{\psi_G(\underline{y})} + h(s).$$

Theorem 73 can be easily obtained from Theorem 75. Indeed, for any admissible segment $\underline{t}\colon I \to U'$, we have

$$\begin{aligned}\lim_{\alpha'\to 0} \alpha' \mathcal{F}(\underline{t}(\alpha')) &= \lim_{\alpha'\to 0} \left[\frac{\phi_G(\underline{\mathbf{p}}^G, (-\log |t_e(\alpha')|^{\alpha'})_e)}{\psi_G((-\log |t_e(\alpha')|^{\alpha'})_e)} + \alpha' h(s) \right] \\ &= \frac{\phi_G(\underline{\mathbf{p}}^G, \underline{Y})}{\psi_G(\underline{Y})},\end{aligned}$$

where we use that ϕ_G/ψ_G is homogeneous of degree one, and that h is bounded.

REMARK 76. To compute the quantum field theory amplitude for "off shell" momenta as a limit of heights, the surprising "on shell" condition in Theorem 73 can be avoided by simply taking momenta $\underline{\mathbf{p}}^G = \underline{\mathbf{p}}_1^G = \underline{\mathbf{p}}_2^G$ and disjoint multisections σ_1, σ_2 which have the same intersection data with components of the curve at infinity, by using (10) and the above limit calculation.

The proofs of Theorems 74 and 75 are based on the Hodge theoretic interpretation of the Archimedean height. Since both sides of the equality (10) are bilinear in the momenta, we can reduce to the case of integer-valued divisors. Then $\mathfrak{A}_s$ and $\mathfrak{B}_s$ define a biextension mixed Hodge structure $H_{\mathfrak{B}_s,\mathfrak{A}_s}$ with graded pieces $\mathbb{Z}(1), H^1(\mathcal{C}_s, \mathbb{Z}(1)), \mathbb{Z}(0)$, see [**124**]. The moduli space of such biextensions is the $\mathbb{C}^\times$-bundle associated to the Poincaré line bundle over $J(\mathcal{C}_s) \times \widehat{J(\mathcal{C}_s)}$, and one recovers the archimedean height by evaluating its canonical metric at $H_{\mathfrak{B}_s,\mathfrak{A}_s}$. As s varies, the biextensions $H_{\mathfrak{B}_s,\mathfrak{A}_s}$ fit together into an admissible variation of mixed Hodge structures over U. We are able to write down explicitly the period map

$$\widetilde{\Phi}\colon \widetilde{U} \longrightarrow \mathcal{H}_g \times \mathrm{Row}_g(\mathfrak{C}) \times \mathrm{Col}_g(\mathfrak{C}) \times \mathfrak{C},$$

where $\widetilde{U}$ is the universal cover of U and $\mathbb{H}_g$ the Siegel upper half-space. If $\mathcal{P}^\times$ denotes the Poincaré bundle over the universal family of abelian varieties and their duals, $\widetilde{\Phi}$ descends to the map $\Phi\colon U \to \mathcal{P}^\times$ which sends s to $H_{\mathfrak{B}_s,\mathfrak{A}_s}$. Then (a weak version of) the nilpotent orbit theorem for variations of mixed Hodge structures (c.f. [**207, 146, 205**]) allows us to describe the asymptotic of the height pairing.

We note that the asymptotic of the height pairing between algebraic cycles gives rise to interesting questions, and has been studied by other authors, see, e.g., [**124, 130, 136, 166, 206, 60**]. In particular, an analogue of theorem 74 for families of curves over a one-dimensional base was established by Holmes and de Jong in [**136**]. The question of obtaining an explicit description of the asymptotics, in terms of a non-archimedean data, as we do for the case of dimension one, is quite interesting that we are currently investigating.

5.5. The exchange graph and the stability of the ratio of the two Symanzik polynomials

The proof of Theorem 74, once the period map is explicitly written down and the nilpotent orbit theorem applied, is reduced to an interesting stability phenomenon concerning the ratio of the two Symanzik polynomials, which we show to be surprisingly related to a combinatorial result about the exchange properties between spanning trees and spanning 2-forests of a given graph. In this section, we describe the results of that paper. It is interesting to mention that the combinatorial results and the stability theorem described below, as well as the definition of Symanzik polynomials, can be generalised to higher dimensional simplicial complexes and matroids as it has been shown by my student Matthieu Piquerez [**209**]. The result in this section can be thus regarded as belonging to the general framework of the *geometry of polynomials and rational functions associated to discrete combinatorial structures* along other recent works such as [**44, 57, 221**].

5.5.1. Determinantal representation of the Symanzik polynomials. Let $G = (V, E)$ be a finite connected graph on the set of vertices V of size n and with the set of edges $E = \{e_1, \dots, e_m\}$ of size m. Denote by g the genus of G, which is by definition the integer $g = m - n + 1$.

Let R be a ring of coefficients (that we will later assume to be either $\mathbb{R}$ or $\mathbb{Z}$), and consider the free R-module $R^E \simeq R^m = \left\{ \sum_{i=1}^m a_i e_i \mid a_i \in R \right\}$ of rank m generated by the elements of E. For any element $a \in R^E$, we denote by a_i the coefficient of e_i in a.

Any edge e_i in E gives a bilinear form of rank one $\langle . , . \rangle_i$ on R^m by the formula

$$\langle a, b \rangle_i := a_i b_i.$$

Let $\underline{y} = \{y_i\}_{e_i \in E}$ be a collection of elements of R indexed by E, and consider the symmetric bilinear form $\alpha = \langle . , . \rangle_{\underline{y}} := \sum_{e_i \in E} y_i \langle . , . \rangle_i$. In the standard basis $\{e_i\}$ of R^E, α is the diagonal matrix with y_i in the i-th entry, for $i = 1, \dots, m$. We denote by $Y := \mathrm{diag}(y_1, \dots, y_m\}$ this diagonal matrix.

Let $H \subset R^E$ be a free R-submodule of rank r. The bilinear form α restricts to a bilinear form $\alpha_{|H}$ on H. Fixing a basis $B = \{\gamma_1, \dots, \gamma_r\}$ of H over R, and denoting by M the $r \times m$ matrix with row vectors γ_i written in the standard basis $\{e_i\}$ of R^E, the restriction $\alpha_{|H}$ can be identified with the symmetric $r \times r$ matrix MYM^τ so that for two vectors $c, d \in R^r \simeq H$ with $a = \sum_{j=1}^r c_j \gamma_j$ and $b = \sum_{j=1}^r d_j \gamma_j$, we have

$$\alpha(a, b) = cMYM^\tau d^\tau.$$

The Symanzik polynomial $\psi(H, \underline{y})$ associated to the R-submodule $H \hookrightarrow R^E$ is defined as

$$\psi(H, \underline{y}) := \det(MYM^\tau).$$

Note that since the coordinates of MYM^τ are linear forms in $y_1, \dots, y_m$, $\psi(H, \underline{y})$ is a homogeneous polynomial of degree r in y_is.

For a different choice of basis $B' = \{\gamma'_1, \dots, \gamma'_r\}$ of H over R, the determinant gets multiplied by an element of $R^{\times 2}$. It follows that $\psi(H, \underline{y})$ is well-defined up to an invertible element in $R^{\times 2}$. In particular, if $R = \mathbb{Z}$, the quantity $\psi(H, \underline{y})$ is independent of the choice of the basis and is therefore well-defined.

We now fix an orientation on the edges of the graph. We have a boundary map $\partial : R^E \to R^V$, $e \mapsto \partial^+(e) - \partial^-(e)$, where ∂^+ and ∂^- denote the head and the tail of e, respectively. The homology of G is defined via the exact sequence

$$0 \to H_1(G, R) \to R^E \xrightarrow{\partial} R^V \to R \to 0. \tag{11}$$

The homology group $H = H_1(G, R)$ is a free submodule of R^m free of rank g, the genus of the graph G, for any ring R. In particular, fixing a basis B of $H_1(G, \mathbb{Z})$, the polynomial

$$\psi_G(\underline{y}) := \psi(H, \underline{y})$$

is independent of the choice of B. Writing M for the $g \times m$ matrix of the basis B in the standard basis $\{e_i\}$ of R^E, one has by definition $\psi_G(\underline{y}) = \det(MYM^\tau)$. It follows from the Kirchhoff's matrix-tree theorem [**153**] that

$$\psi_G(\underline{Y}) = \sum_{T \in \mathcal{ST}} \prod_{e \notin T} Y_e,$$

which is the form of the first Symanzik polynomial given at the beginning of this section.

The exact sequence (11) yields an isomorphism $R^E/H \simeq R^{V,0}$, where $R^{V,0}$ consists of those $x \in R^V$ whose coordinate sum to zero.

Let now $\mathbf{p} \in R^{V,0}$ be a non-zero element, and let ω be any element in $\partial^{-1}(\mathbf{p})$. Denote by $H_\omega = \partial^{-1}(R.\mathbf{p}) = H + R.\omega$, and note that H_ω is a free R-module of rank $g+1$ which comes with the basis $B_\omega = B \sqcup \{\omega\}$.

The second Symanzik polynomial of $(G, \underline{\mathbf{p}})$ is defined by

$$\phi_G(\underline{\mathbf{p}}, \underline{y}) := \psi(H_\omega, \underline{y})$$

for the element $\omega \in R^E$ with $\mathbf{p} = \partial(\omega)$. The polynomial $\phi_G(\underline{\mathbf{p}}, \underline{y})$ is homogeneous of degree $g+1$ in y_i's, independent of the choice of the element $\omega \in \partial^{-1}(\underline{\mathbf{p}})$, c.f. [**ABBF**]. Writing N for the $(g+1) \times m$ matrix for the the basis B_ω in the standard basis of R^E, we see that

$$\phi_G(\underline{\mathbf{p}}, \underline{y}) = \det(NYN^\tau).$$

We have the following expression for the second Symanzik polynomial, c.f. e.g. [**65**] or [**209**],

$$\phi_G(\underline{\mathbf{p}}, \underline{y}) = \sum_{F \in \mathcal{SF}_2} q(F) \prod_{e \notin E(F)} y_e\,,$$

which is precisely the form of the second Symanzik polynomial given previously.
The definition can be extended to $\mathbf{p} \in \mathbb{R}^D$ using the Minkowski bilinear form on $\mathbb{R}^D$ [**ABBF**].

5.5.2. Statement of the main theorem. Let U be a topological space and $y_1, \ldots, y_m : U \to \mathbb{R}_{>0}$ be m continuous functions. Let $\mathbf{p} \in (\mathbb{R})^{V,0}$ be a fixed vector, and let $\psi_G(\underline{y}) : U \to \mathbb{R}_{>0}$ and $\phi_G(\mathbf{p}, \underline{y}) : U \to \mathbb{R}_{>0}$ be the real-valued functions on U defined by the first and second Symanzik polynomials.

Notation. We will use the following terminology in what follows: for two real-valued functions F_1 and F_2 defined on U, we write $F_1 = O_{\underline{y}}(F_2)$ if there exist constants $c, C > 0$ such that $|F_1(s)| \leq c|F_2(s)|$ on all points $s \in U$ which verify $y_1(s), \ldots, y_m(s) \geq C$.

Let $A : U \to \mathrm{Mat}_{m\times m}(\mathbb{R})$ be a matrix-valued map taking at $s \in U$ the value $A(s)$. Assume that A verifies the following two properties

(i) A is a bounded function, i.e., all the entries $A_{i,j}$ of A take values in a bounded interval $[-C, C]$ of $\mathbb{R}$, for some positive constant $C > 0$.
(ii) The two matrices $M(Y + A)M^\tau$ and $N(Y + A)N^\tau$ are invertible on U.

One might view the contribution of A as a perturbation of the standard scalar product on the edges of the graph given by the (length) functions $y_1, \ldots, y_m$, which can be further regarded as changing the geometry of the graph, seen as a discrete metric space. The main result of our paper [**Ami4**] is the following.

THEOREM 77 ([**Ami4**]). *Assume $A : U \to \mathrm{Mat}_{m\times m}(\mathbb{R})$ verifies the condition (i) and (ii) above. The difference $\frac{\det(N(Y+A)N^\tau)}{\det(M(Y+A)M^\tau)} - \frac{\det(NYN^\tau)}{\det(MYM^\tau)}$ is $O_{\underline{y}}(1)$.*

This result might appear somehow surprising, given that the rational functions which appear in the expression above are of degree one. Moreover, simple examples of rational functions of negative degree such as y_1/y_2^2 show that depending on the relative size of the different parameters, the behaviour at infinity can be very irregular. E.g., in the example y_1/y_2, if y_1 grows at any rate faster than y_2^2, then the ratio is unbounded at infinity. The content of the theorem is thus a strong stability theorem at infinity of the ratio of the two Symanzik polynomials.

We note that a similar result has been proved using different tools in a recent paper of Burgos, Holmes and de Jong [**62**] in the setting of what is called *normlike functions*. On the other hand, as we proviously mentioned, the result holds in much more generality as it has been worked out by Matthieu Piquerez [**209**], using the combinatorial approach of [**Ami4**] that we describe now.

5.5.3. The exchange graph. Our proof of the above theorem is indirect and completely combinatorial. We define an auxiliary graph which encodes the exchange properties between the spanning trees and spanning 2-forests in G. We call this graph the exchange graph. Exchange properties between spanning trees are well-understood and part of the more general theory of matroids. The exchange property between spanning trees of a given graph assert that for the distinct spanning trees T_1 and T_2 of a graph G, there exists an edge e_1 of T_1 and an edge e_2 of T_2 such that $T_1 + e_2 - e_1$ is a spanning tree of G. In particular, this shows that if we define a graph with vertices consisting of all the spanning trees of G and with edges consisting of all the pairs $\{T_1, T_2\}$ with $T_2 = T_1 - e + f$ for two edges $e \in E(T_1)$ and $f \notin E(T_1)$, then we have a connected graph. The idea of the exchange graph is similar. We consider the exchange properties between pairs of spanning trees and spanning 2-forests.

Let $G = (V, E)$ be a connected multigraph with vertex set V and edge set E. To exchange graph $\mathscr{H}$ of G is defined as follows. The vertex set $\mathscr{V}$ of $\mathscr{H}$ is the disjoint union of two sets $\mathscr{V}_1$ and $\mathscr{V}_2$, where

$$\mathscr{V}_1 := \Big\{ (F,T) \,\big|\, F \in \mathcal{SF}_2(G), T \in \mathcal{ST}(G)\,,\, E(F) \cap E(T) = \emptyset \Big\},$$

and

$$\mathscr{V}_2 := \Big\{ (T,F) \,\big|\, T \in \mathcal{ST}(G), F \in \mathcal{SF}_2(G)\,,\, E(F) \cap E(T) = \emptyset \Big\}.$$

There is an edge in $\mathscr{E}$ connecting $(F,T) \in \mathscr{V}_1$ to $(T', F') \in \mathscr{V}_2$ if there is an edge $e \in E(T)$ such that $F' = T - e$ and $T' = F + e$.

Our result in [**Ami4**] gives a complete description of the connected components of $\mathscr{H}$.

Let $\mathscr{H}_0 = (\mathscr{V}_0, \mathscr{E}_0)$ be a connected component of $\mathscr{H}$. Write $\mathscr{V}_0 = \mathscr{V}_{0,1} \sqcup \mathscr{V}_{0,2}$ with $\mathscr{V}_{0,i} \subset \mathscr{V}_i$, for $i = 1, 2$. Note that both $\mathscr{V}_{0,1}$ and $\mathscr{V}_{0,1}$ are non-empty. Let $(F,T) \in \mathscr{V}_{0,i}$. Let $G_0 = (V, E_0)$ be the spanning subgraph of G having the edge set $E_0 = E(T) \cup E(F)$. By definition of the edges in $\mathscr{H}$, and connectivity of $\mathscr{H}_0$, we have for all $(A,B) \in \mathscr{V}_0$, $E(A) \cup E(B) = E(G_0)$. We refer to G_0 as the spanning subgraph of G associated to the connected component $\mathscr{H}_0$ of $\mathscr{H}$.

Recall that for a subset $X \subset V$ of the vertices of a (multi)graph $G = (V, E)$, we denote by $G[X]$ the induced graph on X: it has vertex set X and edge set all the edge of E with both end-points lying both in X.

Note that for any subset $X \subset V$, the induced subgraph $G_0[X]$ has at most $2|X| - 2$ edges. We call a subset X of vertices of G_0 *saturated* with respect to G_0 if the induced subgraph $G_0[X]$ has precisely $2|X| - 2$ edges. A *saturated component* X of G_0 is a maximal subset of G for inclusion which is saturated with respect to G_0.

Here is our theorem.

THEOREM 78 ([**Ami4**]). *Let G be a multigraph.*

(1) *The exchange graph $\mathscr{H}$ is connected if and only if the following two conditions hold:*
  (i) *the edge set of G can be partitioned as $E(G) = E(T) \sqcup E(F)$ for a spanning tree T and a spanning 2-forest F of G.*
  (ii) *any non-empty subset X of V saturated with respect to G consists of a single vertex.*

(2) *More generally, there is a bijection between the connected components* $\mathscr{H}_0$ *of* $\mathscr{H}$ *and the pair* $(G_0; \{T_{1,1}, T_{1,2}, \dots, T_{r,1}, T_{r,2}\})$ *where*
  (i) G_0 *is a spanning subgraph of* G *which is a disjoint union of a spanning tree* T *and a spanning forest* F *of* G.
  (ii) *denoting the maximal subsets of* V *saturated with respect to* G_0 *by* $X_1, \dots, X_r$, *then* $T_{j,1}$ *and* T_{j_2} *are two disjoint spanning trees on the vertex set* X_j, *and* $E(G_0[X]) = E(T_{j,1}) \sqcup E(T_{j,2})$, *for* $j = 1, \dots, r$.

Under this correspondence, the vertex set of $\mathscr{H}_0$ *consists of all the vertices* $(A, B) \in \mathscr{V}$ *which verify* $E(A) \cup E(B) = E(G_0)$, *and for all* $j = 1, \dots, r$, $A[X_j] = T_{j,1}$ *and* $B[X_j] = T_{j,2}$.

We refer to [**Ami4**] for a proof of this theorem, and to [**209**] for a generalisation of the result to matroids.

The proof of Theorem 77 uses Cauchy-Binet formula to expand the determinants in terms of spanning trees and spanning 2-forests. The terms in the nominator which are not trivially bounded by the denominator can be reorganised into a sum of terms over the connected components of the exchange graph. The classification theorem above then shows that this terms vanish globally. Details can be found in [**Ami4**] and [**209**] (which contains more general statements as well).

Proof of Theorem 74. The proof of the theorem in the case the genus of the dual graph G coincides with the genus of the Riemann surface follows directly by the explicit formula obtained in [**ABBF**] for the variation of the archimedean height pairing, in the form needed to apply Theorem 77. The proof in the case the genus of G is strictly smaller than the genus of the Riemann surface can be obtained by a tricky argument from the above case. Details can be found in [**Ami4**]. □

CHAPTER 6

Chow ring of products of graphs

In this chapter we present the results of [**Ami2**] where we study the structure of a Chow ring associated to a product of graphs. This ring naturally arises from the Gross-Schoen desingularisation of a product of regular proper semi-stable curves over discrete valuation rings, for the corresponding family of dual graphs, and can be viewed as the universal combinatorial part of the Chow ring of products of semistable curves, when the family of dual graphs is fixed.

We prove a localisation theorem, which describes the Chow ring of the product of any family of graphs as an inverse limit of the Chow ring of hypercubes. This shows that the Chow rings of products of graphs form a sheaf for the topology generated by the products of subrgaphs.

We provide a complete description of the degree map, leading to a complete description of the degree of intersections between divisors in the special fibre of the regular semistable model of any product of curves.

We prove vanishing theorems in the Fourier dual description of the Chow ring of the hypercube, which, combined with the work of Johannes Kolb, leads to an analytic formula for the arithmetic intersection number of adelic metrised line bundles on products of curves over complete discretely valued fields, generalsing a previous result of Shou-Wu Zhang in his work on Gross-Schoen cycles and dualising sheaves.

Similar in sprite results on the combinatorial part of the Chow ring of semistable models exist for few other families, notably for wonderful compactifications of the complement of hyperplane arrangements, by the work of Feitchner-Yuzvinsky [**106**] and Adiprasito-Huh-Katz [**4**].

6.1. Definition of the combinatorial Chow ring

Let R be a complete discrete valuation ring with an algebraically closed residue field k and fraction field K, and let X be a smooth proper curve over K. By semi-stable reduction theorem, replacing K with a finite extension if necessary, we can find a regular proper strict semi-stable model $\mathfrak{X}$ of X over the valuation ring. Denote by $\mathfrak{X}_s$ the special fibre of $\mathfrak{X}$, and let $G = (V, E)$ be the dual graph of $\mathfrak{X}_s$. For each vertex $v \in V$, denote by X_v the corresponding irreducible component of $\mathfrak{X}_s$. The intersection products $X_v.X_u$ are described by the graph G as follows:

$$\forall\ u, v \in V, \qquad X_u.\, X_v = \begin{cases} \text{number of edges } \{u, v\} \text{ in } G & \text{if } u \neq v, \\ -\mathrm{val}_G(v) & \text{if } u = v, \end{cases} \tag{12}$$

where $\mathrm{val}_G(v)$ is the valence of v in G. The intersection products satisfy the following two sets of relations:

($\mathscr{A}1$) For all $u, v \in V$, $X_v X_u = 0$ if $\{u, v\} \notin E$;
($\mathscr{A}2$) For all $u \in V$, $X_u(\sum_{v\in V} X_v) = 0$.

Consider the polynomial ring $Z(G) = \mathbb{Z}[C_v \,|\, v \in V]$ on variables C_v, and define the ideal $\mathscr{I}_{\mathrm{rat}} \subseteq Z(G)$ of elements *rationally equivalent to zero* as the ideal generated by the polynomials $C_u C_v$, for $\{u, v\} \notin E$, and $C_u(\sum_{v\in V} C_v)$ for $u \in V$. Define the Chow ring $\mathrm{Chow}_{\mathrm{GS}}(G)$ of the graph G by $\mathrm{Chow}_{\mathrm{GS}}(G) := Z(G)/\mathscr{I}_{\mathrm{rat}}$, and note that we have a morphism of graded rings $\mathrm{Chow}_{\mathrm{GS}}(G) \to \mathrm{Chow}^c_{\mathfrak{X}_s}(\mathfrak{X})$, where $\mathrm{Chow}^c_{\mathfrak{X}_s}(\mathfrak{X})$ denotes the subring of the Chow ring with support $\mathrm{Chow}_{\mathfrak{X}_s}(\mathfrak{X})$ of $\mathfrak{X}$ with support in $\mathfrak{X}_s$ generated by the irreducible components of the special fibre (see [**113**, Chapter 17] or [**117**, Section 8] for the definition of Chow rings with support).

For a graph consisting of a single edge $e = \{u, v\}$ on two vertices, we have $\mathrm{Chow}_{\mathrm{GS}}(e) = \mathbb{Z}[C_u, C_v]/(C_u^2 + C_u C_v, C_v^2 + C_u C_v) \simeq \mathbb{Z} \oplus \mathbb{Z}C_u \oplus \mathbb{Z}C_v \oplus \mathbb{Z}C_u C_v$. For a general graph G, the structure of $\mathrm{Chow}_{\mathrm{GS}}(G)$ is completely described by an exact sequence of the form

$$0 \to \mathrm{Chow}_{\mathrm{GS}}(G) \to \prod_{e\in E} \mathrm{Chow}_{\mathrm{GS}}(e) \to \prod_{\{e_1,e_2\}\in L(G)} \mathbb{Z}, \tag{13}$$

where $L(G)$ denotes the line graph of G .(Recall that the line graph of a give graph $G = (V, E)$ is the graph on vertex set E and with edge set consisting of all the pairs $\{e, e'\} \subset E$ with e and e' incident edges in G.)

We have a (local) degree map $\deg : \mathrm{Chow}_{\mathrm{GS}}(G) \to \mathbb{Z}$ which is defined as follows. For any edge $e = \{u, v\} \in E$, define the map $\deg_e : \mathrm{Chow}_{\mathrm{GS}}(e) \to \mathbb{Z}$ by sending an element x of $\mathrm{Chow}_{\mathrm{GS}}(e)$ to the coefficient of $C_u C_v$ in x. The degree map deg is then the composition of the embedding $\mathrm{Chow}_{\mathrm{GS}}(G) \hookrightarrow \prod_{e\in E} \mathrm{Chow}_{\mathrm{GS}}(e)$ with the map $\sum_{e\in E} \deg_e$. By definition, the degree map coincides with the intersection pairing (12), i.e., for all $u, v \in V$, we have $\deg(C_u C_v) = X_u X_v$.

We now provide a generalisation of the above picture for the products of (an arbitrary number) of proper smooth curves over a complete discretely valued field.

So let $X_1, \dots, X_d$ be proper smooth curves over K, and, replacing K with a finite extension if necessary, consider a regular strict semi-stable model $\mathfrak{X}_i$ of X_i over the valuation ring for each i. Starting from the product $\mathfrak{X}_1 \times_{\mathrm{Spec}(R)} \cdots \times_{\mathrm{Spec}(R)} \mathfrak{X}_d$, the Gross-Schoen desingularisation procedure [**121**] provides a regular proper semi-stable model $\mathfrak{X}$ of the product $X = X_1 \times \cdots \times X_d$ over the valution ring R. The desingularisation depends on the choice of a total order on the components of the special fibre of each $\mathfrak{X}_i$.

Denote by $G_1 = (V_1, E_1), \dots, G_d = (V_d, E_d)$ the dual graphs of the special fibres of $\mathfrak{X}_1 \dots, \mathfrak{X}_d$, respectively, and suppose that a total order $\leq_i$ on the vertex set V_i is given for each i. The dual complex of the special fibre of the Gross-Schoen model $\mathfrak{X}$ of X is a triangulation of the product $\mathscr{G} = G_1 \times \cdots \times G_d$. When $\mathcal{G}$ is given by its natural cubical structure with cubes corresponding to the elements of the product $\mathscr{E} = E_1 \times \cdots \times E_d$, the triangulation consists of the union of the standard triangulation of these d-dimensional cubes, compatible with the fixed total orders on the vertex set of each graph G_i. This is defined as follows.

First, for each $1 \leq i \leq d$, we orient the edges of G_i with respect to the total order $\leq_i$ in such a way that any edge $\{u, v\} \in E_i$ gets orientation uv with $u <_i v$.

Let $\mathcal{E} = E_1 \times \cdots \times E_d$, and for each $\mathbf{e} = (e_1, \dots, e_d) \in \mathcal{E}$, for oriented edges $e_1 \in E_1, \dots, e_d \in E_d$, denote by $\square_{\mathbf{e}}$ the product $e_1 \times \cdots \times e_d$. We identify $\square_{\mathbf{e}}$ with the d-dimensional cube $\square^d$ with vertices $\{0, 1\}^d$ via the identification of each oriented edge $e_i = u_i v_i$ with $\{0, 1\}$, identifying thus u_i with 0 and v_i with 1. We endow the hypercube $\square^d$ with its standard simplicial structure. Namely, identify $\square^d$ with the vertex set of the hypercube $[0,1]^d$, and for each element σ of the symmetric group $\mathfrak{S}_d$ of order d, define

$$\Delta_\sigma := \Big\{ (x_1, \dots, x_d) \in [0,1]^d \,\Big|\, 0 \leq x_{\sigma(1)} \leq \cdots \leq x_{\sigma(d)} \leq 1 \Big\}.$$

The non-degenerate d-simplices of $\square^d$ are the vertices of Δ_σ, for any element $\sigma \in \mathfrak{S}_d$.

The vertex set $\mathcal{G}_0$ of the simplicial set $\mathcal{G}$ is the product $\mathcal{G}_0 = V_1 \times \cdots \times V_d$ of the vertex sets, whose elements are in bijection with irreducible components of the special fibre $\mathfrak{X}_s$ of $\mathfrak{X}$: for an element $\mathbf{v} \in \mathcal{G}_0$, we denote by $X_{\mathbf{v}}$ the corresponding irreducible component of $\mathfrak{X}_s$.

Consider the Chow ring with support $\mathrm{Chow}_{\mathfrak{X}_s}(\mathfrak{X})$. The intersection products between $X_{\mathbf{v}}$, for $\mathbf{v} \in \mathscr{V}$, in $\mathrm{Chow}_{\mathfrak{X}_s}(\mathfrak{X})$ verify three types of equations ($(\mathscr{R}1), (\mathscr{R}2), (\mathscr{R}3)$ below), given by Kolb in [**157, 158**], two of which are higher dimensional analogous of the relations ($\mathscr{A}1$) and ($\mathscr{A}2$). This leads to the definition of a Chow ring $\mathrm{Chow}_{\mathrm{GS}}(\mathcal{G})$ for the product of graphs, that we describe next.

Denote by $Z(\mathcal{G})$ the polynomial ring with coefficients in $\mathbb{Z}$ generated by the vertices of $\mathcal{G}$, namely,

$$Z(\mathcal{G}) := \mathbb{Z}[C_{\mathbf{v}} \,|\, \mathbf{v} \in \mathscr{G}_0],$$

where the variables $C_{\mathbf{v}}$ are associated to the vertices (0-simplices) of $\mathcal{G}$. We view $Z(\mathcal{G})$ as a graded ring where each variable $C_{\mathbf{v}}$ is of degree one.

The graded ideal $\mathscr{I}_{\mathrm{rat}}$ of all the elements of $Z(\mathcal{G})$ which are *rationally equivalent to zero* is defined as the ideal generated by the following three types of generators:

($\mathscr{R}1$) $C_{\mathbf{v}_1} C_{\mathbf{v}_2} \dots C_{\mathbf{v}_k}$ for $k \in \mathbb{N}$ and elements $\mathbf{v}_j \in \mathcal{G}_0$ such that $\mathbf{v}_1, \dots, \mathbf{v}_k$ do not form a simplex in $\mathcal{G}$;

($\mathscr{R}2$) $C_{\mathbf{u}} \Big(\sum_{\mathbf{v} \in \mathcal{G}_0} C_{\mathbf{v}} \Big)$ for any vertex $\mathbf{u} \in \mathcal{G}_0$; and

($\mathscr{R}3$) $C_{\mathbf{u}} C_{\mathbf{w}} \Big(\sum_{\mathbf{v} \in \mathcal{G}_0 : v_i = u_i} C_{\mathbf{v}} \Big)$ for any pair of vertices $\mathbf{u}, \mathbf{w} \in \mathcal{G}_0$ and any index $1 \leq i \leq d$ with $u_i \neq w_i$.

Definition 79. The combinatorial (Gross-Schoen) Chow ring of $\mathcal{G}$ is the graded ring $\mathrm{Chow}_{\mathrm{GS}}(\mathcal{G}) := Z(\mathcal{G}) / \mathscr{I}_{\mathrm{rat}}$.

The ring $\mathrm{Chow}_{\mathrm{GS}}(\mathcal{G})$ is the universal graded commutative ring with generators indexed by vertices of $\mathcal{G}$ and verifying relations ($\mathscr{R}1$), ($\mathscr{R}2$), ($\mathscr{R}3$) above; in particular we get a well-defined map

$$\alpha_{\mathfrak{X}} : \mathrm{Chow}_{\mathrm{GS}}(\mathcal{G}) \to \mathrm{Chow}_{\mathfrak{X}_s}(\mathfrak{X}), \tag{14}$$

for $\mathfrak{X}$ the Gross-Schoen desingularisation of the products of curves $\mathfrak{X}_1 \times \cdots \times \mathfrak{X}_d$ (where for each i, the dual graph of the special fibre of $\mathfrak{X}_i$ is isomorphic to G_i).

Remark 80. There are other different types of cohomological rings one can associate to a product of graphs, e.g., the Stanley ring of the product of graphs (with its natural cubical structure) [**133**], the tropical Chow ring of products of (metric) graphs [**11, 226**], the Chow ring of matroids [**4, 106**], and the tropical homology groups [**139**].

6.2. Statement of the main results

We describe the structure of the above Chow ring through the structure Theorem 82 which gives an alternative description of the additive structure of the graded pieces of the Chow ring, the localisation Theorem 81, which is a generalisation of the exact sequence (13) to products of graphs, the calculation of the degree map, which is a generalisation of (12) to higher dimension, and a set of vanishing theorems. We now discuss these results.

6.2.1. Localisation. Theorem 81 provides a generalisation of the exact sequence (13) to products of graphs, showing that the calculations in the ring $\mathrm{Chow}_{\mathrm{GS}}(\mathcal{G})$ can be reduced to calculation in the Chow ring of the hypercubes of dimension d, namely, the products of d copies of the complete graph K_2 on two vertices.

Recall first that a homomorphism of graphs $f : H \to G$ is a map $f : V(H) \to V(G)$ such that for any edge $\{u, v\} \in E(H)$, either $f(u) = f(v)$ or $\{f(u), f(v)\} \in E(G)$.

Let $H_1, \dots, H_d$ be d simple connected graphs. Define $\mathscr{H} = H_1 \times \cdots \times H_d$ with the induced simplicial structure. Suppose that for each $i = 1, \dots, d$, a homomorphism of graphs $f_i : H_i \to G_i$ is given (such that f_i respects also the two fixed orderings on the vertex set of H_i and G_i). The product of f_i leads to a morphism of simplicial sets $f : \mathscr{H} \to \mathcal{G}$, and induces a morphism of graded rings $f^* : Z(\mathcal{G}) \to Z(\mathcal{H})$, defined on the level of generators by sending $C_{\mathbf{v}}$ for $\mathbf{v} \in \mathcal{G}_0$ to

$$f^*(C_{\mathbf{v}}) = \sum_{\substack{\mathbf{u} \in \mathscr{H}_0 \\ f(\mathbf{u}) = \mathbf{v}}} C_{\mathbf{u}}.$$

It further induces a well-defined map of Chow rings $f^* : \mathrm{Chow}_{\mathrm{GS}}(\mathcal{G}) \to \mathrm{Chow}_{\mathrm{GS}}(\mathcal{H})$.

Let now $G_1 = (V_1, E_1), \dots, (G_d, E_d)$ be a collection of d simple connected graphs, and $\mathcal{G} = \prod_i G_i$, as above. For $\mathbf{e} \in \mathcal{E} = E_1 \times \cdots \times E_d$, let $\square_{\mathbf{e}} = e_1 \times \cdots \times e_d$. Regarding each edge e_i as a subgraph of G isomorphic to K_2, and applying the functoriality to the inclusions of the subgraph $e_i \hookrightarrow G_i$, we get a map $\iota_{\mathbf{e}}^* : \mathrm{Chow}_{\mathrm{GS}}(\mathcal{G}) \to \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}}) \simeq \mathrm{Chow}_{\mathrm{GS}}(\square^d)$ associated to the inclusion map of simplicial sets

$$\iota_{\mathbf{e}} : \square_{\mathbf{e}} \hookrightarrow \mathcal{G}.$$

In addition for any face $\square_{\mathbf{x}}$ of $\square_{\mathbf{e}}$, which is a cube of smaller dimension obtained by taking a product of subgraphs H_i of G_i of the form $H_i = e_i$ or H_i a vertex of e_i, we get similarly a map of Chow rings $\mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}}) \to \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{x}})$.

With this preparation, the localisation theorem can be stated as follows.

THEOREM 81 ([**Ami2**]). *The map of graded rings*

$$\prod_{\mathbf{e} \in \mathcal{E}} \iota_{\mathbf{e}}* : \mathrm{Chow}_{\mathrm{GS}}(\mathcal{G}) \to \bigoplus_{\mathbf{e} \in \mathcal{E}} \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}})$$

is injective and identifies $\mathrm{Chow}_{\mathrm{GS}}(\mathcal{G})$ *as the inverse limit of the Chow ring of cubes of* $\mathcal{G}$ *for the diagram of maps induced by the inclusion of cubes.*

We refer to [**Ami2**] for a more explicit statement. Endowing the simplicial set $\mathcal{G}$ with the cubical topology with a basis of open sets consisting of the products of subgraphs of G_i, the localisation theorem ensures that the Chow rings form a sheaf for the coverings of $\mathcal{G}$ with open sets whose union covers all the simplices of $\mathcal{G}$.

To our surprise, the proof of the theorem turns out to be quite non-trivial and tricky. It is an indirect argument based on an alternative description of the Chow groups in terms of non-degenerate simplices of the product $\mathcal{G}$ and specific relations, taking into account the cubical structure of $\mathcal{G}$.

6.2.2. Alternative description of the additive structure of $\mathrm{Chow}_{\mathrm{GS}}(\mathcal{G})$**.** The total orders $\leq_i$ on the vertex sets V_i, $i = 1, \dots, d$, induce a partial order $\leq$ on $\mathcal{G}_0$ defined by saying $\mathbf{u} = (u_1, \dots, u_d) \leq \mathbf{v} = (v_1, \dots, v_d)$ if $u_i \leq_i v_i$ for each i.

Let $\mathbf{u} < \mathbf{v}$ be two elements of $\mathcal{G}_0$ such that $\{\mathbf{u}, \mathbf{v}\}$ forms a one-simplex. This means that for each i, we have $u_i = v_i$ or $\{u_i, v_i\} \in E_i$. Denote by $I(\mathbf{u}, \mathbf{v})$ the set of all indices i with $u_i < v_i$, and let $\mathbf{e}_{\mathbf{u},\mathbf{v}}$ be the cube of dimension $|I(\mathbf{u}, \mathbf{v})|$ formed by all the vertices $\mathbf{z} = (z_1, \dots, z_d)$ in $\mathcal{G}_0$ with $z_i \in \{u_i, v_i\}$, i.e., $\mathbf{e}_{\mathbf{u},\mathbf{v}} = \prod_{i=1}^{d} \{u_i, v_i\}$.

For any integer $k \in \mathbb{N}$, denote by $\mathcal{G}_k^{nd}$ the set of all non-degenerate k-simplices of $\mathcal{G}$. Each element σ of $\mathcal{G}_k^{nd}$ is a sequence $\mathbf{u}_0 < \mathbf{u}_1 < \cdots < \mathbf{u}_k$ of vertices $\mathbf{u}_i \in \mathcal{G}_0$ such that $\{\mathbf{u}_j, \mathbf{u}_{j+1}\}$ is a 1-simplex of $\mathcal{G}$, for any $0 \leq j \leq k-1$.

Let σ be a non-degenerate k-simplex of $\mathcal{G}$. For two indices $1 \leq i, j \leq d$ lying both in $I(\mathbf{u}_t, \mathbf{u}_{t+1})$ for some $0 \leq t \leq k-1$, define

$$\widetilde{R}_{\sigma,i,j} := \sum_{\substack{\mathbf{w} \in \mathbf{e}_{\mathbf{u}_t,\mathbf{u}_{t+1}} \\ w_i = u_{t,i}}} C_{\mathbf{w}} - \sum_{\substack{\mathbf{w} \in \mathbf{e}_{\mathbf{u}_t,\mathbf{u}_{t+1}} \\ w_j = u_{t,j}}} C_{\mathbf{w}}.$$

For any k-simplex $\sigma \in \mathcal{G}_k$, denote by C_σ the product (with multiplicity) of $C_{\mathbf{v}}$ over all vertices of σ, i.e., $C_\sigma := \prod_{\mathbf{v} \in \sigma} C_{\mathbf{v}}$. Using ($\mathscr{R}1$) and ($\mathscr{R}3$), one verifies that $C_\sigma \widetilde{R}_{\sigma,i,j} \in \mathscr{I}_{\mathrm{rat}}$.

For any $k \in \mathbb{N}$, denote by $\mathbb{Z}\langle \mathcal{G}_k^{nd} \rangle$ the $\mathbb{Z}$-submodule of $Z(\mathcal{G})$ generated by all elements C_σ for $\sigma \in \mathcal{G}_k^{nd}$. By definition of the simplicial structure, one sees that for an element $\sigma \in \mathcal{G}_k^{nd}$ consisting of vertices $\mathbf{u}_0 < \cdots < \mathbf{u}_k$, and for $i, j \in I(\mathbf{u}_t, \mathbf{u}_{t+1})$ as above, the product $C_\sigma \widetilde{R}_{\sigma,i,j}$ lies in $\mathbb{Z}\langle \mathcal{G}_k^{nd} \rangle$. Denote by $\mathscr{I}_k^{nd}$ the $\mathbb{Z}$-submodule of $\mathbb{Z}\langle \mathcal{G}_k^{nd} \rangle$ generated by all the elements $C_\sigma \widetilde{R}_{\sigma,i,j}$, for $\sigma \in \mathcal{G}_k^{nd}$ and $i, j \in I(\mathbf{u}_t, \mathbf{u}_{t+1})$ as above. We have the following theorem.

THEOREM 82 ([**Ami2**]). *For any non-negative integer k, we have*

$$\mathrm{Chow}_{\mathrm{GS}}^{k+1}(\mathcal{G}) \simeq \mathbb{Z}\langle \mathcal{G}_k^{nd} \rangle / \mathscr{I}_k^{nd}.$$

The existence of a surjection from $\mathbb{Z}\langle \mathcal{G}_k^{\mathrm{nd}} \rangle / \mathscr{I}_k^{nd}$ to $\mathrm{Chow}_{\mathrm{GS}}^{k+1}(\mathcal{G})$ is a consequence of the combinatorial moving lemma, which states that non-degenerate simplices generate the Chow ring. For the proof of the injectivity of this map, we define a filtration on $Z^{k+1}(\mathcal{G})$, and then we show by induction that it induces a trivial filtration on the kernel of the above map. It is not clear if this filtration has any geometric origin.

6.2.3. Combinatorics of the degree map. Let $\square^d = \{0, 1\}^d$ be the d-dimensional hypercube, which is the d-fold product of the complete graph K_2 on two vertices $0 < 1$ with its standard simplicial structure. It follows from the structure theorem that the Chow ring $\mathrm{Chow}_{\mathrm{GS}}(\square^d)$ is of rank one in graded degree $d+1$, i.e., $\mathrm{Chow}_{\mathrm{GS}}^{d+1}(\square^d) \simeq \mathbb{Z}$, generated by C_σ for any non-degenerate d-simplex σ of $\square^d$. This leads to a well-defined degree map

$$\deg : \mathrm{Chow}_{\mathrm{GS}}^{d+1}(\square^d) \to \mathbb{Z}.$$

Combining this with the localisation theorem, and the vanishing of $\mathrm{Chow}_{\mathrm{GS}}^{i}(\square^d)$ in degree $i \geq d+2$, which follows for example from the structure theorem, or the moving lemma, we infer that for any collection of simple connected graphs $G_1 = (V_1, E_1), \dots, G_d = (V_d, E_d)$, we have $\mathrm{Chow}^{d+1}_{\mathrm{GS}}(\mathscr{G}) \simeq \mathbb{Z}^{|\mathscr{E}|}$. Therefore we get a degree map $\deg : \mathrm{Chow}^{d+1}_{\mathrm{GS}}(\mathcal{G}) \simeq \mathbb{Z}^{|\mathscr{E}|} \to \mathbb{Z}$, by aditionning the coordinates in $\mathbb{Z}^{|\mathscr{E}|}$.

Our next result gives a combinatorial formula for the value of the degree map. Combined with the map $\alpha_{\mathfrak{X}}$ in (14), this results in a concrete effective description of the local degrees in the Chow ring $\mathrm{Chow}^{c}_{\mathfrak{X}_s}(\mathfrak{X})$ for the Gross-Schoen desingularisation $\mathfrak{X}$ of a product of semi-stable R-curves $X_1, \dots, X_d$, generalizing (12) to higher dimension.

Since $\mathrm{Chow}^{d+1}_{\mathrm{GS}}(\mathcal{G})$ is generated by monomials, we can restrict to the case of a monomial, and by localisation theorem, and the definition of the degree map, it will be enough to treat the case of the hypercube $\square^d$.

By definition of the simplicial structure, each (possibly degenerate) d-simplex σ of $\square^d$ is of the form $\mathbf{v}_1^{n_1}\mathbf{v}_2^{n_2}\dots\mathbf{v}_k^{n_k}$ with $\mathbf{v}_1 < \mathbf{v}_2 < \dots < \mathbf{v}_k$, and $n_i \geq 1$ with $\sum_i n_i = d+1$. We have $0 \leq |\mathbf{v}_1| < \dots < |\mathbf{v}_k| \leq d$, where for any $\mathbf{v} \in \square^d$, we denote by $|\mathbf{v}|$ the *length* of $\mathbf{v}$ defined as the number of coordinates of $\mathbf{v}$ equal to one. Let $[d] := \{0, 1, \dots, d\}$. We have

THEOREM 83 ([**Ami2**]). *Notations as above, let $C_\sigma = C_{\mathbf{v}_1}^{n_1} \dots C_{\mathbf{v}_k}^{n_k}$. One of the two following cases can happen.*

(1) If there exists an $1 \leq i < k$ such that $n_1 + \dots + n_i > |\mathbf{v}_{i+1}|$, then $C_\sigma = 0$. Similarly, if there exists an $k \geq i \geq 2$ such that $n_i + \dots + n_k > d - |\mathbf{v}_{i-1}|$, then $C_\sigma = 0$.

(2) Otherwise, there exists a sequence of integers $y_0, x_1, y_1, x_2, y_2, \dots, x_{k-1}, y_{k-1}, x_k$ verifying the following properties

- $|\mathbf{v}_1| = y_0$.
- *For all $i = 2, \dots, k$, $|\mathbf{v}_i| = |\mathbf{v}_{i-1}| + x_i + y_i + 1$.*
- $n_i = y_{i-1} + x_i + 1$ *for all $i = 1, \dots, k$,*

and, in this case, we have

$$\deg(C_\sigma) = (-1)^{d+1-k}\binom{y_0 + x_1}{y_0}\binom{x_1 + y_1}{x_1}\binom{y_1 + x_2}{y_1}\dots\binom{x_{k-1} + y_{k-1}}{y_{k-1}}\binom{y_{k-1} + x_k}{x_k}.$$

6.2.4. Fourier transform and a vanishing theorem. Identifying the points of $\square^d$ with the elements of the vector space $\mathbb{F}_2^d$, it is possible to give a dual description of the Chow ring of the hypercube using the Fourier duality. So let $\langle\, ,\rangle$ be the scalar product on $\mathbb{F}_2^d$ defined by

$$\forall\, \mathbf{u}, \mathbf{v} \in \mathbb{F}_2^d, \qquad \langle \mathbf{v}, \mathbf{u}\rangle := \sum_{i=1}^{d} v_i.u_i \in \mathbb{F}_2.$$

For $\mathbf{w} \in \mathbb{F}_2^d$, define $F_{\mathbf{w}}$ by

$$F_{\mathbf{w}} := \sum_{\mathbf{v}\in\square^d} (-1)^{\langle \mathbf{v}, \mathbf{w}\rangle} C_{\mathbf{v}}.$$

By Fourier duality, we have for any $\mathbf{v} \in \mathbb{F}_2^d$,

$$C_{\mathbf{v}} = \frac{1}{2^d}\sum_{\mathbf{v}\in\square^d} (-1)^{\langle \mathbf{v}, \mathbf{w}\rangle} F_{\mathbf{w}}.$$

It follows that the set $\{F_{\mathbf{w}}\}_{\mathbf{w}\in\square^d}$ forms another system of generators for the Chow ring $\mathrm{Chow}_{\mathrm{GS}}(\square^d)[\frac{1}{2}]$ localised at 2, that we call the Fourier dual of the set $\{C_{\mathbf{v}}\}_{\mathbf{v}\in\square^d}$.

The generators $\{F_{\mathbf{w}}\}$ in $\mathrm{Chow}_{\mathrm{GS}}(\square^d)$ verify three types of relations which give a complete set of relations with coefficients in $\mathbb{Z}[\frac{1}{2}]$ [**Ami2**].

We now describe a criterion guaranteeing the vanishing of a monomial of the form $F_{\mathbf{w}_0}\dots F_{\mathbf{w}_d}$, for elements $\mathbf{w}_0,\dots,\mathbf{w}_d\in\mathbb{F}_2^d$. Let $\mathcal{P}=\{P_1,\dots,P_k\}$ be a partition of $\{1,\dots,d\}$ into k disjoint non-empty sets. For each $\mathbf{w}_i$, denote by $\alpha(\mathbf{w}_i,\mathcal{P})$ the number of indices $1\le i\le k$ such that there exists $j\in P_i$ with $\mathbf{w}_j=1$.

THEOREM 84 (Vanishing theorem [**Ami2**]). *If* $\sum_{i=0}^d\alpha(\mathbf{w}_i,\mathcal{P})<d+k$, *then we have* $F_{\mathbf{w}_0}\dots F_{\mathbf{w}_d}=0$ *in the Chow ring.*

This property was conjectured by Kolb and is required in [**158**] in order to get the analytic description of the local degree map, that we briefly describe in the next section.

Note that, contrary to Theorem 83, we do not have a complete description of the degree in terms of the generators $F_{\mathbf{w}}$.

6.3. Analytic description of the local intersection numbers

Let X be a smooth proper curve over a complete discretely valued field K with an algebraically closed residue field. Recall from Chapter 2 that the Berkovich analytification X^{an} of X is a compact path-wise connected Hausdorff topological space which deformation retracts to a compact metric graph Γ. If X admits a regular semi-stable model $\mathfrak{X}$ over the valuation ring of K, the metric graph Γ has a model (G,ℓ) given by the dual graph $G=(V,E)$ of $\mathfrak{X}$ and the edge length $\ell:E(G)\to\mathbb{R}$ given by $\ell(e)=1$ for all edges $e\in E$.

Any Cartier divisor D on $\mathfrak{X}$ with support in the special fibre $\mathfrak{X}$ gives a map $f:V\to\mathbb{Z}$, that we can extend to Γ by linear interpolation on interior points of the intervals in Γ corresponding to the edges of G. For two Cartier divisors $D_1,D_2\in\mathrm{Chow}^1_{\mathfrak{X}_s}(\mathfrak{X})$ with functions $f_1,f_2:\Gamma\to\mathbb{R}$, the degree map given by the pairing (12) gives a number $\deg(D_1D_2)$, which can be described analytically as

$$\deg(D_1D_2)=\langle f_1,f_2\rangle_{\mathrm{Dir}}=-\int_\Gamma f_1'f_2'. \tag{15}$$

Here $\langle .,.\rangle_{\mathrm{Dir}}$ denotes the Dirichlet pairing on piecewise smooth functions on Γ [**25, 248**] (and Chapter 7).

By an approximation argument involving semi-stable models over finite base field extensions and viewing Cartier divisors with support in the special fibres of semi-stable models as piecewise linear functions on Γ, one can continuously extend the (degree) pairing between divisors to the full class of piecewise smooth functions on Γ such that the equation above remains valid for this more general class of functions [**248**].

Motivated by applications in arithmetic geometry, Zhang derived in [**250**] a generalisation of the analytic formula (15) for the degree pairing in the case of a 2-fold product of a smooth proper curve X over K. The formula allowed to reduce the geometric Bogomolov conjecture to a conjecture on harmonic analysis on metric graphs, which was then solved by Cinkir [**72**].

Kolb [**158**] later generalised Zhang's formula to d-fold products of X assuming the validity of the vanishing Theorem 84. We state his result in the more general setting of a product of smooth proper curves $X_1,\dots,X_d$.

Let $X_1, \dots, X_d$ be smooth proper curves over K, that we suppose (up to passing to a finite extension of K), to have regular strict semi-stable models $\mathfrak{X}_1, \dots, \mathfrak{X}_d$ over the valuation ring R. Denote by $G_1, \dots, G_d$ the dual graphs of the special fibres of $\mathfrak{X}_1, \dots, \mathfrak{X}_d$, respectively, and let $\mathcal{G}$ be the product $G_1 \times \dots \times G_d$ with its simplicial structure. The Gross-Schoen desingularisation gives a regular proper strict semi-stable model $\mathfrak{X}$ of the product $X = X_1 \times \dots \times X_d$ with special fibre $\mathfrak{X}_s$ having a dual complex isomorphic to $\mathcal{G}$. The geometric realisation of $\mathcal{G}$ is a locally affine space which embeds in the Berkovich analytification X^{an} of X (by a general theorem of Berkovich [**34**]). Each Cartier divisor D with support in the special fibre $\mathfrak{X}_s$ induces a metric on the trivial line bundle corresponding to a piecewise affine function on the geometric realisation of $\mathcal{G}$ (for metric on lines bundles see e.g. [**249**]). The intersection pairing given by the degree map induces a multi-linear pairing between piecewise affine functions (by passing to finite extensions of K if necessary), and we have

$$\langle f_{D_0}, \dots, f_{D_d} \rangle = \deg(D_0 \dots D_d).$$

This pairing can be viewed as the local contribution to the intersection product of metrised line bundles in non-Archimedean Arakelov theory. It is useful to extend this pairing to a larger class of metrised line bundles. By approximation, for each of the piecewise smooth functions f_i, one may take a sequence of piecewise linear functions converging to f_i, and extend the pairing as the limit of the pairing between piecewise linear functions. This has been carried out in great detail in [**158**]. The well-definedness of the extension as well as the analytic generalisation of the Formula (15) is guaranteed if the vanishing condition in Theorem 84 holds. To state the theorem, we need to introduce some notations.

For each graph G_i, denote by Γ_i the metric graph associated to (G, ℓ) with length function $\ell \equiv 1$, the constant function. For each $n \in \mathbb{N}$, denote by $G_i^{(n)} = (V_i^{(n)}, E_i^{(n)})$ the n-th subdivision of G_i, where each edge e is subdivided into n edges. The pair $(G_i^{(n)}, \ell^{(n)})$ with length function $\ell^{(n)} \equiv 1/n$ is a model of the same metric graph Γ_i. A total order on the vertex set of G_i naturally extends to a total order on the vertex set of $G_i^{(n)}$ such that the vertices of $G_i^{(n)}$ on each edge e of G form a monotone sequence. Denote by $\mathcal{G}^{(n)}$ the simplicial set on the product $G_1^{(n)} \times \dots \times G_d^{(n)}$. This provides a triangulation of the topological space $\mathscr{T} = \Gamma_1 \times \dots \times \Gamma_d$. The space $\mathscr{T}$ has a natural affine structure induced by the cubes $\square_{\mathbf{e}} \simeq [0,1]^d$ for each $\mathbf{e} \in \mathscr{E}^{(n)} = E_1^{(n)} \times \dots \times E_d^{(n)}$. Define the space $\mathcal{C}^\infty_\Delta(\mathscr{T})$ as the space of functions $f : \mathscr{T} \to \mathbb{R}$ which are smooth on simplices of $\mathcal{G}$ [**158**]. This means, for any cube $\square_{\mathbf{e}} \simeq [0,1]^d$, the restriction of f to each triangle Δ of $[0,1]^d$ can be extended to a smooth function in a neighborhood of Δ.

For each $f \in \mathcal{C}^\infty_\Delta(\mathscr{T})$, denote by $f^{(n)}$ the piecewise affine function on $\mathscr{T}$ obtained by interpolating the values of f on the vertices to all the interior points of σ, on each simplex σ of $\mathcal{G}^{(n)}$.

The graphs $G_i^{(n)}$ are the dual graphs of a semi-stable model $\mathfrak{X}_i^{(n)}$ of $X_{K'}$ for an appropriate finite extension K' of K, and the simplicial set $\mathcal{G}^{(n)}$ corresponds to the dual complex of the Gross-Schoen desingularisation of the product $\mathfrak{X}_1^{(n)} \times \dots \mathfrak{X}_d^{(n)}$. Looking at $\mathbb{R}$-Cartier divisors with support in the special fibre $\mathfrak{X}_s^{(n)}$ as real valued functions defined on the vertices of $\mathcal{G}^{(n)}$, the degree map in the ring $\mathrm{Chow}_{\mathfrak{X}_s}^{\mathfrak{X}^{(n)}}$ leads to a pairing $\langle f_0^{(n)}, \dots, f_d^{(n)} \rangle$ for any collection of functions $f_0, \dots, f_d \in \mathcal{C}^\infty_\Delta(\mathscr{T})$. With these preliminaries, combining our Theorem 84 with the results in [**158**], we get the following generalisation of Equation (15).

THEOREM 85. *For any collection of functions $f_0, \dots, f_d \in \mathcal{C}^\infty_\Delta(\mathscr{T})$, the limit*

$$\langle f_0, \dots, f_d\rangle := \lim_{n\to\infty} \langle f_0^{(n)}, \dots, f_d^{(n)}\rangle$$

exists, and admits the following analytic development

$$\langle f_0, \dots, f_d\rangle = \sum_{\substack{\textit{partitions}\\ \mathcal{P} \textit{ of } [d]}} \frac{1}{2^{|\mathcal{P}|+d}} \sum_{\substack{\mathbf{w}_0,\dots,\mathbf{w}_d \in \mathbb{F}_2^d\\ \sum \alpha(\mathbf{w}_i,\mathcal{P}) = d+|\mathcal{P}|}} \deg\Big(\prod_{i=0}^d F_{\mathbf{w}_i}\Big) \int_{\mathrm{Diag}_\mathcal{P}} \prod_{i=0}^d D^{\mathbf{w}_i}_{\alpha(\mathbf{w}_i,\mathcal{P})}(f_i).$$

In the above formula the generalised diagonal $\mathrm{Diag}_\mathcal{P}$ is the union of the generalised diagonals $\mathrm{Diag}^{\mathbf{e}}_\mathcal{P}$ in the hypercubes $\square_\mathbf{e} \simeq \square^d = [0,1]^d$ consisting of all the points $(x_1, \dots, x_d) \in [0,1]^d$ which verify $x_i = x_j$ for all $i, j \in [d]$ belonging to the same element of the partition $\mathcal{P}$. The term $D^{\mathbf{w}_i}_{\alpha(\mathbf{w}_i,\mathcal{P})}(f_i)$ is a *partial derivative of f_i of order $\alpha(\mathbf{w}_i, \mathcal{P})$ in the direction of $\mathbf{w}_i$ and along the generalised diagonal* $\mathrm{Diag}_\mathcal{P}$. For example, for the partition $\mathcal{P}$ of $[d]$ into singletons, we have $\alpha(\mathbf{w}, \mathcal{P}) = |\mathbf{w}|$ for any $\mathbf{w} \in \mathbb{F}_2^d$, and on any cube $\square_\mathbf{e} \simeq [0,1]^d$, we have $D^{\mathbf{w}_i}_{\alpha(\mathbf{w}_i,\mathcal{P})} = (\frac{\partial}{\partial x_1})^{w_1} \dots (\frac{\partial}{\partial x_d})^{w_d}$. We omit the formal definition and refer to [**158**] for more details.

Let us finish by mentioning that a general approach to non-Archimedean Arakelov geometry using Berkovich theory and tropical geometry has been proposed in the recent works of Chambert-Loir and Ducros [**66**] and Gubler and Künnemann [**123**]. It would be interesting to formulate the above results in this framework, and to obtain generalisation to other analytic varieties.

CHAPTER 7

Eigenvalue estimates in graphs and combinatorial Li-Yang-Yau inequality

The result presented in this chapter concern the spectral geometry of graphs, and have their source of motivation by the results and problems encountered in the previous chapters.

The spectrum of the Laplacian of a finite graph is a source of information about the structural properties of the graph and is used in a large variety of applications to other domains, see e.g. [**192, 137, 69**] for a discussion of applications.

The study of the spectrum of a finite graph is in many ways related to the spectral theory of Riemannian manifolds. Starting from the pioneering works of Burger and Brooks [**59, 61**], which connect the eigenvalues of a covering space to the ones of their Cayley-Schreier graphs, and that of Colin de Verdière and Colbois on degenerations of Riemannian manifold to (metric) graphs [**78, 76**], finite graphs have been used as a way to get control on the eigenvalues of Riemannian manifolds. Bounds on graph eigenvalues are usually used to prove bounds for Riemannian manifolds, and results in geometric analysis have been a source of inspiration to state and prove corresponding results concerning finite graphs, essentially from scratch. These proofs, interestingly, usually bear some high level similarity to the ones used in Riemannian geometry.

In [**AmiCo**], taking a different route, we show how eigenvalue bounds for surfaces combined with basic spectral theory of (singular) surfaces and a suitable version of the transfer principle allows to obtain eigenvalue estimates for graphs in terms of their geometric genus, i.e., the minimum genus of an orientable surface on which the graph can be embedded. The obtained results are tight and improve several previous results in the literature. The transfer principle has other applications that we briefly discuss in this section. This concerns uniform upper and lower bounds on the eigenvalues of the continuous Laplacian on metric graphs in terms of the eigenvalues of their simple graph models, as well as a generalisation of the mesh partitioning results of Miller-Teng-Thurston-Vavasis [**190**] and Spielman-Teng [**228**] to arbitrary meshes.

On the other hand, we prove in [**AmiKool**] a combinatorial version of the Li-Yang-Yau inequality [**170**], linking the divisorial gonality of the graph, as defined in the previous chapters, to its volume and its spectral gap. The link between the two concepts is provided by tree-decompositions of graphs, a fundamental concept in Robertson-Seymour graph minor theory, which might be viewed as a relaxed version of the topological degree.

7.1. Spectral geometry of graphs

We start by first recalling some basic fundamental results concerning the eigenvalues of general graphs, and then restrict to some special families of graphs. For more details concerning these materials, see e.g., [**86, 137**].

For any graph G on n vertices, denote by $\lambda_0(G) = 0 < \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_{n-1}(G)$ all the eigenvalues of the graph Laplacian $\Delta = \Delta_G$. Recall that Δ is the positive semidefinite operator defined on the space of real valued functions on the vertices of G by

$$\Delta(f)(v) = \sum_{u:uv\in E} f(v) - f(u),$$

for any function $f : V \to \mathbb{R}$.

The eigenvalues of a bounded degree graph provide information on the existence of good clusterings of that graph. For clusterings in two classes, this is the statement of the celebrated discrete Cheeger inequality by Alon and Milman [**70, 12**]: a sparse balanced cut may be found if and only the second eigenvalue, or Fiedler value, is small. More precisely, let S be a subset of V. The (edge) boundary of S, denoted by $B(S)$, is the set of edges $E(S, S^c)$ between a vertex in S and a vertex in its complementary $S^c = V \setminus S$. Its size is denoted by $b(S)$. The expansion of a subset S of vertices is by definition $b(S)/|S|$. The (edge) expansion of G is defined as follows:

$$exp(G) = \min_{S\subset V, |S|\leq \frac{|V|}{2}} \frac{b(S)}{|S|}\,.$$

By definition, the expansion is bounded by minimum degree of G.

The following theorem of Alon-Milman shows that the spectral gap of G, which is by definition the first non-trivial eigenvalue λ_1 of the Laplacian Δ, controls the expansion factor of G if G is regular.

THEOREM 86 (Alon-Milman [**12**]). *Let G be a d-regular graph. Then*

$$\frac{\lambda_1}{2} \leq \exp(G) \leq \sqrt{2d\lambda_1}\,.$$

More recent results [**168, 176**] show that similar statements hold for k-way clusterings, whose optimal quality is shown to relate to the k-th eigenvalue. While this connection does in general get looser as k grows, it is about as tight as in the two-way case when the density of states near k is large enough. The proofs of these results are constructive and show that specific spectral clustering algorithms find a clustering whose quality is controlled by the spectrum of the graph.

In particular, upper bounds on the eigenvalues of a class of graphs directly translate into efficient clustering algorithms with quality guarantees. This motivated a series of works in natural families of graphs starting with [**228**], which we discuss a bit later.

For general graphs, the following classical theorem of Alon-Boppana provides an upper bound on the spectral gap for a regular graph.

THEOREM 87 (Alon-Boppana [**198**]). *Let G be a d-regular graph. We have*

$$\lambda_1 \leq d - 2\sqrt{d-1} + o(1)\,.$$

On the other hand, Friedman [**111**] has proved that for any ϵ, random d-regular graphs have asymptotically almost surely $|\lambda_1 - d + 2\sqrt{d-1}| \leq \epsilon$. (See also the recent work of Bordenave [**45**] for a different proof of this result and its extension to random lifts, see also the next remark). So the bound in the Alon-Boppana theorem is asymptotically almost surely tight.

REMARK 88. We note by passing that a graph is called Ramanujan if for all the non-trivial eigenvalues, $|d - \lambda_i(G)| \leq 2\sqrt{d-1}$ [**86**]. Until recently it was unknown if an infinite family of Ramanujan graphs existed in all fixed degrees d; constructions were known for $d = q + 1$ for q a prime power [**86**]. An elegant recent paper of Marcus, Speilman and Srivastava [**184**] solved this problem, by showing the existence of an infinite family of bipartite Ramanujan graphs of any given degree d, obtained recursively by random 2-lifts of a previously constructed Ramanujan graphs. A group-representation-theoretic generalisation of [**184**] is discussed simultaneously in the master thesis of my student Maurice Fuhr [**112**] and the recent paper of [**125**], which contains a generalisation of the matching polynomial.

7.1.1. Tree-decomposition, minors, and graph minor theorem. We now review some basic terminology on tree-decompositions of finite graphs, and discuss the eigenvalues of minor-closed classes of graphs.

Let $G = (V, E)$ be a connected graph. A *tree-decomposition* of G is a pair $(T, \mathcal{X})$ where T is a finite tree on a set of vertices I, and $\mathcal{X} = \{X_i : i \in I\}$ is a collection of subsets of V, subject to the following three conditions:

(1) $V = \cup_{i \in I} X_i$,
(2) for any edge e in G, there is a set $X_i \in \mathcal{X}$ which contains both end-points of e,
(3) for any triple i_1, i_2, i_3 of vertices of T, if i_2 is on the path from i_1 to i_3 in T, then $X_{i_1} \cap X_{i_3} \subseteq X_{i_2}$.

Note that the point (3) in the above definition simply means that the subgraph of T induced by all the vertices i which contain a given vertex v of the graph G is connected. In other words, a tree-decomposition of a graph G consists of representing G as an *intersection graph* of a collection of subtrees of a given tree, i.e., choosing a tree T, associating a subtree T_v of T to each vertex of v such that if two vertices are adjacent in G, then the corresponding subtrees share a common vertex.

The *width* of a tree-decomposition $(T, \mathcal{X})$ is defined as $w(T, \mathcal{X}) = \max_{i \in I} |X_i| - 1$. In the alternative picture, this is the maximum number of subtrees which all share a common vertex of T, deduced by one. The *tree-width* of G denoted by $tw(G)$ is the minimum width of any tree-decomposition of G.

Tree-width is a subtle invariant of a given graph in terms of calculability. There is a useful duality theorem concerning the tree-width, resulting in a min-max theorem, which allows in practice to bound the tree-width of graphs.

The dual notion for tree-width is a *bramble* (named by Reed [**214**]): a bramble in a finite graph G is a collection of connected subsets of $V(G)$ such that the union of any two of these subsets forms again a connected subset of $V(G)$. (To be more precise, we should say the graph induced on these subsets is connected.) This means for any two members of a bramble, there exists an edge such that each of the two element contain an endpoint of that edge.

The *order* of a bramble is the minimum size of a subset of vertices which intersect any set in the bramble. The bramble number of G, denoted by $bn(G)$, is the maximum order of a bramble in G. The duality theorem then is the following statement.

THEOREM 89 (Seymour-Thomas [**224**]). *For any graph G, $tw(G) = bn(G) - 1$.*

For a game theoretic interpretation of this result in terms of a search game between a fugitive and a group of searchers in the graph see [**224**]. More general forms of the duality theorem can be found in [**AMNT, 92**].

Example 90. Let H be an $n \times n$ grid, which has n^2 vertices enumerated with pairs (i, j), $1 \leq i, j \leq n$. Let H_0 be the upper left $(n-1) \times (n-1)$ sub-grid of H. Let C be the last column of H, and let L be the last line of H from which we remove the vertex (n, n), i.e., $L = \{(n, 1), \ldots, (n, n-1)\}$. Consider the collection $\mathcal{B}$ consisting of all the crosses of H_0, and both C and L. A cross is the union of any line and any column. It is easy to see that $\mathcal{B}$ has order $n+1$, which then implies that $tw(H) \geq n$. Actually, we have the equality $tw(H) = n$, as one can easily construct a tree-decomposition of H of width n. This example shows that grid graphs can have very large tree-width. Since all grid graphs are planar, the example shows that the tree-width is an unbounded function on the class of planar graphs.

The other important notion in graph theory is the notion of *minor* in a graph. A graph H is a *minor* of another graph G, and we write $H \preceq G$, if H is isomorphic to a graph H' which can be obtained from G by a sequence of operations consisting in
- contracting an edge of G, or
- removing an edge of G.

The main theorem concerning the notion of graph minors is the following theorem of Robertson and Seymour, which is a finiteness theorem with some mathematical logic flavour.

Theorem 91 (Robertson-Seymour [**215**]). *Let $\mathcal{F}$ be a family of graphs which is stable under minors, i.e., if $G \in \mathcal{F}$ and H is a minor of G, then H belongs to $\mathcal{F}$. Then there is a finite number of graphs (possibly empty if $\mathcal{F}$ contains all finite graphs) $H_1, \ldots, H_k$ such that G belongs to $\mathcal{F}$ if and only if G does not contain any of H_i as minor.*

We say a graph $G = (V, E)$ can be embedded in a surface S if there is a way to draw G on S in such a way that there is no crossing between the edge of G in the drawing. Since graphs embeddable on a fixed surface S form a family of graphs stable under minors, the above theorem provides a far reaching generalisation of Kuratowski theorem which characterises planar graphs as the family of graphs which do not contain the complete graph on five vertices K_5, and the complete bipartite graph $K_{3,3}$ on two parts of size three each. On the other hand, describing the family of forbidden minors H_i explicitly is a quite difficult problem. Beside the example of the planar graphs, and the graphs embeddable on the projective plane, there are no such explicit results in the literature.

Back now to the tree-width, it is easy to see that tree-width is minor monotone, in the sense that if $H \preceq G$, then $tw(H) \leq tw(G)$. Since grid minors have arbitrary large tree-width, it follows that bounded tree-width graphs cannot have large grid minors. In fact, Robertson and Seymour prove that tree-width is a bounded function on the class of graphs with forbidden H-minor if and only if H is planar.

7.1.2. Eigenvalue estimates in minor closed families. Let H be a given graph. Consider the family $\mathcal{F}_H$ of all connected graphs G which do not contain H as minor. Note that $\mathcal{F}_H$ is minor closed. The following theorem shows that graphs in $\mathcal{F}_H$ are far from being an expander.

Theorem 92 (Kelner-Lee-Price-Teng [**151**]). *There is a constant $h = h(H)$ such that for any graph G in $\mathcal{F}_H$ and any $1 \leq k$, we have $\lambda_k(G) \leq \frac{h d_{\max} k}{|G|}$ where $d_{\max}$ is the maximum valence of vertices in G and $|G|$ is the number of vertices in G.*

The paper [**151**] contains also a similar result for the case of graphs of bounded geometric genus, where roughly speaking, h is replaced by g, the genus of the surface on which all the graphs can be embedded. We discuss this and other previous results in the next section. As we will show later in Theorem 95, a more precise estimate holds in this case.

To end this subsection, consider now the class of bounded tree-width graphs. A graph of tree-width bounded by some constant N does not contain a grid of size $N \times N$ as minor, as was noted previously. It follows that

PROPOSITION 93. *Notations as above, there exists an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that for a graph G of tree-width $tw(G)$, one has $\lambda_k(G) \leq f(tw(G)) d_{\max} k/|G|$.*

For λ_1, we have the following more precise result.

THEOREM 94 (Chandran-Subramanian [**68**]). *For any graph $G = (V, E)$, the following holds*

$$\lambda_1 \leq \frac{12\big(tw(G) + 1)\big) d_{\max}}{|G|}.$$

More precise estimates in terms of the tree-width for higher eigenvalues are not known.

7.1.3. Eigenvalue estimates in graphs and surfaces: a parallel history. As we now outline, eigenvalue bounds for surfaces of bounded genus and graphs of bounded geometric genus have a somehow parallel history.

Motivated by the problem of clustering in planar graphs, Spielman and Teng [**228**] proved the bound

$$\lambda_1(G) = \mathcal{O}(\frac{1}{n})$$

for the spectral gap of a bounded degree planar graph on n vertices. Their method uses a suitably centered circle packing representation of the graph. Kelner extended this result to an $\mathcal{O}((g+1)/n)$ bound for geometric genus g graphs [**150**], for graphs which can be embedded in an orientable surface of genus g. His argument uses Riemann-Roch theorem for Riemann surfaces to find a circle packing representation of the graph. The approach cannot work directly and a subdivision process has to be first performed so that a conclusion may be reached. As we mentioned in the last section, recently, Kelner, Lee, Price and Teng proved an $\mathcal{O}((g+1)\log(g+1)^2 k/n)$ upper bound for the k-th eigenvalue [**151**]. Their bound comes from a metric graph partitioning result by Klein, Plotkin and Rao [**154**], which is applied to a suitably uniformized metric on the graph, found by solving a multicommodity flow problem.

Eigenvalue bounds for manifolds have a similar story. Hersch [**131**] first proved an $\mathcal{O}(\frac{1}{\mathrm{vol}(M)})$ bound for the Neumann value of the sphere $\mathbb{S}^2$ equipped with a Riemannian metric. Yang and Yau [**247**] then showed that for genus g surfaces an $\mathcal{O}((g+1)/\mathrm{vol}(M))$ bound holds, and Li and Yau improved the latter result by replacing the genus with the finer conformal invariant they defined [**170**]. (See also Section 7.3.1 for the discussion of these results.)
It is interesting to notice that these proofs are quite similar at a high level to the ones later used in the graph setting. Conformal uniformisation was used in place of circle packing representations, but the very same topological argument for centring the packing in the discrete case was used in the manifold case as well.

For higher eigenvalues, Korevaar [**160**] established an $O((g+1)k/\mathrm{vol}(M))$ for genus g surfaces, and Hassannezhad [**128**] improved this to $O((g+k)/\mathrm{vol}(M))$ by combining the two methods of constructing disjoint capacitors of Grigor'yan, Netrusov and Yau [**120**], and Colbois and Maerten [**77**].

7.2. Eigenvalue estimates in graphs of bounded geometric genus

This section presents the main result of [**AmiCo**]. We extend the result of [**128**] to the graph setting using a suitable transfer method, c.f. Theorem 98. In addition to providing a uniform arguably more conceptual proof of the results of [**151, 150, 228**], this should make the above mentioned existing similarities between the methods used in the spectral theory of surfaces and graphs more transparent.

7.2.1. Statement of the main theorem on eigenvalues. Let $G = (V, E)$ be a finite simple connected graph. Recall that for two vertices $u, v \in V$, we write $u \sim v$ if the two vertices u and v are connected by an edge in G. The valence of a vertex v of G is denoted by d_v^G, or simply d_v if there is no risk of confusion and the graph G is understood from the context, and we denote by $d_{\max}$ the maximum degree of vertices of the graph. Denote by n the number of vertices and by g the geometric genus of G.

Denoting by $C(G)$ the vector space of all real valued functions f defined on the set of vertices of G, and let Δ be the Laplacian of G.

We now define the *normalised Laplacian* $\mathcal{L}$ of G as follows. Let S be the linear operator on $C(G)$ whose matrix in the standard basis of $C(G)$ is diagonal with entries the valences of the vertices of G, i.e., for any $f \in C(G)$

$$\forall\, v \in V(G), \quad S(f)(v) = d_v f(v).$$

The normalised Laplacian is the operator $S^{-1/2} \Delta S^{-1/2}$.

We denote as before by

$$\lambda_0(G) = 0 < \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_{n-1}(G)$$

the set of eigenvalues of Δ, which we call the standard spectrum of G, and by

$$\lambda_0^{\mathrm{nr}}(G) = 0 < \lambda_1^{\mathrm{nr}}(G) \leq \cdots \leq \lambda_{n-1}^{\mathrm{nr}}(G)$$

the set of all eigenvalues of the normalised Laplacian $\mathcal{L}$, which we call the normalised spectrum. The standard and normalised spectrum of G are easily seen to satisfy the inequalities $d_{\min} \lambda_k^{\mathrm{nr}}(G) \leq \lambda_k(G) \leq d_{\max} \lambda_k^{\mathrm{nr}}(G)$ for any k.

Here is the main theorem of [**AmiCo**].

THEOREM 95 ([**AmiCo**]). *There exists a universal constant C such that the eigenvalues of the normalized Laplacian of any graph G on n vertices satisfy:*

$$\forall k \in \mathbb{N}, \qquad \lambda_k^{\mathrm{nr}}(G) \leq C \, \frac{d_{max}(g+k)}{n},$$

where $d_{\max}$ and g are the maximum valence and the geometric genus of G, respectively.

The linear dependance in the maximum degree is clearly optimal, as can be seen by considering star graphs, which have lower bounded Fiedler value. The above result also implies a similar bound for the eigenvalues of the standard Laplacian, at the expense of an extra d_{max} factor. The dependence in g and k in the theorem is tight, at least when g is sufficiently high as the following remark shows.

REMARK 96. It is shown in [**75**] that for large g, there are area one and genus g Riemannian surfaces S with

$$\lambda_k(S) \geq \frac{4\pi}{5}(g-1) + 8\pi(k-1) - \epsilon$$

for any $\epsilon > 0$. Now, the classical Brooks-Burger method implies the existence of a bounded degree graph G of genus g with n vertices such that $\lambda_k(G) \geq C\lambda_k(S)/n$. Hence, at least for large enough n and g, there are graphs whose eigenvalues match the behaviour of the estimate in Theorem 95.

We note that informally, the improvement over [**151**] means that the asymptotic behaviour of graphs' eigenvalues do not depend on the geometric genus of the graph. This fact, which may be seen as a one-sided discrete form of Weyl's law for surfaces, is consistent with the intuition that at a small scale, bounded genus graphs behave like planar graphs. Note however that the result in [**151**] also applies to graphs in any fixed proper minor-closed family, where as we explained, the genus g is replaced with a parameter h depending on the family, while the stronger bounds of Theorem 95 cannot be extended to minor-closed classes, as we show in the following explicit examples.

Remark 97. The following example shows that the strong estimates as in Theorem 95 cannot hold for more general classes of graphs closed under taking minor.

Recall that the Cartesian product $G_1 \Box G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ has vertex set $V_1 \times V_2$ and there is an edge between (v_1, v_2) and (u_1, u_2) in $V_1 \times V_2$ if either $u_1 = v_1$ and $\{u_2, v_2\} \in E_2$, or $u_2 = v_2$ and $\{u_1, v_1\} \in E_1$. The Laplacian eigenvalues of $G_1 \Box G_2$ are of the form $\lambda_i(G_1) + \lambda_j(G_2)$ for $i = 1, \dots, |V_1|$ and $j = 1, \dots, |V_2|$.

Let d be a fixed large enough integer, and for any $\ell \in \mathbb{N}$, consider the Cartesian product $C_{2l} \Box G$ of a cycle $C_{2\ell}$ of length $2l$ with a d-regular graph G on t vertices, for an integer $t \in \mathbb{N}$.

For any fixed $t \in \mathbb{N}$, we get in this way a family of graphs by varying ℓ and G. All these graph are of treewidth bounded by some $f(t)$ for a (linear) function f of t. Bounded treewidth graphs form a minor-closed family, so all these graphs belong to a fixed proper minor-closed family $\mathcal{F}_t$. For G a random d-regular graph on t vertices, and for the l^{th} eigenvalue of $C_{2l} \Box G \in \mathcal{F}_t$, for $l \in \mathbb{N}$, we have $\lambda_l(C_{2\ell} \Box G) = \Omega(\frac{tl}{|C_{2l} \Box G|})$ with high probability as t tends to infinity. This shows that there do not exist in general constants $h = h(\mathcal{F}_t)$ and $C = C(\mathcal{F}_t)$ associated to $\mathcal{F}_t$ ensuring that the inequality $\lambda_k(G) \leq C\, d_{max}^2\,(g_t + k)/n$ hold for any graph $G \in \mathcal{F}_t$ on n vertices, and for any $k \in \mathbb{N}$ (unlike what happens for the class of bounded genus graphs). In particular, the strong estimates as in Theorem 95 cannot hold for general minor-closed classes of graphs.

7.2.2. The transfer principle. We now explain the transfer theorem used to prove Theorem 95.

7.2.2.1. *Two-fold covers and their associated discrete Laplacians.* Let M be a measured topological space, and denote by μ the measure on M. A *2-fold cover* of M is a finite collection $\mathcal{U} = (U_v)_{v \in V}$, for a finite index set V, of open subsets U_v of non-zero measure such that almost every point in M is covered by exactly two subsets. To any 2-fold cover of a measured space we associate a discrete Laplacian as follows:

We first form a graph $G = (V, E)$ on the set of vertices V and with edges $\{u, v\} \in E$ for two vertices u, v such that $\mu(U_v \cap U_u) \neq 0$. We define a weight function $\omega : E \to \mathbb{R}$ which to any edge $e = \{u, v\}$ of G, associates the weight $\omega(e) = \mu(U_u \cap U_v)$. The weighted valence d_v^ω of a vertex v of G is defined by

$$d_v^\omega = \sum_{u : u \sim v} \mu(U_u \cap U_v).$$

The discrete Laplacian associated to the 2-fold cover $\mathcal{U}$ denoted by $\mathcal{L}_{\mathcal{U}}$ is the normalized graph Laplacian associated to the weighted graph (G, ω). This is defined from the weighted

Laplacian by normalizing using the weighted valence (as in the previous section). Formally, define the weighted Laplacian $\Delta_{\mathcal{U}} : C(G) \to C(G)$ by

$$\forall v \in V, \quad \Delta_{\mathcal{U}}(f)(v) = \sum_{u:u\sim v} \Big(f(v) - f(u)\Big) w(\{u,v\}),$$

for any $f \in C(G)$. Let $S_{\mathcal{U}}$ be the diagonal operator with entries the weighted valence d_v^ω of vertices $v \in V$, i.e., for any $f \in C(G)$,

$$\forall\, v \in V, \quad S_{\mathcal{U}}(f)(v) = d_v^\omega\, f(v).$$

Then we let $\mathcal{L}_{\mathcal{U}} := S_{\mathcal{U}}^{-\frac{1}{2}} \Delta_{\mathcal{U}} S_{\mathcal{U}}^{-\frac{1}{2}}$. Denote by $\lambda_k(\mathcal{L}_{\mathcal{U}})$ the k-th smallest eigenvalue of $\mathcal{L}_{\mathcal{U}}$.

When (M,μ) carries a natural notion of Laplacian, it is possible to relate the eigenvalues of the Laplacian on M to the eigenvalues of $\mathcal{L}_{\mathcal{U}}$ for any 2-fold cover $\mathcal{U}$. More precisely, let the measured space (M,μ) belong to any of the following three classes:

- ($\mathscr{C}1$) a smooth manifold with a smooth Riemannian metric $\mathfrak{g}$, and μ the measure associated to the metric $\mathfrak{g}$;
- ($\mathscr{C}2$) a compact smooth surface with a conformal class of smooth Riemannian metrics $\mathfrak{g}$, and μ a Radon measure absolutely continuous with respect to $\mu_{\mathfrak{g}}$;
- ($\mathscr{C}3$) a metric graph with μ the Lebesgue measure.

In any of the above cases, we can define a Laplacian on (M,μ) c.f. [**AmiCo, 156, 25, 248**], and we denote by $\lambda_k(M,\mu)$, or simply $\lambda_k(M)$ if there is no risk of confusion, the eigenvalues for the corresponding Laplacian. The transfer principle can be stated as follows.

Theorem 98 ([**AmiCo**]). *Let (M,μ) be a measured space as in ($\mathscr{C}1$), ($\mathscr{C}2$), or ($\mathscr{C}3$) above. Assume all the elements in a 2-fold cover $\mathcal{U}$ of M have Neumann value at least η. Then for all positive integers k we have:*

$$\lambda_k(\mathcal{L}_{\mathcal{U}}) \leq 2\frac{\lambda_k(M)}{\eta}.$$

The main difference with the classical versions of the transfer principle [**61, 59, 185**] is that we discretize the continuous Laplacian as a weighted normalized graph Laplacian instead of a combinatorial one, which allows for a closer connection between the two. The above mentioned results take as input a partition of M, while this theorem is expressed in terms of two-fold covers, which adds more flexibility.

In order to prove Theorem 95, we apply the above theorem in the case where (M,μ) is a measured surface equipped with a conformal class of smooth Riemannian metrics $\mathfrak{g}$.

7.2.3. Further applications of the transfer principle. In this section, we briefly discuss two more applications of the transfer principle.

7.2.3.1. *Eigenvalues of the Laplacian on metric graphs.* Consider a graph $G = (V,E)$ where each edge e in E has been assigned a length $\ell_e > 0$, and let Γ be the corresponding metric graph. Let $S(\Gamma)$ be the space of piecewise smooth function on Γ.

The metric graph Γ has a natural Lebesgue measure denoted by dx. Recall from Chapter 4 that the Laplacian of Γ is the (measure valued) operator Δ on Γ which to a function $f \in S(\Gamma)$ associates the measure

$$\Delta(f) := -f''dx - \sum_p \sigma_p \delta_p,$$

where dx is the Lebesgue measure on Γ, δ_p is the Dirac measure at p, and σ_p is the sum of the slopes of f along the (outgoing) unit tangent vectors at p.

Define the space $\mathrm{Zh}(\Gamma)$ as the space of all functions $f \in S(\Gamma)$ such that $f'' \in L^1(\Gamma, dx)$. The inner product $(,)$ and the Dirichlet pairing $(,)_{\mathrm{Dir}}$ on $\mathrm{Zh}(\Gamma)$ are defined as follows.

$$(f,g) = \int_\Gamma fg\ dx.$$

$$(f,g)_{\mathrm{Dir}} := \int_\Gamma f\Delta(g) = \int_\Gamma g\Delta(f) = \int_\Gamma f'g'\,dx = (f',g').$$

A function f in $\mathrm{Zh}(\Gamma)$ is an eigenfunction on Γ with eigenvalue λ if for any function $g \in \mathrm{Zh}(\Gamma)$, we have $(f,g)_{\mathrm{Dir}} = \lambda(f,g)$. The eigenvalues of Δ are all nonnegative and, assuming Γ is connected, they form a discrete subset $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \dots$ of $\mathbb{R}$. In addition, λ_k has the following (usual) variational characterisation:

$$\lambda_k = \inf_{\substack{\Lambda_{k+1}\subset \mathrm{Zh}(\Gamma)\\ \dim(\Lambda_{k+1})=k+1}} \ \sup_{f\in\Lambda_{k+1}} \frac{(f,f)_{\mathrm{Dir}}}{(f,f)} \tag{16}$$

Let (G,ℓ) be a simple graph model of Γ. For any vertex v of G, denote by $S_G(v)$ the metric star with center v in G, which is by definition the metric graph with the vertex set $v \cup N_G(v)$ and with edge set the set of all edges in G incident on v, where the length of each edge $e \in S_v$ is equal to ℓ_e. Here $N_G(v)$ refers to the set of neighbor vertices of v in G. The family of all the stars S_v, for $v \in V(G)$, forms a 2-fold cover $\mathcal{S}$ of Γ. Denote by $\lambda_k^{nr}(G,\ell)$ the k-th eigenvalue of $\mathcal{L}_{\mathcal{S}}$. Applying Theorem 98, we get the existence of a universal constant C such that for any k, we have

$$\lambda_k^{nr}(G,\ell) \le Cl_{max}^2\lambda_k(\Gamma),$$

where l_{max} is the length of the longest edge in G.

We now show that under certain natural conditions, it is possible to achieve eigenvalue lower bounds closely matching the above upper bound. For a simple graph model (G,ℓ) of Γ denote by $\ell_{\min,G}$ the minimum length of edges e in $E(G)$, and call the model *length-balanced* if the length of all edges $e \in E(G)$ satisfy $\ell_e \le 2\ell_{\min,G}$. We have the following stronger theorem.

THEOREM 99 ([**AmiCo**]). *There are absolute constants c_1, c_2 such that for any length-balanced simple graph model (G,ℓ) of Γ on n vertices, and for any $k \le n-1$, we have*

$$\frac{c_2}{d_{\max}}\ell_{\min,G}^2\,\lambda_k(\Gamma) \le \lambda_k^{\mathrm{nr}}(G,\ell) \le c_1\ell_{\min,G}^2\,\lambda_k(\Gamma).$$

The theorem can be regarded as a uniform quantitative complement to results of Faber [**105**] on the spectral convergence of finite graphs to metric graphs.

7.2.3.2. *Anisotropic mesh partitioning.* We now discuss a practical application of our transfer theorem to the mesh partitioning problem in scientific computing. Parallelsing finite elements computations requires to split the base mesh in such a way that communication between different pieces is minimised. This is naturally formalised as a (possibly multi-way) sparsest cut problem, which we may want to solve using spectral clustering. Guarantees for such methods in this setting were proved by Miller-Teng-Thurston-Vavasis and Spielman-Teng [**190, 228**]. More precisely, these papers show that spectral partitioning provides good cuts for meshes in d-dimensional Euclidean space provided that all d-simplices in the mesh are *well-shaped*, i.e. not too far from being equilateral.

It is not hard to design a 2-fold cover of a general mesh such that our transfer result provides guarantees for spectral clustering applied to anisotropic meshes. Specifically, let T be a triangulation of a domain $D \subset \mathbb{R}^d$. Performing a barycentric subdivision of all d-simplices gives a triangulation T'. For a d-simplex σ of T, let now U_σ be the interior of the union of σ with the $d+1$ d-simplices of T' that share a facet with σ. The collection of U_σ forms a 2-fold cover $\mathcal{U}$ of the domain, and the corresponding Laplacian $\mathcal{L}_\mathcal{U}$ is defined using weights w_{σ_1,σ_2} that are proportional to the sum of the volumes of σ_1 and σ_2. Hence, assuming that neighbouring d-simplices in T have volumes within a ratio of $\kappa > 1$, we see that the eigenvalues of $\mathcal{L}_\mathcal{U}$ and those of the normalised Laplacian of the dual graph of T are also within a ratio of κ.

The Neumann value of U_σ can be shown to be at least $C^{-1}\kappa^{-1}\epsilon^{-2}$ for some universal constant $C > 0$, where ϵ is the maximum diameter of simplices in T. Therefore, Theorem 98 applied to the cover $\mathcal{U}$ yields that the Fiedler value of the dual graph of T is at most $2C\kappa^2\lambda_1(D)\epsilon^2$. By Cheeger's inequality, a suitable spectral partitioning algorithm gives a balanced cut of size at most $\kappa C'\sqrt{\lambda_1(D)}/\epsilon$, for some constant C'. If d-simplices in T are nearly equilateral, then we have an upper bound on κ, and thus we get $\epsilon \simeq (\mathrm{vol}(D)/n)^{1/d}$, where n is the number of simplices in T. We so recover in this case the $n^{1/d}$ behaviour proved in [**190, 228**] for the size of the cut. However, the methods used in these works do not seem to apply to the case of general anisotropic meshes.

7.3. Li-Yang-Yau inequality

In this section, we discuss Yang-Li-Yau type inequalities for the geometric and divisorial gonality in general metric graphs, implying a similar inequality for curves over non-Archimedean fields.

7.3.1. Classical version. We first recall the Li-Yau inequality [**170**]. Let M be a compact surface with a Riemannian metric g. We denote by $d\mu$ the volume form corresponding to its metric, and by $\mu(M)$ the total volume of M. Consider the sphere $\mathbb{S}^2$ with its standard metric g_0, and let $\phi : M \to \mathbb{S}^2$ be a non-degenerate conformal map. The group of conformal diffeomorphisms of $\mathbb{S}^2$, denoted by $\mathrm{Diff}_c(\mathbb{S}^2)$ acts on the set of non-degenerate conformal maps from M to $\mathbb{S}^2$ in a natural way. Define $\mu_c(M,\phi)$ as the supremum volume of M with the respect to the volume forms induced on M from S^2 by the conformal maps in the orbit of ϕ, i.e.,

$$\mu_c(M,\phi) := \sup_{\psi \in \mathrm{Diff}_c(\mathbb{S}^2)} \int_M |\nabla(\psi\circ\phi)|^2 d\mu.$$

The conformal area $\mu_c(M)$ of M (with respect to the conformal structure on M induced by the metric g) is by definition the infimum of $\mu_c(M,\phi)$ over non-degenerate conformal maps $\phi : M \to \mathbb{S}^2$, i.e., $\mu_c(M) := \inf_\phi \mu_c(M,\phi)$.

THEOREM 100 (Li-Yau [**170**]). *Denote by $\lambda_1 > 0$ the first non-zero eigenvalue of the Laplacian of (M,g). Then $\lambda_1\mu(M) \le 2\mu_c(M)$.*

This refines earlier results of Hersch [**131**] and Szegö [**231**]. Furthermore, as a corollary, it gives the following previous result of Yang and Yau [**247**].

Let M be a Riemann surface, equipped with a metric of constant curvature in its conformal class, λ_1 and μ the first non-trivial eigenvalue of the Laplacian and the volume of

M, respectively. Denote by $\gamma(M)$ the gonality of M, the minimum degree of a (branched) covering $M \to \mathbb{P}^1(\mathbb{C})$.

For a conformal map of positive degree d from M to N, one has $\mu_c(M) \leq d\mu_c(N)$, it follows that

$$\lambda_1 \, \mu(M) \leq 2\gamma(M) \, \mu_c(\mathbb{S}^2).$$

Observing that $\mu_c(\mathbb{S}^2) = 4\pi$, one obtains

THEOREM 101 (Yang-Yau [**247**]). *For any Riemann surface M,*

$$\lambda_1 \, \mu(M) \leq 8\pi\gamma(M).$$

This result has been quite useful for applications in arithmetic geometry, for instance in the study of rational points of bounded degree on smooth proper curves over a number field K, see for example [**1, 97**]. Indeed a theorem of Frey [**110**] (based on the results of Faltings) implies that curves of large gonality have only finite number of points defined over finite extensions of K of bounded degree, and the Yang-Li-Yau inequality above provides a practical lower bound on the gonality in terms of geometric invariants of a complexification of the curve.

Analogous type of estimates for smooth proper curves defined e.g. over global function fields can be obtained by working with the analytification of the curve over any place of the global field, which deformation retracts to a finite metric graph.

7.3.2. Yang-Li-Yau for metric graphs. We state two types of Yang-Li-Yau inequalities, one concerning the geometric gonality and one concerning the divisorial gonality of metric graphs, as defined in the previous section.

7.3.2.1. *Combinatorial version I: geometric gonality.* Recall from Chapter 3 that the geometric gonality $\gamma_{gm}(C)$ of a tropical curve is by definition the minimum degree of a finite tropical map to the tropical $\mathbb{TP}^1$.

Let C be a tropical curve with combinatorial type a graph G with set of vertices V and set of edges E. Denote by λ_1 be the first non-trivial eigenvalue of the discrete Laplacian Δ of G. We have

THEOREM 102 (Cornelissen-Kato-Kool [**84**]). *There is a universal constant A such that for any tropical curve C and any finite graph model G of C, we have the following inequality*

$$\gamma_{gm}(C) \geq A \frac{\lambda_1}{d_{\max}} |G|,$$

where $d_{\max}$ denotes the maximum valence, and $|G|$ is the number of vertices in G.

We give an alternative shorter proof of the above theorem. First, we have the following basic proposition relating the geometric gonality of a tropical curve with combinatorial type G to the tree-width of G.

PROPOSITION 103. *For any tropical curve C with finite graph model $G = (V, E)$, we have the inequality $2\gamma_{gm}(C) \geq tw(C)$.*

PROOF. Let $\phi : C \to \mathbb{TP}^1$ be a morphism of degree $\gamma_{gm}(C)$. Consider the restriction of ϕ to a finite harmonic morphism from a metric graph representative Γ of C with a model graph $\widetilde{G}$ on vertex set $\widetilde{V}$ and edge set $\widetilde{E}$ such that G is a subgraph of $\widetilde{G}$, and denote by T the image of Λ in $\mathbb{TP}^1$, so T is a finite tree. Let I_1 be a vertex set for T which contains $\phi(V)$, and E_1

be the corresponding set of edges. For each edge e in T_1 take a point in the interior of e, and let I be the new vertex set for T obtained by adding to I_1 all these new vertices.

A tree-decomposition $(T, \mathcal{X})$ of G can be defined as follows. For each vertex i in I, consider the preimage $\phi^{-1}(i)$ of i. This set consists of some (possibly empty) vertices $v_1, \dots, v_s$ of G and some (possibly empty) points $x_1, \dots, x_l$ in the interior of some edges $e_1 = u_1w_1, \dots, e_l = u_lw_l$ of G. Define $X_i = \{v_1, \dots, v_s, u_1, w_1, \dots, u_l, w_l\}$. Since ϕ is of degree $\gamma(C)$, $|\phi^{-1}(i)| \leq \gamma_{gm}(C)$ and thus, X_i has cardinality at most $2\gamma_{gm}(C)$. It is easy to check that $(T, \mathcal{X} = \{X_i\}_{i\in I})$ is a tree-decomposition of G. This proves the proposition. □

Proof of Theorem 102. This follows from the above proposition and the bound given in Theorem 94. □

As another corollary, note that if a graph G is a model of a tropical curve with bounded geometric gonality, then the tree-width of G is bounded, and thus, G cannot contain a large grid as minor. Combined with Proposition 103, one obtains the following corollary.

COROLLARY 104. *For any tropical curve C of combinatorial type G, one has*

$$f(2\gamma(C)) \geq \lambda_k(G).|G|\frac{d_{\max}}{k},$$

for the function f given in Proposition 93.

In particular, if in a family of tropical curves C_i of combinatorial type G_i, $d_{\max}$ is bounded and for some constant k, $\lambda_k(G_i).|G_i|$ tend to infinity, then one has $\gamma_{gm}(C_i) \to \infty$.

Theorem 102 has applications in arithmetic geometry. For example, it can be used to obtain a linear lower bound in the genus for the gonality of Drinfeld modular curves, which allowed to lower bound the modularity of elliptic curves over function fields, to obtain finiteness results of rational points of bounded degree on Drinfeld modular curves, and to get uniform bounds on isogenies and torsion points of Drinfeld modules, c.f. [**84**] for more details.

7.3.2.2. *Combinatorial version II: divisorial gonality.* Let now Γ be a metric graph. Let Δ be the (continuous) Laplacian of Γ, and denote by λ_1 the first non-trivial eigenvalue of Δ. Denote by $\mu(\Gamma)$ the total length of Γ, and by $d_{\max}$ the maximum valency of points of Γ (which is the maximum degree of any simple graph model of Γ). For a simple graph model G of Γ, let $\ell_{\min}(G)$ be the minimum length of edges in G, and define $\ell_{\min}(\Gamma)$ as the maximum of $\ell_{\min}(G)$ over all simple graph models G of Γ.

Denote by γ_{div} the divisorial gonality of Γ, which we recall, is the smallest integer d such that there exists a divisor of degree d and rank one on Γ. Since the fibers of any tropical morphism of degree d from the tropical C defined by Γ to $\mathbb{TP}^1$ is a divisor of degree d and rank at least one, it follows that $\gamma_{gm}(C) \geq \gamma_{\mathrm{div}}(\Gamma)$. The following theorem we proved in [**AmiKool**] is thus a refinement of Theorem 102.

THEOREM 105 ([**AmiKool**]). *There exists a constant C such that for any compact metric graph Γ of total length $\mu(\Gamma)$ with first non-trivial eigenvalue $\lambda_1(\Gamma)$ of the Laplacian Δ, the following holds*

$$\gamma_{\mathrm{div}}(\Gamma) \geq C\frac{\lambda_1(\Gamma)\ell_{\min}(\Gamma)\mu(\Gamma)}{d_{\max}}.$$

Here $\ell_{\min}$ is the minimum edge length in Γ.

As a corollary of Theorem 105, and the specialisation inequality, c.f. Chapter 2, we get

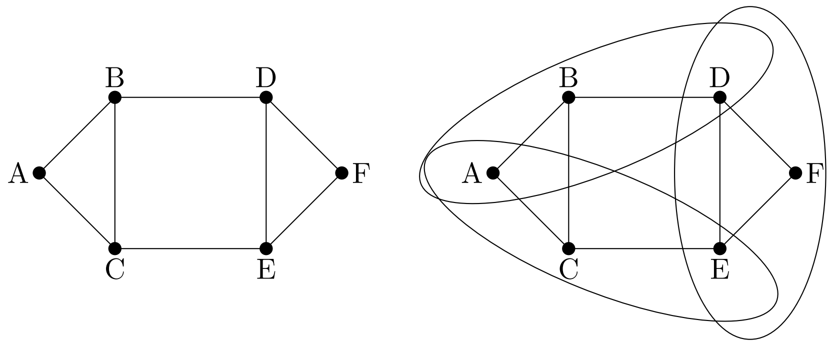

FIGURE 1. An example of a graph together with a strong bramble $\mathcal{F} = \{\{A, B, D\}, \{A, C, E\}, \{D, E, F\}\}$. The minimum hitting set is $\{D, E\}$, and the strong bramble number is 2.

THEOREM 106. *Let X be a smooth proper curve over a non-Archimedean field K, and let Γ be a metric graph associated to X. We have*

$$\gamma(X) \geq C\frac{\mu(\Gamma)\ell_{\min}(\Gamma)\lambda_1(\Gamma)}{d_{\max}}.$$

Here $\gamma(X)$ is the gonality of X, which is the minimum degree of a dominant map from X to $\mathbb{P}^1$, and C is the constant provided by Theorem 105.

The proof of Theorem 105 is based on a variant of tree-decomposition and its dual structure, introduced in our paper which we call weak tree-decompositions and strong brambles/topological bramble.

A *topological bramble* (or simply top-bramble) in a metric graph Γ is a finite family $\mathcal{F}$ of non-empty closed connected metric subgraphs of Γ such that any two elements X and Y in $\mathcal{F}$ have a non-empty intersection. The order of a top-bramble $\mathcal{F}$ is the minimum size of a hitting set for $\mathcal{F}$, i.e., the minimum size of a set $S \subset \Gamma$ such that $S \cap X \neq \emptyset$ for any $X \in \mathcal{F}$. The topological bramble number of Γ denoted by $tbn(\Gamma)$ is the maximum order of any topological bramble on Γ. We prove the following theorem.

THEOREM 107 ([**AmiKool**]). *Let Γ be a metric graph. The divisorial gonality of Γ is lower bounded by its top-bramble number $tbn(\Gamma)$.*

Let now $G = (V, E)$ be a finite simple graph on vertex set V and with edge set E. We previously gave the definition of a bramble in Section 7.1.1, and mentioned that it provides a dual notion for tree-width. A strong bramble is a specific kind of bramble in a finite graph defined as follows. A strong bramble in G is a finite collection $\mathcal{F}$ of connected subsets of G such that for any two elements B and C, one has $B \cap C \neq \emptyset$. Figure 1 illustrates an example of a graph with a strong bramble.

Note in particular that for any two elements B and C in $\mathcal{F}$, the union $B \cup C$ is obviously connected. In other words, a strong bramble is a bramble. The order of a strong bramble is defined as its order as a bramble, i.e., the minimum size of a hitting set for $\mathcal{F}$ in V. The strong bramble number of a finite graph G, denoted by $sbn(G)$, is the maximum order of any strong bramble in G.

It is proved in [**AmiKool**] that the topological bramble number of a metric graph Γ is the supremum of the strong bramble number of any simple graph model of Γ.

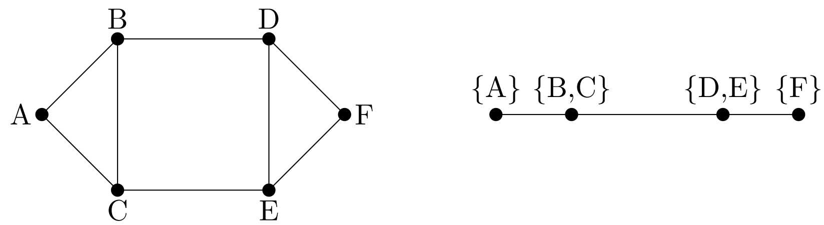

FIGURE 2. The same graph as above with a weak tree-decomposition of width 2.

We provide in [**AmiKool**] a dual notion to strong brambles, which we call weak tree-decompositions.

A weak tree-decomposition of a connected grap $G = (V, E)$ is a pair $(T, \mathcal{S})$ where T is a finite tree on a set of nodes I, and $\mathcal{S} = \{S_i : i \in I\}$ is a collection of subsets of V, subject to the following three conditions:

(1) $\cup_{i \in I} S_i = V$,
(2) for any edge e in G with extremities v and w, there is an edge $\{i, j\}$ in T such that $\{v, w\} \subset S_i \cup S_j$.
(3) for any vertex v in G, the set of nodes i of T with $v \in S_i$ form a connected subtree of T.

This is illustrated in the example given in Figure 2.

Note that the only difference with the usual definition of a tree-decomposition is in point (2) where we impose a weaker condition. In particular, it might happen that an edge e of G is not necessarily contained in any set $S_i \in \mathcal{S}$. The width of a weak tree-decomposition $(T, \mathcal{S})$ is defined as $w(T, \mathcal{S}) = \max_{i \in I} |S_i|$. The weak tree-width of G, denoted by $wtw(G)$, is the minimum width of any weak tree-decomposition of G. Similar as in the definition of tree-width, the weak tree-width is defined in such a way that trees themselves have weak tree-width equal to one.

The following theorem is a duality theorem, in the spirit of the duality theorems in graph minor theory, which relates strong brambles to weak tree-decompositions. It does not seem to follow from the generalised forms of duality established in [**AMNT, 92**].

THEOREM 108 ([**AmiKool**]). *A finite graph G has a weak tree-decomposition of width k if and only if there is no strong bramble of order strictly larger than k in G. In other words, $wtw(G) = sbn(G)$.*

The proof of Theorem 105 is then the combination of all the above results with Theorem 94.

The following theorem is an immediate corollary of the above discussion.

THEOREM 109 ([**AmiKool**]).
- *The divisorial gonality of a random Erdös-Rényi graph $G(n, p)$ is $\Theta(n)$ asymptotically almost surely in the range $p >> \frac{1}{n}$. More generally, the divisorial gonality of any metric graph whose model is a random $G(n, p)$ is $\Theta(n)$.*
- *The divisorial gonality of a random d-regular graph is $\Theta(n)$ asymptotically almost surely, for $d \geq 3$.*

CHAPTER 8

Trees: random walk, explosion, and logarithmic factorials

This chapter summarises the result of the three papers [**ADGO1, ADGO2, Ami3**]. All the three works are related to the properties of random walks on trees. The first two papers study the explosion phenomena in branching random walks, while the last work is related to the concept of generalised factorials as defined in an arithmetic concept by Bhargava. Generalising the work of Bhargava, we associate to any finite or infinite tree, a characteristic sequence of numbers that we call logarithmic factorials of the tree. The relation to random walk on the tree then comes in to the picture when we study the growth of this characteristic numbers. In fact as we will show, the way the characteristic numbers are defined provide an algorithm to sample the harmonic measure on the boundary of the tree, in case such a harmonic measure exists.

8.1. Explosion in heavy-tailed branching random walks

A branching random walk consists of a process in which particles execute random walks while also branching. In the case the random walk is executed on the space of real numbers, the process is more formally described as follows. Consider a process which starts by a single particle positioned at 0. The particle moves to another point of $\mathbb{R}$, according to a displacement distribution W, and dies at this point while generating $k \in \mathbb{N}$ other particles accordingly to an offspring distribution Z. Each of these particles in turn executes a walk in $\mathbb{R}$ according to the displacement distribution W, then dies and generates new particles according to Z. This procedure is repeated all over the integer values of time. We will assume that W is non-negative (in which case, the process is also called an age-dependent process).

Considering the tree-picture of the above process, let T_Z refer to a random Galton-Watson tree with offspring distribution Z, and let Z_n be the number of children at level n. Each child's birthdate is equal to the birthdate of the parent, plus a random copy of W, where $W \geq 0$ is the displacement random variable. The collection of birthdates is the standard branching random walk described above.

Let M_n be the smallest or leftmost birthdate in the n-th generation. We study the behaviour of M_n when n goes to infinity. Namely, we consider the event that $\lim_{n\to\infty} M_n < \infty$. This event is called *explosion*. (To cover the case of extinction, we can enlarge the definition of M_n defining $M_n = \infty$ if $Z_n = 0$; thus extinction implies that no explosion occurs.) In the tree view, with generations in the Galton-Watson process, the occurrence of an explosion is equivalent to the existence of an infinite path along which the (increasing) sequence of birthdates has a finite limit.

The literature on explosion is partially surveyed by Vatutin and Zubkov [**238**]. The early work deals with exponentially distributed weights: in this case, there is no explosion almost surely if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n \sum_{r=0}^{n} \mathbb{P}\{Z > r\}} < \infty$$

(see [**127**, V. 6], [**220, 93**]). This condition cannot be simplified; Grey [**118**] showed that there does not exist any fixed function $\psi \geq 0$ such that explosion would be equivalent to $\mathbb{E}\{\psi(Z)\} = \infty$.

Some general properties of the event of explosion were obtained in [**222**] by considering the generating functions of the number of particles born before time t, parametrized by t, and looking at the nonlinear integral equation satisfied by these generating functions. By using this analytic approach, and under a strong regularity condition on the distribution function F_W of the displacement W, Sevast'yanov [**222, 223**], Gel'fond [**115**], and Vatutin [**236, 237**] obtain necessary and sufficient conditions on the event of explosion. The regularity condition used in these works force F_W to behave in a polynomial manner near the origin. As soon as we leave that polynomial oasis, Vatutin's condition is violated. Examples include $F_W(w) \sim \exp(-1/w^\alpha)$ and $F_W(w) \sim 1/\log^\alpha(1/w)$ for $\alpha > 0$.

We provided in [**ADGO1**] a classification theorem for the displacement random variables in heavy-tailed branching random walks in $\mathbb{R}$ for which an explosion happens. In the rest of this section, we explain the results of [**ADGO1**].

8.1.1. Reduction to the case of an infinite mean. In the process of studying the event of explosion, we first consider the case where the offspring distribution has finite mean. The different cases described below show that we can either trivially solve the problem, or reduce to the most interesting case of an infinite mean.

Consider a Galton-Watson process with offspring distribution Z satisfying $0 < \mathbb{E}\{Z\} < \infty$. We still assume $\mathbb{P}\{Z = 1\} < 1$. Let W be a weight (or displacement) distribution on the edges of the Galton-Watson tree.

Consider first the case where $\mathbb{P}\{W = 0\} = 1$. In this case, explosion is equivalent to the event that the Galton-Watson tree is infinite, i.e., the survival of the Galton-Watson process. In that case, if $\mathbb{E}\{Z\} \leq 1$, there is no survival, and if $\mathbb{E}\{Z\} > 1$, there is a positive probability of survival [**16**]. From now on we will assume that $\mathbb{P}\{W = 0\} < 1$ and assume that the Galton-Watson process is supercritical.

In the case of a supercritical Galton-Watson process, under the assumption $\mathbb{E}\{Z\} < \infty$, the results of Hammersley [**126**], Kingman [**152**], and Biggins [**39**] show the existence of a constant γ such that conditional on the non-extinction of the process, M_n/n tends to γ almost surely. This shows that the random variables M_n, conditional on survival, behave linearly in n, i.e., $M_n = \gamma n + o(n)$. One consequence of the Hammersley-Kingman-Biggins theorem is that if $\gamma > 0$, then explosion never happens. Now define

$$H := \mathbb{E}\{Z\}\mathbb{P}\{W = 0\}.$$

It can be shown that $\gamma = 0$ if and only if $H \geq 1$. We consider in fact three cases: $H < 1$, $H > 1$ and $H = 1$.

• Case I: $H < 1$.

Here, as stated above, explosion occurs with probability zero.

• Case II: $H > 1$.

In this case, explosion happens with probability one. To see this, take a sub-Galton-Watson tree by keeping only children for which $W = 0$. This tree is supercritical and thus survives with some positive probability ρ. It follows that with positive probability, there is an infinite path of length zero. It can be shown that conditional on survival, explosion is a 0-1 event, and so we we infer that it happens with probability one. A theorem of Dekking and Host [**87**] ensures the existence of an almost surely finite random variable M such that M_n converges a.s. to M. Under the extra condition $\mathbb{E}\{Z^2\} < \infty$, they determine stronger results on the limit distribution M.

• Case III: $H = 1$.

This threshold case is the most intriguing—it was already considered in an earlier pioneering work of Bramson [**54**], and the work of Dekking and Host [**87**]. In this case, the occurrence of explosion is a delicately balanced event that depends upon the behavior of the distribution of W near the origin and on the distribution of Z.

Bramson's main theorem is the following result on the behaviour of M_n under the assumption that there exists a $\delta > 0$ such that $\mathbb{E}\{Z^{2+\delta}\} < \infty$. For any fixed λ, define $\sigma_{\lambda,n} = p + (1-p)e^{-\lambda^n}$ where $p = \mathbb{P}\{W = 0\} < 1$. Then explosion happens if and only if there exists some $\lambda > 1$ such that $\sum_{n=1}^{\infty} F_W^{-1}(\sigma_{\lambda,n}) < \infty$. In the case of no explosion, and conditional on the survival of the branching process, the following convergence result on the asymptotic of M_n holds. Almost surely, we have

$$\lim_{n\to\infty} \frac{M_n}{\sum_{k=1}^{s(n)} F_W^{-1}(\sigma_{2,k})} = 1, \tag{17}$$

where $s(n) = \lceil \log\log n / \log 2 \rceil$. We refer to [**87**] for a generalisation of Bramson's theorem to the case of $\mathbb{E}\{Z^2\} < \infty$, under some extra mild conditions.

Following Bramson [**54**], we first transform the tree T_Z into a new tree T' as follows. The roots are identical. First consider the sub-Galton-Watson tree rooted at the root of T_Z consisting only of children (edges) that have zero weight. This subtree is critical. For any distribution of Z satisfying the threshold condition, note that the size S of the sub-Galton-Watson tree is a random variable $S \ge 1$ with $\mathbb{E}\{S\} = \infty$. In some cases, we know more—for example, when $\mathrm{Var}\{Z\} = \sigma^2 \in (0,\infty)$, then $\mathbb{P}\{S \ge k\} \sim \sqrt{2/\pi\sigma^2 k}$ as $k \to \infty$ (see, e.g., the book of Kolchin [**159**]). All of the nodes in S are mapped to the root of the new tree T'. The children of that root in T' are all the children of the mapped nodes in T_Z that had $W \neq 0$.

Let X_i be the number of vertices of degree i in the sub-Galton-Watson tree. The number of children of the root of T_Z is distributed as

$$\zeta = \sum_{i=0}^{\infty} \sum_{j=1}^{X_i} \zeta_{i,j},$$

where $\zeta_{i,1}, \zeta_{i,2}, \dots$ are i.i.d. random variables having distribution of a random variable ζ_i. In addition the distribution of ζ_i is given by

$$\mathbb{P}\{\zeta_i = k\} = c_i \binom{k+i}{i} (1 - \mathbb{P}\{W=0\})^k \, \mathbb{P}\{W=0\}^i \, \mathbb{P}\{Z = k+i\},$$

where c_i is a normalizing constant. Note that $\sum_{i\ge 0} X_i = S$.

For each child of the root in T', repeat the above collapsing procedure. It is easily seen that T' itself is a Galton-Watson tree with offspring distribution ζ. The moment generating function $G_\zeta(s)$ of ζ is easily seen to satisfy the functional equation

$$G_\zeta(s) = G_Z\Big((1 - \mathbb{P}\{W = 0\})s + \mathbb{P}\{W = 0\}G_\zeta(s)\Big). \tag{18}$$

Furthermore, the displacement distribution is W conditional on $W > 0$. Finally, one can verify that $\mathbb{E}\{\zeta\} = \infty$. More importantly, explosion occurs in T_Z if and only if explosion happens in T'. We have thus reduced the explosion question to one for a new tree in which the expected number of children is infinite, and in which W does not have an atom at zero.

Observe that the transformation described in Case III is valid whenever W has an atom at the origin. In particular, this construction can also be used to eliminate an atom at the origin when $\mathbb{P}\{W = 0\} > 0$ and $\mathbb{E}\{Z\} = \infty$. In this case, we still have $\mathbb{E}\{\zeta\} = \infty$.

It follows from the above discussion that in the study of the the event of explosion, we need to consider only the (most interesting) case where

$$\mathbb{E}\{Z\} = \infty,\ \mathbb{P}\{W = 0\} = 0.$$

All our results below are concerned only with this case.

Remark 110. Before moving to the infinite mean case, let us mention that more precise information on the behaviour and convergence to infinity of M_n can be obtained in the finite mean case under extra conditions. Recall that in the finite mean case, $M_n = \gamma n + o(n)$ for some $\gamma \geq 0$. McDiarmid showed in [**187**] that $M_n - \gamma n = O(\log n)$ if $\mathbb{E}\{Z^2\} < \infty$ and W has an exponential upper tail. Recently, Hu and Shi [**138**] proved that if the displacements are bounded and $\mathcal{E}\{Z^{1+\epsilon}\} < \infty$ for any $\epsilon > 0$, then conditional on survival, $(M_n - \gamma n)/\log n$ converges in probability but, interestingly, not almost surely. This work and the recent work of Aïdekon-Shi [**6**] provide Seneta-Heyde norming results [**40**] in the boundary case. Under the extra assumption that Z is bounded, Addario-Berry and Reed [**3**] calculate $\mathbb{E}\{M_n\}$ to within $O(1)$ and prove exponential tail bounds for $\mathbb{P}\{|M_n - \mathcal{E}\{M_n\}| > x\}$. Extending these results, Aïdekon [**5**] proves the convergence of M_n centered around its median for a large class of branching random walks. Tightness results in general can be found in Bachmann [**18**] and Bramson-Zeitouni [**56, 55**].

8.1.2. A simple necessary condition for explosion. There is a rather obvious necessary condition for explosion. Let Y_i be the minimum weight edge at level i in the tree. Then the sum of weights along any infinite path is certainly at least $\sum_{i=1}^{\infty} Y_i$. We say that a fixed weighted tree is *min-summable* if this sum is bounded; if a tree is not min-summable, it cannot have an exploding path.

For any fixed, infinite, rooted tree T, and distribution W on the nonnegative reals, let T^W denote a random weighted tree obtained by weighting each edge with an independent copy of W. For a fixed tree T and weight distribution W, it follows easily from Kolmogorov's 0-1 law that explosion and min-summability of T^W are both 0-1 events. Thus, we make the following definitions.

Definition 111. For any infinite rooted tree T,

(i) let $\mathcal{W}_{\text{EX}}(T)$ be the set of weight distributions so that T^W contains an exploding path almost surely, and
(ii) let $\mathcal{W}_{\text{MS}}(T)$ be the set of weight distributions so that T^W is min-summable almost surely.

In this new notation, the observation above is simply that $\mathcal{W}_{\text{EX}}(T) \subseteq \mathcal{W}_{\text{MS}}(T)$, for any tree T. Unsurprisingly, in general $\mathcal{W}_{\text{EX}}(T)$ may be strictly contained within $\mathcal{W}_{\text{MS}}(T)$. For example, consider an infinite binary tree T and a uniform weight distribution W on $[0,1]$. Except with probability at most $\exp(-2^{i/2})$ the minimum of 2^i copies of W is at most $2^{-i/2}$. Thus, with positive probability $\sum_{i\geq 1} Y_i \leq \sum_{i\geq 1} 2^{-i/2} < 3$, and so $W \in \mathcal{W}_{\text{MS}}(Z)$. On the other hand, we may easily prove that $W \notin \mathcal{W}_{\text{EX}}(Z)$.

8.1.3. Statement of the main results. It may appear that, aside from some trivial cases, $\mathcal{W}_{\text{MS}}(T)$ should always strictly contain $\mathcal{W}_{\text{EX}}(T)$. However, somewhat counter-intuitively, this is not the case; there are examples of trees with generation sizes growing very fast (double exponentially) for which $\mathcal{W}_{\text{EX}}(T) = \mathcal{W}_{\text{MS}}(T)$. Consider for example the tree T defined as follows: all nodes of generation n have 2^{2^n} children. In this case, for a given weight distribution W, the distribution of the sum of minimum weights of levels is

$$\sum_{n\geq 1} \min_{1\leq i\leq 2^{(2^n-1)}} W_n^i,$$

where each W_n^i is an independent copy of W. Also the path constructed by the simple greedy algorithm which, starting from root, adds at each step the lowest weight edge from the current node to one of its children, has total weight distributed as

$$\sum_{n\geq 1} \min_{1\leq i\leq 2^{2^{(n-1)}}} W_n^i.$$

The property of these sums being finite almost surely is clearly equivalent, so that $\mathcal{W}_{\text{EX}}(T) = \mathcal{W}_{\text{MS}}(T)$. Our main result is that this phenomenon is in fact quite general in trees obtained by a Galton-Watson process with a heavy tailed offspring distribution. To formalise this, let us call the distribution Z *plump* if for some positive constant ϵ the inequality

$$\mathbb{P}\{Z \geq m^{1+\epsilon}\} \geq \frac{1}{m} \tag{19}$$

holds for all m sufficiently large. Equivalently, Z is plump if its distribution function F_Z satisfies $F_Z^{-1}(1-1/m) \geq m^{1+\epsilon}$ for m sufficiently large. Note that $\mathbb{E}Z = \infty$ for any plump Z.

THEOREM 112 (Equivalence Theorem [**ADGO1**]). *Let Z be a plump distribution. Let T be a random Galton-Watson tree with offspring distribution Z, but conditioned on survival. Then we have $\mathcal{W}_{\text{EX}}(T) = \mathcal{W}_{\text{MS}}(T)$ with probability.*

We now state a second form of the equivalence theorem. For this, we extend the definition of $\mathcal{W}_{\text{EX}}$ and $\mathcal{W}_{\text{MS}}$ to Galton-Watson offspring distributions. So let Z be an offspring distribution and W a weight distribution. We define

$$\mathcal{W}_{\text{EX}}(Z) := \Big\{ W \,|\, W \in \mathcal{W}_{\text{EX}}(T_Z) \text{ almost surely conditioned on survival} \Big\}, \qquad \text{and}$$

$$\mathcal{W}_{\text{MS}}(Z) := \Big\{ W \,|\, W \in \mathcal{W}_{\text{MS}}(T_Z) \text{ almost surely conditioned on survival} \Big\}.$$

The alternative (though slightly weaker) formulation of the Equivalence Theorem can now be stated as follows:

THEOREM 113 (Equivalence Theorem-Second Version [**ADGO1**]). *For a plump distribution Z,*

$$\mathcal{W}_{\mathrm{EX}}(Z) = \mathcal{W}_{\mathrm{MS}}(Z).$$

Min-summability is clearly a simpler kind of condition than explosion; in particular, it only depends on the generation sizes Z_n rather than the full structure of the tree T_Z. Indeed, the Equivalence Theorem becomes more interesting if one observes that it is possible to derive the following quite explicit necessary and sufficient condition for min-summability.

THEOREM 114 ([**ADGO1**]). *Given a plump offspring distribution Z, let $m_0 > 1$ be large enough such that the condition* (19) *holds for all $m \geq m_0$. Define the function $h : \mathbb{N} \to \mathbb{R}^+$ as follows:*

$$h(0) = m_0 \qquad \textit{and} \qquad h(n+1) = F_Z^{-1}(1 - 1/h(n)) \quad \textit{for all } n \geq 1. \tag{20}$$

Then for any weight distribution W, $W \in \mathcal{W}_{\mathrm{MS}}(Z)$, and hence also $W \in \mathcal{W}_{\mathrm{EX}}(Z)$, if and only if $\sum_n F_W^{-1}(h(n)^{-1}) < \infty$.

One may wonder now if there is a way to weaken the condition given in (19) such that the equivalence theorem still remains valid. We show that this condition is to some extent the best we can ask for. More precisely, we prove

THEOREM 115 (Sharpness of Condition (19) [**ADGO1**]). *Let $g : \mathbb{N} \to \mathbb{N}$ be an increasing function satisfying*

$$g(m) = m^{1+o(1)}.$$

Then there is an offspring distribution Z satisfying $\mathbb{P}\{Z \geq g(m)\} \geq 1/m$ for all $m \in \mathbb{N}$, but for which $\mathcal{W}_{\mathrm{EX}}(Z) \neq \mathcal{W}_{\mathrm{MS}}(Z)$.

Another natural question to ask is the growth rate of M_n to infinity in the case there is a.s. no exploding path. Although there is no reason to expect a convergence theorem in the case of no explosion for general plump distributions in the absence of any smoothness condition on the tails of Z, we show that a slightly stronger plumpness property allows to obtain a precise information on the rate of convergence to infinity of M_n. To explain this, note that the plumpness assumption on Z is equivalent to $1 - F_Z(k) \geq k^{-\eta}$ for $\eta = \frac{1}{1+\epsilon}$ and for all k sufficiently large. Consider now the stronger smoothness condition

$$1 - F_Z(k) = k^{-\eta}\ell(k), \tag{21}$$

where ℓ is any continuous and bounded function which is nonzero at infinity. We have

THEOREM 116 (Limit Theorem [**ADGO1**]). *Let Z satisfy the smoothness condition, and let W be any weight distribution with $W \notin \mathcal{W}_{\mathrm{EX}}(Z)$. Then a.s. conditional on survival,*

$$\lim_{n\to\infty} \frac{M_n}{\sum_{k=1}^n F_W^{-1}\Big(\exp\big(-(1+\epsilon)^k\,\big)\Big)} = 1$$

for all $\epsilon > 0$.

To see the meaning of Condition (21), we remark that applying a Tauberian theorem, c.f. Feller [**108**, XIII. 5, Thm. 5]), this condition becomes equivalent to the following condition

$$K_Z(s) := 1 - G_Z(1-s) \sim a\, s^\eta \ell(\frac{1}{s})$$

near $s = 0$ for some $a > 0$, where G_Z is the moment generating function of Z. Going back to Case III of the finite mean case, and the transformation described there, we observe that the use of the functional Equation (18) allows to translate the smoothness condition above, imposed on the modified offspring distribution ζ of infinite mean (obtained after the transformation), to a smoothness condition on Z, the original distribution of finite mean. In particular,

$$K_\zeta(s) = 1 - G_\zeta(1-s) \sim a s^{1/(1+\epsilon)}\big(1 + O(s^\beta)\big) \qquad \text{for } s \text{ near zero,}$$

for some $a, \epsilon, \beta > 0$ is equivalent to a condition of the form

$$K_Z(s) \sim \mathbb{E}\{Z\}\, s - c\, s^{1+\epsilon}(1 + O(s^\delta)) \qquad \text{for } s \text{ near zero,} \tag{22}$$

for some $c, \delta > 0$. We note that Condition (22) assumes some regularity on the tails of Z but the variance could be infinite; thus, the above result can be regarded as a strengthening of the pioneering theorem of Bramson [**54**], that we mentioned in the beginning of this section.

8.2. Explosion and linear transit times in infinite trees

Following the notation we introduced in the previous section, assume now that the tree T is deterministic. Let i.i.d. random weights w_e be assigned to the edges of T, and let, as above, $M_n(T)$ denote the minimum weight of a path from the root to a node of the nth generation. In the context of first passage percolation, looking at the weight of an edge as the transition time between the two corresponding nodes, $M_n(T)$ is the first passage time to the nth generation. We now consider the possible behaviour of $M_n(T)$ with particular focus on the following cases: we say, as before, that T is *explosive* if

$$\lim_{n\to\infty} M_n(T) \,<\, \infty\,,$$

and say that T *exhibits linear growth* if

$$\liminf_{n\to\infty} \frac{M_n(T)}{n} \,>\, 0\,.$$

Pemantle and Peres introduced in [**208**] the concept of stochastic dominance between trees and proved that amongst trees with a given sequence of generation sizes, explosion is most likely in the case that the tree is spherically symmetric. Recall that a tree T is called spherically symmetric if all the vertices at generation n have the same number $f(n)$ of children, for some function $f : \mathbb{N}\cup\{0\} \to \mathbb{N}$. Pemantle and Peres proved that for a spherically symmetric tree T with a non-decreasing generation sizes f, and with weights w_e independent exponential random variables of mean one, the probability of the event of explosion is 0 or 1 according to whether the sum $\sum_{n=0}^{\infty} f(n)^{-1}$ is infinite or finite. They also showed that the same statement holds for weight random variables with distribution G satisfying $\lim_{t\to 0} G(t)t^{-\alpha} = c > 0$ for some $\alpha > 0$. Furthermore, they asked if the same simple explosion criterion holds for general edge weight distributions, under reasonable assumptions.

One of our aims in [**ADGO2**] is to answer, essentially completely, this question of Pemantle and Peres. To present the result, we need to introduce some notations.

Let $f : \mathbb{N}\cup\{0\} \to \mathbb{N}$, and let T_f denote the spherically-symmetric tree in which each node v of generation n has $f(n)$ children. Given a distribution function $G : [0,\infty) \to [0,1]$, let T_f^G denote the randomly weighted tree obtained by giving each edge of T_f an i.i.d. weight distributed according to G.

Definition 117. Given the distribution G and non-decreasing function $f : \mathbb{N} \cup \{0\} \to \mathbb{N}$, we say that f is *G-explosive* if T_f^G is explosive almost surely.

It is easy to see that a sufficient condition for f being G-explosive is (a slightly stronger version of) the "local min-summability" condition: apply a greedy algorithm to construct an infinite path in the tree T_f^G by starting from the root, and by choosing recursively for the end vertex v_n of the already constructed path up to level n, the minimum weight edge $v_n v_{n+1}$ among the $f(n)$ adjacent edges to level $n+1$. This motivates the following definition.

Definition 118. Given the distribution G and a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$, we say that f is *G-small* if

$$\sum_{n \geq 0} G^{-1}(f(n)^{-1}) \, < \, \infty \, . \tag{23}$$

One may now interpret Pemantle and Peres [**208**, Page 193] as asking the following.

Question 119 (Pemantle-Peres [**208**]). *For which G does the equivalence*

$$f \textit{ is } G\textit{-small} \Leftrightarrow f \textit{ is } G\textit{-explosive}$$

hold in the class of non-decreasing functions $f : \mathbb{N} \cup \{0\} \to \mathbb{N}$?

They speculated that perhaps the equivalence holds provided G is continuous and strictly increasing. The following definition encodes robust versions of the properties of being continuous and strictly increasing.

Definition 120. The distribution G is *controlled near* 0 if

$$1 \, < \, \liminf_{x \to 0} \frac{G(cx)}{G(x)} \, \leq \, \limsup_{x \to 0} \frac{G(cx)}{G(x)} \, < \, \infty \, ,$$

for some constant $c > 1$.

Note that in particular, if $\lim_{x \to 0} G(x) x^{-\alpha}$ exists and is positive, for some constant $\alpha > 0$, then G is controlled near zero. Thus, the following generalises the result of Pemantle-Peres [**208**] and, in spirit, confirms the validity of their speculation.

Theorem 121 ([**ADGO2**]). *If G is controlled near* 0*, then the equivalence*

$$f \textit{ is } G\textit{-small} \Leftrightarrow f \textit{ is } G\textit{-explosive}$$

holds in the class of non-decreasing functions $f : \mathbb{N} \to \mathbb{N}$.

On the other hand, we give examples that demonstrate that the "controlled near 0" condition cannot be significantly weakened. For example, we show that

Theorem 122 ([**ADGO2**]). *Let G be any weight distribution satisfying either*

$$\limsup_{i \to \infty} \frac{G(x_i)}{G(x_i / c)} \, < \, \limsup_{i \to \infty} \frac{G(c x_i)}{G(x_i)} \, = \, \infty \, , \tag{24}$$

or

$$1 \, = \, \liminf_{i \to \infty} \frac{G(x_i)}{G(x_i / c)} \, < \, \liminf_{i \to \infty} \frac{G(c x_i)}{G(x_i)} \, , \tag{25}$$

for some constant $c > 0$ and decreasing sequence $x_i : i \geq 1$ with limit 0*. Then there exists a function $f : \mathbb{N}_0 \to \mathbb{N}$ which is G-explosive but not G-small.*

In order to prove Theorem 121, we consider a class of infinite weighted trees related to the Poisson-weighted infinite tree, the PWIT, introduced by Aldous [**9, 10**], and determine precisely which trees in this class of generalised PWITs have linear growth almost surely [**ADGO2**].

The linear growth property in this class of generalised PWITs has other interesting applications. To describe this, consider the following method for constructing a random real tree. Given a sequence $\ell(i) : i \in \mathbb{N}$, define the real tree A_ℓ recursively as follows. Let $A_\ell(1)$ consist of a closed segment of length $\ell(1)$ rooted at one end, and for each $i \geq 1$, define $A_\ell(i+1)$ by attaching one end of a closed segment of length $\ell(i+1)$ to a uniformly randomly chosen point of the tree $A_\ell(i)$. Let

$$A_\ell^o := \bigcup_{i \geq 1} A_\ell(i)$$

and define A_ℓ as the completion of A_ℓ^o. The random real tree A_ℓ is referred to as the random real tree given by the stick breaking process obtained by cutting the positive real line according to the segment lengths sequence ℓ.

Note that in the case where the sequence $\ell(i)$ is the length of the segment $[P^\lambda(i), P^\lambda(i+1)]$ given by an inhomogeneous Poisson point process P^λ with intensity $\lambda(t) = t$ on $\mathbb{R}^+$, the random real trees A_ℓ is precisely the *continuum random tree* constructed by Aldous [**8**].

The behavior of such trees, in the case of deterministic lengths $\ell(i)$ obeying a limit law, has been studied recently by Curien and Haas [**85**].

For a real tree A, we denote by $d(A)$ the depth of A, i.e., the supremum of distances from points of A to the root. We denote by $\mathrm{diam}(A)$ the diameter of A. Note that $d(A) \leq \mathrm{diam}(A) \leq 2d(A)$.

It can be easily shown that the random real tree A_ℓ given by the stick breaking process is compact almost surely if and only the diameter of A_ℓ is finite almost surely, c.f. [**85**]

The following is then a very natural problem: *Classify all sequences $\ell(i), i \in \mathbb{N}$, for which we have $d(A_\ell) < \infty$, or equivalently, $\mathrm{diam}(A_\ell) < \infty$, almost surely.*

As an application of our result on linear growth of P$^\lambda$WIT, we answer this question completely for those length sequences $\ell(i)$ which are deterministic and decreasing [**ADGO2**].

THEOREM 123 ([**ADGO2**]). *Let $\ell(i), i \in \mathbb{N}$, be a decreasing sequence. Then $d(A_\ell) < \infty$ almost surely if and only if $\sum_{n \geq 1} \ell(2^n) < \infty$.*

The constant 2 as the base of the exponential may be replaced by any constant $c > 1$ without affecting the summability condition. The question in the case of general edge lengths, without the monotonicity condition, is still open.

8.3. Logarithmic tree factorials

In a series of two papers [**36, 37**], Bhargava proposed a nice and somehow surprising generalisation of the familiar notion of factorials of integers $n!$ to any subset of an arbitrary Dedekind ring. He used the concept to obtain several algebraic and arithmetic applications, leading to generalisations of much earlier results of Pólya [**210, 211**] and Ostrowski [**203**] on the structure of the ring of integer valued polynomials, those of Mahler [**183**] and Amice [**13**] on approximation of analytic functions by polynomials, and non-archimedean interpolation. We refer the interested reader to Bhargava's papers [**36, 37**] and the survey [**38**] for the discussion of these results and several interesting open questions.

In this section, we give an overview of the results of [**Ami3**], where we present a combinatorial generalisation of Bhargava's factorials. This consists in associating a sequence of *characteristic numbers* to any (finite or infinite) tree, that we might call *logarithmic factorials* of the tree. We discuss this generalisation and some of their basic properties. Then we relate the growth of these characteristic numbers to the transience or recurrence of the random walk on the tree.

8.3.1. Definition of tree factorials. Let T be a rooted tree with root $\mathfrak{r}$, and let $\ell : E(T) \to \mathbb{R}_+$ be a length function on the edges of T. Denote by Γ the metric realisation of the pair (T, ℓ), which is a rooted metric tree with root $\mathfrak{r}$. We call the unite length function $\ell \equiv 1$ which assigns value one to all the edges of a tree T the *standard* length function.

We orient T away from the root, and, by an abuse of the notation, denote by $E(T)$ the set of oriented edges of T. For any vertex of T, we denote by $[\mathfrak{r}, v]$ the oriented path (resp. segment) from $\mathfrak{r}$ to v in T (resp. Γ).

Consider the boundary ∂T of T, which is by definition, the set of all infinite oriented paths in T with starting vertex at the root $\mathfrak{r}$, and define the *extended boundary* $\widetilde{\partial} T$ as the union of ∂T with the set of all oriented paths in T from the root $\mathfrak{r}$ to a leaf of T. For any pair (T, ℓ) with metric realisation Γ, define $\widetilde{\partial}(T, \ell) = \widetilde{\partial}\Gamma = \widetilde{\partial} T$. For any point $\rho \in \widetilde{\partial} T$, we denote by $E(\rho)$ the set of all the edges of T which are in ρ.

Any two different elements of ∂T have a finite number of edges in common. So we can define a non-negative real-valued *intersection pairing* $\langle\, ,\rangle$ on $\widetilde{\partial}\Gamma$ as follows. For any two points $\rho, \tau \in \widetilde{\partial}\Gamma$, with $\rho \neq \tau$ if both ρ and τ both belong to the boundary of T, let

$$\langle \rho, \tau \rangle := \ell(\rho \cap \tau) = \sum_{e \in E(\rho) \cap E(\tau)} \ell(e).$$

Consider the following *greedy procedure* in choosing a sequence of elements $\rho_0, \rho_1, \dots$ in $\widetilde{\partial}\Gamma$. Let $\rho_0 \in \widetilde{\partial}\Gamma$ be any arbitrary element of the extended boundary. Proceeding inductively on $n \in \mathbb{N}$, assume that $\rho_0, \dots, \rho_{n-1} \in \widetilde{\partial}\Gamma$ have been chosen, and choose ρ_n, if possible, arbitrarily among the set of all elements $\rho \in \widetilde{\Gamma} \setminus \{\rho_0, \dots, \rho_{n-1}\}$ which minimises the sum $\sum_{j=0}^{n-1} \langle \rho, \rho_j \rangle$. Define $a_n := \sum_{j=0}^{n-1} \langle \rho_n, \rho_j \rangle$. Then we have, somehow surprisingly, that

THEOREM 124 ([**Ami3**]). *For any pair (T, ℓ) consisting of a rooted tree T and a length function ℓ on T, the sequence $\{a_n\}$ constructed above only depends on the metric realisation Γ of (T, ℓ).*

We call the number a_n both the (T, ℓ) and Γ-*factorial* of n, and denote it by $n!_{(T,\ell)}$ or $n!_\Gamma$. We call the sequence $\{\rho_n\}$ in the construction above a *factorial-defining sequence* for (T, ℓ) and Γ. When ℓ is the standard length function, we simple write $n!_T$ for the factorials of the pair (T, ℓ).

Our definition above is an extension to arbitrary (metric) trees of the (logarithmic) factorial sequence associated by Bhargava to subsets of the ring of valuation of a local field, that we now recall [**36, 37**] (note that Bhargava instead uses the terminology of P-ordering and p-sequence).

Let K be a local field with discrete valuation val, with ring of valuation R, with maximal ideal $\mathfrak{m}$, and with residue field $\kappa = R/\mathfrak{m}$, which is thus a finite field. Let S be a subset of R. The logarithmic factorial sequence associated to S is obtained as follows. Choose $s_0 \in S$

arbitrary. Proceeding inductively, and assuming $s_0, \dots, s_{n-1}$ are already chosen, choose s_n among all $s \in S$ which minimises the quantity $\mathrm{val}\big(\prod_{j=0}^{n-1}(s-s_j)\big)$. Define

$$n!_S := \mathrm{val}\big(\prod_{j=0}^{n}(s_n - s_0)\dots(s_n - s_{n-1})\big).$$

To any subset $S \subset R$ of K as above, one can associate its *adelic tree* T_S, which is a rooted locally finite tree with vertices of valence bounded by $|\kappa|+1$, as follows. For each integer $h \in \mathbb{N}_*$, consider the projection $\phi_h : R \to R/\mathfrak{m}^h$, and define $V_h = \phi_h(S)$. The rooted tree T_S has vertex set $\sqcup_{h=0}^{\infty} V_h$, and has as root the unique element of V_0. The edge set of T_S is defined as follows. For any h, there exists a map $\pi_h : R/\mathfrak{m}^{h+1} \to R/\mathfrak{m}^h$, and we have $\phi_h = \pi_h \circ \phi_{h+1}$. A vertex u in V_h is adjacent to a vertex $v \in V_{h+1}$ if and only if $\pi_h(v) = u$.

In the case $S = R$, the tree T_R is the $|\kappa|$-regular tree, in which every vertex has $|\kappa|$ descendants, and obviously, for any subset $S \subset R$, the tree T_S is a subtree of T_R. Consider the closure $\overline{S}$ of S in K. The factorials of the tree T_S, as defined above, coincide with the factorials of the subset $\overline{S} \subset R$. Since the factorials of S and $\overline{S}$ are all easily seen to be equal, we get the following proposition.

PROPOSITION 125. *Let K be a local field with valuation ring R. Let S be a subset of R with adelic tree T_S. We have $n!_S = n!_{T_S}$, where $n!_S$ denotes the Bhargava's S-factorial of n.*

The proof given by Bhargava of the well-definedness of the factorial sequence $n!_S$ is somehow indirect and goes through the ring of integer valued polynomials on S. Our proof of Theorem 124 is direct and somehow simpler. We give an alternative local definition of a sequence associated to a pair (T, ℓ), show by induction that it is well-defined and only depends on the metric realisation Γ, and then prove the equivalence of that definition with the definition given above. Thus our proof leads to an alternative combinatorial proof of the well-definedness of the factorial sequence associated to a subset of local fields.

Note that we have not made so far any finiteness assumption on the valence of vertices of T. In fact, as we will explain in a moment, we can always reduce to the case of locally finite trees with a *capacity function* on leaves, so we next define such objects.

8.3.2. Locally finite trees with a capacity function on leaves. Let T be a locally finite rooted tree and let ℓ be a length function on $E(T)$. Denote by $L(T)$ the set of all leaves of T. By a *capacity function* on T we mean a function $\chi : L(T) \to \mathbb{N} \cup \{\infty\}$. We modify the definition of the factorial sequence given in the previous section by taking into account the capacity of leaves of T as follows. Assuming for an integer $n \in \mathbb{N}$ that $\rho_0, \dots, \rho_{n-1}$ are chosen, we choose ρ_n, if possible, among those $\rho \in \widetilde{\partial}\Gamma$ which minimises the sum $\sum_{j=0}^{n-1}\langle \rho, \rho_j\rangle$, and which verify the capacity condition that, when ρ is a leaf of T, the number of times ρ appears in the sequence $\rho_0, \dots, \rho_{n-1}$ is strictly less than the capacity of ρ. So in the sequence $\rho_0, \rho_1, \dots$ each leaf of T can appear at most as many times as its capacity. We define

$$a_n := \sum_{j=0}^{n-1}\langle \rho_n, \rho_j\rangle. \tag{26}$$

Then we have the following Theorem.

THEOREM 126 ([**Ami3**]). *The sequence $\{a_n\}$ only depends on the pair (Γ, χ), where Γ is the metric realisation of the pair (T, ℓ).*

We call a_n the (Γ, χ) or (T, ℓ, χ)-factorial of n, and denote it by $n!_{(T,\ell,\chi)} = n!_{(\Gamma,\chi)}$. When ℓ is the standard length function, we simply write $n!_{(T,\chi)}$.

Let S be a subset of the valuation ring R of a local field K. Let $h \in \mathbb{N}$. In the adelic tree T_S of S consider the subtree $T_{S,h}$ of all the vertices at distance at most h from the root $\mathfrak{r}$ of T_S. Define the capacity function χ_h on leaves of $T_{S,h}$ as follows. For any leaf v of $T_{S,h}$, consider the subtree $T_{S,v}$ of T which consists of v and all its descendants, and define $\chi_h(v)$ as the number of elements in the extended boundary of $T_{S,v}$. We have the following proposition.

PROPOSITION 127. *Notations as above, we have $n!_{(T,\chi_h)} = n!_{S,h}$, where $n!_{S,h}$ denotes the sequence of order h associated to S in the terminology of Bhargava* [**37**].

Thus, the factorials in the presence of a capacity function generalises factorials of order h for subsets of local fields in the terminology of [**37**].

8.3.3. Reduction to locally finite trees. Let (T, ℓ) be a pair consisting of a tree T and a length function ℓ on T. We define the *locally finite component* T_0 of T as follows. Consider the set V_0 of all vertices v of T with the property that all the interior vertices of the oriented path $[\mathfrak{r}, v]$ have bounded valence in T. So, for example, if the root $\mathfrak{r}$ has infinite valence, then V_0 consists of a single vertex $\mathfrak{r}$. Define the subtree T_0 of T as the tree induced by T on V_0. For any leaf of T_0 which is a vertex of valence infinity in T, define the capacity $\chi_0(v)$ of v to be infinity. For other leaves of T_0, which are thus also leaves of T, define $\chi_0(v) = 1$. Let ℓ_0 be the restriction of ℓ to the edges of T_0. Then we have

PROPOSITION 128. *Notation as above, we have for all n, $n!_{(T,\ell)} = n!_{(T_0,\ell_0,\chi_0)}$.*

Let now T be a locally finite tree, ℓ a length function on T, and χ a capacity function. Define the tree T_1 by adding $\chi(v)$ disjoint infinite paths to any leaf v of T, and extend ℓ to a length function ℓ_1 on T_1 by assigning arbitrary lengths to the new edges of T_1. It is easy to see that for any n, we have $n!_{(T,\ell,\chi)} = n!_{(T_1,\ell_1)}$.

Therefore, there is no restriction in assuming the tree T is locally finite, and, if necessary, a capacity function χ is given.

8.3.4. Growth of the factorial sequence and equidistribution. Let T be a locally finite rooted tree and ℓ be a length function on T. Denote by Γ the metric realisation of (T, ℓ). The factorial sequence satisfies the following logarithmic version of the fact that $n!m!$ divides $(n+m)!$.

PROPOSITION 129. *Notations as above, we have for all m, n in $\mathbb{N}$,*

$$(m+n)!_\Gamma \geq m!_\Gamma + n!_\Gamma.$$

Combining this property with Fekete's lemma [**107**], we get the convergence of the sequence $\frac{1}{n} n!_\Gamma$ to a quantity that we denote by $H(\Gamma)$, or sometimes $H(T, \ell)$, in $\mathbb{R} \cup \{\infty\}$.

The quantity $H(\Gamma)$ is an invariant of Γ that we characterise in [**Ami3**]. We first describe a necessary and sufficient condition for the finiteness of $H(T, \ell)$.

Define the *conductance* $c : E(T) \to \mathbb{R}_+$ given by $\forall uv \in E(T)$, $c(u,v) = c(v,u) := \frac{1}{\ell(uv)}$.

Consider the random walk $RW(T, \ell)$ on T which starts at the root $\mathfrak{r}$, and which has probability of going from a vertex u of the tree to any of its neighbors v in the tree given by $p_{uv} := \frac{c(u,v)}{\sum_{w \sim u} c(u,w)}$.

The following theorem relates the finiteness of the limit of logarithmic factorials to the transience of the random walk on the tree, and leads to an alternative characterisation of transient trees in terms of the sequence of factorial numbers associated to the tree. We say that a pair (T,ℓ) is *weakly complete* if any infinite oriented path P in T which entirely consists of valence two vertices is of infinite length in the metric realisation Γ of (T,ℓ). Note that this is the case if the length function is ϵ-away from zero for some $\epsilon>0$, i.e., if the lengths of all edges are at least ϵ.

THEOREM 130 ([**Ami3**]). *Let T be an infinite locally finite rooted tree and ℓ a length function on T. Assume that the pair (T,ℓ) is weakly complete. The following two statements are equivalent.*

- *The random walk $RW(T,\ell)$ is transient.*
- *The limit $H(T,\ell)$ is finite.*

Equivalently, the random walk $RW(T,\ell)$ is recurrent if and only if $H(T,\ell)=\infty$.

In the presence of a capacity function χ on the leaves of T, the normalized factorials $\frac{1}{n}n!_{(\Gamma,\chi)}$ still converge to a parameter $H(\Gamma,\chi)$, and the theorem above still holds if the values of χ are all finite as can be easily observed by the transformation (T_1,ℓ_1) of (T,ℓ,χ) described in the previous section. (Indeed, in this case, we will always have $H(T,\ell)=H(T,\ell,\chi)$.) On the other hand, when χ takes value ∞ at some leaves of T, then the value of $H(T,\ell,\chi)$ is always finite.

We now turn to the question of determining the value of $H(T,\ell)$. By the previous theorem, we can assume that the random walk $RW(T,\ell)$ is transient. We have the following theorem. Let T be a locally finite tree and ℓ a length function on T so that the random walk $RW(T,\ell)$ on T is transient. Let η be a the unit current flow on T and μ_{har} the corresponding harmonic measure on ∂T [**179, 180, 181**].

THEOREM 131 ([**Ami3**]). *Assume that (T,ℓ) is weakly complete. Then,*

- *any factorial determining sequence $\rho_0,\rho_1,\dots$ of (T,ℓ) is equidistributed in ∂T with respect to the harmonic measure μ_{har}.*
- *we have $H(T,\ell)=\|\eta\|^2$, where $\|\eta\|^2$ is the energy of the unit current flow η on T.*

We note that a consequence of the above theorem is that the algorithm described in the definition of the tree factorials provides an effective way of sampling the harmonic measure on if the tree is transient.

As another corollary, we get the following equidistribution theorem for subsets of local fields. Let us call a subset S of the valuation ring R of a local field K *transient* if the adelic tree T_S of S is transient. For a transient subset S of R, we denote by μ_{har} the corresponding harmonic measure of S which has support in the closure $\overline{S}$ of S in R. Note that a transient set S is a set with *finite logarithmic capacity* and the corresponding harmonic measure is also called the *equilibrium measure* in the literature, see e.g. [**23**]. We have

THEOREM 132 ([**Ami3**]). *Let K be a local field with valuation ring R, and let S be an infinite subset of R. The following two conditions are equivalent.*

- *The subset S of R is transient.*
- *The sequence $\frac{1}{n}n!_S$ converges to a finite $H(S)\in(0,\infty)$.*

Moreover, for a transient subset S of R, any factorial determining sequence $s_0, s_1, s_2, \dots$ of S is equidistributed in $\overline{S}$ with respect to the equilibrium measure, and we have

$$H(S) = \int_{\substack{(x,y)\in\overline{S}\times\overline{S}\\ x\neq y}} \mathrm{val}(x-y)d\mu_{\mathrm{har}}(x)d\mu_{\mathrm{har}}(y)$$
$$= \int_{\overline{S}} \mathrm{val}(x_0-y)d\mu_{\mathrm{har}}(y) \qquad a.s. \text{ for } x_0 \in \overline{S}.$$

In other words, $\exp(H(S))$ *is the logarithmic capacity of the set S.*

We note that in the presence of a capacity function on the leaves of T which takes values infinity, the limit $H(T,\ell,\chi)$ has a similar expression. Indeed, it will be enough to consider the modified tree T_1 obtained by adding a countable number of paths to any leaf v of T with $\chi(v)=\infty$, and define the conductance of all these new edges to be equal to ∞. The random walk on T_1 with these conductances is equivalent to a random walk on T with absorption on the leaves of capacity infinity, and the limit $H(T,\ell,\chi)$ is the squared norm of the unit current flow on T_1. For the special case where T is a finite tree and χ is a function on the leaves of T which takes value infinity at some points of T, we have the following explicit way of calculating $H(T,\ell,\chi)$.

Let $L_0 \subset L(T)$ be the set of all leaves v with $\chi(v)=\infty$, and define the connected graph $G=(V,E)$ obtained by identifying all the vertices in L_0 to a single vertex $\mathfrak{s}$. Let $C^0(G,\mathbb{R})$ be the space of real valued functions on the vertices of G. The length function ℓ induces a length function on the edges of G, to which we can associate a Laplcian operator $\Delta : C^0(G,\mathbb{R}) \to C^0(G,\mathbb{R})$, as we did in the previous chapters. Recall that for any function $f \in C^0(G,\mathbb{R})$, the value of $\Delta(f) \in C^0(G,\mathbb{R})$ at a vertex v of $V(G)$ is given by

$$\Delta(f)(v) = \sum_{\{u,v\}\in E(G)} \frac{1}{\ell_e}\Big(f(u)-f(v)\Big).$$

For a vertex v of G, denote by $\mathbf{1}_v$ the characteric function of v which takes value one at v, and value zero outside v. Let F be the real-valued function on V which solves the Laplace equation $\Delta(F) = \mathbf{1}_{\mathfrak{r}} - \mathbf{1}_{\mathfrak{s}}$. By connectivity of G, up to additioning a constant function, F is unique.

THEOREM 133 ([**Ami3**]). *Notations as above, we have* $H(T,\ell,\chi) = F(\mathfrak{s}) - F(\mathfrak{r})$.

As an immediate corollary, for any $h \in \mathbb{N}$, we get a limit theorem for the factorials of order h associated to subsets of local fields, as defined by Bhargava [**37**].

CHAPTER 9

Unified approach to distance two colouring of graphs on surfaces

In [**AEH**], we introduce a colouring notion called Σ-colouring which generalises both colouring the square of graphs and of cyclic colouring of graphs embedded in a surface. We prove a general result for graphs embeddable in a fixed surface, which implies asymptotic versions of Wegner's and Borodin's Conjecture on the planar version of these two colourings. Furthermore, as a consequence of our results, we prove that the size of a clique in the square of a graph of maximum degree Δ embeddable in some fixed surface is at most $\frac{3}{2}\,\Delta$ plus a constant.

The heart of the proof consists of a result on the global quantitative topology of a graph embedded in a surface, and the geometry of a polytope associated to a graph and the properties of a class of probability distributions associated to the points of this polytope.

In this chapter, we briefly discuss these results.

9.1. Σ-colouring

Given a graph $G = (V, E)$, the *chromatic number* of G, denoted $\chi(G)$, is the minimum number of colours required so that we can properly colour its vertices using those colours [**43**] or [**91**]. If we colour the edges of G, we get the *chromatic index*, denoted $\chi'(G)$. The *list chromatic number* or *choice number* $ch(G)$ is the minimum value k such that if we give each vertex v of G a list $L(v)$ of at least k colours, then we can find a proper colouring in which each vertex gets assigned a colour from its own private list. The list chromatic index $ch'(G)$ is defined analogously for edges.

The *square* G^2 of a graph G is the graph with vertex set $V(G)$, with an edge between any two different vertices that have distance at most two in G. A proper vertex colouring of the square of a graph can also be seen as a vertex colouring of the original graph satisfying:

- vertices that are adjacent receive different colours, and
- vertices that have a common neighbour receive different colours.

Another way to formulate these conditions is as 'vertices at distance one or two must receive different colours'. This is why the name *distance-two colouring* is also used in the literature.

In [**AEH**] we introduce a colouring concept that generalises the concept of colouring the square of a graph, but that also can be used to study different concepts such as *cyclic colouring of plane graphs* (definition will be given later).

For a vertex $v \in V$, let $N(v)$ (or $N_G(v)$ if we want to specify the graph under consideration) be the set of vertices adjacent to v. Suppose that for each vertex $v \in V$, we are given a subset $\Sigma(v) \subseteq N(v)$ of its neighbourhood. We call such a collection a Σ-*system* for G.

A Σ-*colouring* of G is an assignment of colours to the vertices of G so that:

- vertices that are adjacent receive different colours, and
- vertices that appear together in some $\Sigma(v)$ receive different colours.

When additionally each vertex v has its own list $L(v)$ of colours from which its colour must be chosen, we talk about a *list* Σ*-colouring*.

We denote by $\chi(G;\Sigma)$ the minimum number of colours required for a Σ-colouring to exist. Its list variant is denoted by $ch(G;\Sigma)$, and is defined as the minimum integer k such that for each assignment of a list $L(v)$ of at least k colours to vertices $v \in V$, there exists a proper Σ-colouring of G in which all vertices are assigned colours from their own lists.

Notice that we trivially have $\chi(G) = \chi(G;\varnothing)$ and $\chi(G^2) = \chi(G;N_G)$; and the same relations holds for the list variant ($\varnothing$ assigns the empty set to each vertex).

We define the *width* of a Σ-system of G as $\Delta(G;\Sigma) = \max_{v\in V} |\Sigma(v)|$. It is clear that we always need at least $\Delta(G;\Sigma)+1$ colours in a proper Σ-colouring. In the case $\Sigma \equiv N_G$, there exist plenty of graphs G that require $O(\Delta(G)^2)$ colours (where $\Delta(G) = \Delta(G;N_G)$ is the usual maximum degree of G). But for planar graphs, it is known that a constant times $\Delta(G)$ colours is enough (even for list colouring), as we will discuss a bit later.

Following Wegner's Conjecture on colouring the square of planar graphs (see also next subsection), we propose the following conjecture.

CONJECTURE 134 ([**AEH**]). *There exist constants c_1, c_2 and c_3 such that for all planar graphs G and any Σ-system for G, we have*

$$\begin{aligned}
\chi(G;\Sigma) &\le \left\lfloor \tfrac{3}{2}\,\Delta(G;\Sigma)\right\rfloor + c_1; \\
ch(G;\Sigma) &\le \left\lfloor \tfrac{3}{2}\,\Delta(G;\Sigma)\right\rfloor + c_2; \\
ch(G;\Sigma) &\le \left\lfloor \tfrac{3}{2}\,\Delta(G;\Sigma)\right\rfloor + 1, \qquad \text{if } \Delta(G;\Sigma) \ge c_3.
\end{aligned}$$

If $\Sigma \equiv \varnothing$ (hence $\Delta(G;\Sigma) = 0$), then the Four Colour Theorem implies that the smallest possible value for c_1 is four; while the fact that planar graphs are always 5-list colourable but not always 4-list colourable, shows that the smallest possible value for c_2 is five.

Our main result in [**AEH**] is that Conjecture 134 is asymptotically correct: $ch(G;\Sigma) \le \frac{3}{2}\,\Delta(G;\Sigma) + o\big(\Delta(G;\Sigma)\big)$. In fact, we can prove this asymptotic result holds for general surfaces.

THEOREM 135 ([**AEH**]). *For every surface S and real $\varepsilon > 0$, there exists a constant $\beta_{S,\varepsilon}$ such that the following holds for all $\beta \ge \beta_{S,\varepsilon}$. If G is a graph embeddable in S, with a Σ-system of width at most β, then $ch(G;\Sigma) \le \left(\frac{3}{2}+\varepsilon\right)\beta$.*

We now discuss two special consequences of these results. These special versions of Theorems 135 and 144 also show that the term $\frac{3}{2}\,\beta$ is best possible.

9.1.1. Colouring the square of graphs. If G has maximum degree Δ, then a vertex colouring of its square G^2 will need at least $\Delta+1$ colours, and the greedy algorithm shows that it is always possible to find a colouring of G^2 with Δ^2+1 colours. Cages of diameter two, such as the 5-cycle, the Petersen graph and the Hoffman-Singleton graph (see, e.g., [**43**, page 84]), show that there exist graphs that in fact require Δ^2+1 colours.

Regarding the chromatic number of the square of a planar graph, Wegner [**240**] posed the following conjecture (see also the book of Jensen and Toft [**141**, Section 2.18]), suggesting that for planar graphs far less than Δ^2+1 colours suffice.

CONJECTURE 136 (Wegner [**240**]). *For a planar graph G of maximum degree Δ,*

$$\chi(G^2) \le \begin{cases} 7, & \text{if } \Delta = 3, \\ \Delta + 5, & \text{if } 4 \le \Delta \le 7, \\ \left\lfloor \tfrac{3}{2}\,\Delta \right\rfloor + 1, & \text{if } \Delta \ge 8. \end{cases}$$

Wegner also gave examples showing that these bounds would be tight. For even $\Delta \geq 8$, these examples are sketched in Figure 1(a). The graph in the picture has maximum degree $2k$ and

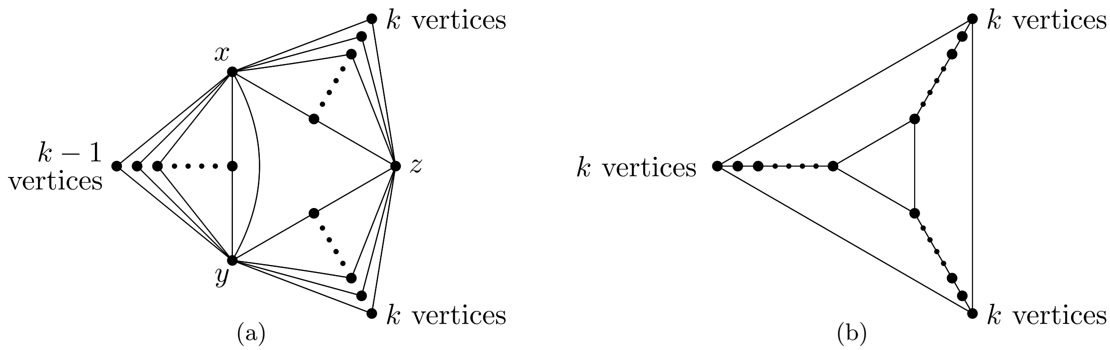

FIGURE 1. (a) A planar graph G with maximum degree $\Delta = 2k$ and $\omega(G^2) = \chi(G^2) = 3k+1 = \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$.
(b) A planar graph H with maximum face order $\Delta^* = 2k$ and $\chi^*(H) = 3k = \left\lfloor \frac{3}{2}\Delta^* \right\rfloor$ (see Subsection 9.1.2).

yet all the vertices except z are pairwise adjacent in its square. Hence to colour these $3k+1$ vertices, we need at least $3k+1 = \frac{3}{2}\Delta + 1$ colours. Note that the same arguments also show that the graph G in the picture has $\omega(G^2) = \frac{3}{2}\Delta + 1$.

Kostochka and Woodall [**161**] conjectured that for every square of a graph, the chromatic number equals the list chromatic number. This conjecture and Wegner's one together imply the conjecture that for planar graphs G with $\Delta \geq 8$, we have $ch(G^2) \leq \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$.

Several upper bounds on $\chi(G^2)$ for planar graphs in terms of Δ have been obtained in [**142, 244, 134, 7, 47, 194**]. The best known upper bound so far is the one obtained by Molloy and Salavatipour [**194**]. It gives $\chi(G^2) \leq \left\lceil \frac{5}{3}\Delta \right\rceil + 78$ for any Δ. As mentioned in [**194**], the constant 78 can be reduced for sufficiently large Δ. For example, it was improved to 24 when $\Delta \geq 241$.

Since $ch(G^2) = ch(G; N_G)$ (i.e., $\Sigma(v) = N_G(v)$ for all $v \in V$), as an immediate corollary of Theorem 135 we obtain.

COROLLARY 137. *Let S be a fixed surface. Then the square of every graph G embeddable in S and of maximum degree Δ has list chromatic number at most $\frac{3}{2}\Delta + o(\Delta)$.*

In fact, the same asymptotic upper bound as in Corollary 137 can be proved even for larger classes of graphs. Additionally, a stronger conclusion on the colouring is possible. For the following result, we assume that colours are integers, which allows us to talk about the 'distance' $|\alpha_1 - \alpha_2|$ between two colours α_1, α_2.

THEOREM 138 (Havet, Van den Heuvel, McDiarmid & Reed [**129**]). *Let k be a fixed positive integer. The square of every $K_{3,k}$-minor free graph G of maximum degree Δ has list chromatic number (and hence clique number) at most $\frac{3}{2}\Delta + o(\Delta)$. Moreover, given lists of this size, there is a proper colouring in which the colours on every pair of adjacent vertices of G differ by at least $\Delta^{1/4}$.*

Note that planar graphs do not have a $K_{3,3}$-minor. In fact, for every surface S, there is a constant k such that no graph embeddable in S has $K_{3,k}$ as a minor. That shows that Theorem 138 is stronger than our Corollary 137. On the other hand, Theorem 138 gives a weaker bound for the clique number than the one we obtain below.

Both Corollary 137 and Theorem 138 can be applied to K_4-minor free graphs, since these graphs are planar and do not have $K_{3,3}$ as a minor. But the best possible bounds for this class are actually known. Lih, Wang and Zhu [**171**] showed that the square of K_4-minor free graphs with maximum degree Δ has chromatic number at most $\left\lfloor \frac{3}{2}\,\Delta \right\rfloor + 1$ if $\Delta \geq 4$ and $\Delta + 3$ if $\Delta = 2, 3$. The same bounds, but then for the list chromatic number of K_4-minor free graphs, were proved by Hetherington and Woodall [**132**].

9.1.2. Cyclic colourings of embedded graphs. Given a surface S and a graph G embeddable in S, we denote by G^S that graph with a prescribed embedding in S. If the surface S is the sphere, we talk about a *plane graph* G^P. The *order* of a face of G^S is the number of vertices in its boundary; the maximum order of a face of G^S is denoted by $\Delta^*(G^S)$.

A *cyclic colouring* of an embedded graph G^S is a vertex colouring of G such that any two vertices in the boundary of the same face have distinct colours. The minimum number of colours required in a cyclic colouring of an embedded graph is called the *cyclic chromatic number* $\chi^*(G^S)$. This concept was introduced for plane graphs by Ore and Plummer [**200**], who also proved that for a plane graph G^P we have $\chi^*(G^P) \leq 2\Delta^*$. Borodin [**46**] (see also Jensen and Toft [**141**, page 37]) conjectured the following.

Conjecture 139 (Borodin [**46**]). *For a plane graph G^P of maximum face order Δ^* we have $\chi^*(G^P) \leq \left\lfloor \frac{3}{2}\,\Delta^* \right\rfloor$.*

The bound in this conjecture is best possible. Consider the plane graph depicted in Figure 1(b): It has $3k$ vertices and has three faces of order $\Delta^* = 2k$. Since all pairs of vertices have a face they are both incident with, we need $3k = \left\lfloor \frac{3}{2}\,\Delta^* \right\rfloor$ colours in a cyclic colouring.

Borodin [**46**] also proved Conjecture 139 for $\Delta^* = 4$. For general values of Δ^*, the original bound $\chi^*(G^P) \leq 2\Delta^*$ of Ore and Plummer [**200**] was improved by Borodin *et al.* [**48**] to $\chi^*(G^P) \leq \left\lfloor \frac{9}{5}\,\Delta^* \right\rfloor$. The best known upper bound in the general case is due to Sanders and Zhao [**217**]: $\chi^*(G^P) \leq \left\lceil \frac{5}{3}\,\Delta^* \right\rceil$.

Although Wegner's and Borodin's Conjectures seem to be closely related, and most of the results approaching these conjectures used the same ideas, there was no direct connection between them prior to our work.

In order to show that our Theorem 135 provides an asymptotically best possible upper bound for the cyclic chromatic number for a graph G with some fixed embedding G^S, we need some extra notation. For each face f of G^S, add a vertex x_f. For any face f of G^S and any vertex v in the boundary of f, add an edge between v and x_f, and denote by G_F the graph obtained from G^S by this construction. Note that the vertex set of G_F consists of $V(G)$ and all the new vertices x_f, for f a face of G^S. Define a Σ-system Σ_F for G_F as follows: For each vertex $v \in V(G)$, let $\Sigma_F(v) = \varnothing$. For each vertex x_f, let $\Sigma_F(x_f)$ be all the neighbours of x_f. Observe that a (list) Σ_F-colouring of G_F colours the vertices of G in a way required for a cyclic (list) colouring of G^S, and that $\Delta(G_F; \Sigma_F) = \Delta^*(G^S)$. In fact, one can show $\chi^*(G^S) \leq \chi(G_F, \Sigma_F) \leq \chi^*(G^S) + 1$.

Using the upper bound on $\chi^*(G^S)$, we get the following corollary of Theorem 135.

COROLLARY 140. *Let S be a fixed surface. Every embedding G^S of a graph G of maximum face order Δ^* has cyclic list chromatic number at most $\frac{3}{2}\Delta^* + o(\Delta^*)$.*

9.1.3. Proof of Theorem 135. The proof of Theorem 135 uses the *discharging method* to prove a structural result on graphs on embedded surfaces. Discharging method is a systematic clever use of the Euler formula on a surface, which was originally used in the proof of the four colour theorem. Very informally, our structural result states that a graph that is maximally embeddable in some fixed surface, either contains one of two fairly simple configurations, or it contains a structure that internally satisfies a specific density-type condition. Since the statement is technical, we omit it.

The structural theorem allows to reduce then the problem to an edge colouring problem for multigraphs as in a previous work of Havet, van den Heuvel, McDiarmid and Reed [**129**], and to apply (a mild extension of) Kahn's results on asymptotic of edge chromatic index [**143, 144**] to conclude. This uses the interesting and still somehow mysterious set-up of hardcore distributions associated to the points of the matching polytope [**145**]. We describe the hardcore distributions and their relations to the edge colouring problem in the next section, omitting the structural result and the way it allows to reduce to an edge colouring problem [**AEH**].

9.2. Matching polytope

We briefly describe the matching polytope of a multigraph. More about this subject can be found in [**218**, Chapter 25].

Let H be a multigraph with m edges. Let $\mathcal{M}(H)$ be the set of all matchings of H, including the empty matching. For each $M \in \mathcal{M}(H)$, let us define the m-dimensional characteristic vector $\mathbf{1}_M$ as follows: $\mathbf{1}_M = (x_e)_{e \in E(H)}$, where $x_e = 1$ for an edge $e \in M$, and $x_e = 0$ otherwise. The matching polytope of H, denoted $\mathcal{MP}(H)$, is the polytope defined by taking the convex hull of all the vectors $\mathbf{1}_M$ for $M \in \mathcal{M}(H)$. Also, for any real number λ, we set $\lambda\,\mathcal{MP}(H) = \{\,\lambda\, x \mid x \in \mathcal{MP}(H)\,\}$.

Edmonds [**96**] gave the following characterisation of the matching polytope.

THEOREM 141 (Edmonds [**96**]). *A vector $\vec{x} = (x_e)$ is in $\mathcal{MP}(H)$ if and only if $x_e \geq 0$ for all x_e and the following two types of inequalities are satisfied:*

- *For all vertices $v \in V(H)$, $\sum_{e:\ v \text{ incident to } e} \ x_e \leq 1$;*
- *for all subsets $W \subseteq V(H)$ with $|W| \geq 3$ and $|W|$ odd, $\sum_{e\,\in\,E(W)} x_e \leq \frac{1}{2}\,(|W| - 1)$.*

9.3. Hardcore distributions

Hardcore distributions are distributions that originally arose in Statistical Physics, and that satisfy very natural conditions and generally provide strong independence properties allowing good sampling from a given family. Given a family of subsets $\mathcal{F}$ of a given set $\mathcal{E}$, a natural way of picking at random an element of $\mathcal{F}$ (or, in an other words, a probability distribution on $\mathcal{F}$) is as follows.

Let us suppose that each element e of $\mathcal{E}$ has been assigned a positive weight λ_e. Then we pick each element $M \in \mathcal{F}$ with probability proportional to $\prod\limits_{e\,\in\,M} \lambda_e$. More precisely, the probability P_M of picking $M \in \mathcal{F}$ at random is given by

$$P_M \;=\; \frac{\prod\limits_{e\,\in\,M} \lambda_e}{\sum\limits_{M'\in\mathcal{F}} \ \prod\limits_{e\,\in\,M'} \lambda_e}.$$

We define the vector $\vec{x} = (x_e)_{e\in\mathcal{E}}$ by setting $x_e = \sum_{M\in\mathcal{F},\, e\in M} P_M$. It is clear that x_e is the probability that a given random element of $\mathcal{F}$ contains the element e. The probability distribution $\{P_M\}$ is called a *hardcore* distribution with *activities* $\{\lambda_e\}$ and *marginals* $\{x_e\}$. The vector $\vec{x}$ is called the *marginal vector* associated with the hardcore distribution $\{P_M\}$.

Given a vector $\vec{x}$, it is not always true that $\vec{x}$ is the marginal vector of some hardcore distribution. Indeed if $\mathcal{P}(\mathcal{F})$ denotes the polytope defined by taking the convex hull of the characteristic vectors of the elements of $\mathcal{F}$, then the marginal vector $\vec{x}$ of a hardcore distribution is in $\mathcal{P}(\mathcal{F})$:

$$\vec{x} \;=\; \sum_{M\in\mathcal{F}} P_M\,\mathbf{1}_M.$$

(Recall that the characteristic vector, $\mathbf{1}_M$, of a given element $M\in\mathcal{F}$ is the $|\mathcal{E}|$-dimensional vector $(y_e)_{e\in\mathcal{E}}$ such that $y_e = 1$ if $e\in M$ and $y_e = 0$ otherwise.)

This provides a necessary condition for a vector to be the marginal vector of a hardcore distribution. It is not difficult to prove that the activities λ_e corresponding to $\vec{x}$, if they exist, are unique.

From now on, let H be a given multigraph. We recall that $\mathcal{M}(H)$ and $\mathcal{MP}(H)$ are the family of matchings and the matching polytope of H, respectively. (So $\mathcal{M}(H)$ will play the role of the family $\mathcal{F}$ from above. And using the notation from above means $\mathcal{MP}(H) = \mathcal{P}(\mathcal{M}(H))$.)

We have the following theorem relating the matching polytope and hardcore distributions.

Theorem 142 (Lee [**167**], Rabinovich *et al.* [**213**]). *For a given real number $0<\delta<1$, suppose $\vec{x}$ is a vector in $(1-\delta)\,\mathcal{MP}(H)$, for some multigraph H. Then there exists a unique family of activities λ_e such that $\vec{x}$ is the marginal vector of the hardcore distribution defined by the λ's. The hardcore distribution $\{P_M\}_{M\in\mathcal{M}(H)}$ is the unique distribution maximising the entropy function*

$$\mathcal{H}(Q_M) \;=\; -\sum_{M\in\mathcal{M}(H)} Q_M\,\log(Q_M)$$

among all the distributions $\{Q_M\}_{M\in\mathcal{M}(H)}$ satisfying $\vec{x} = \sum_{M\in\mathcal{M}(H)} Q_M\,\mathbf{1}_M$.

Kahn and Kayll proved in [**145**] a family of results, resulting in a long-range independence property for the hardcore distributions defined by a marginal vector $\vec{x}$ inside $(1-\delta)\,\mathcal{MP}(H)$, see Kahn's paper [**144**] or the book by Molloy and Reed [**193**] for more details. It seems that these properties are captured by the geometry of the matching polytope, and its associated toric variety.

9.3.1. Hardcore distributions and edge-colouring. The significance of the matching polytope and hardcore distributions, and their relation to list edge-colouring is indicated by the following important result, which can be deduced more or less directly from Kahn's paper [**144**].

Theorem 143. *For all real numbers δ,ν, $0<\delta<1$ and $\nu>0$, there exists a $\Delta_{\delta,\nu}$ such that for all $\Delta\ge\Delta_{\delta,\nu}$ the following holds. If H is a multigraph and L is a list assignment of colours to the edges of H so that*

- *H has maximum degree at most Δ;*
- *for all edges $e\in E(H)$, $|L(e)|\ge\nu\Delta$;*

- *the vector $\vec{x} = (x_e)$ with $x_e = \dfrac{1}{|L(e)|}$ for all $e \in E(H)$ is an element of $(1-\delta)\,\mathcal{MP}(H)$.*

Then there exists a proper edge-colouring of H where each edge gets a colour from its own list.

The proof of the above theorem is based on a randomised algorithm introduced in [**144**]. It proceed by naively colouring the graph by choosing at each step randomly choosen matchings in each colour class subgraph of the graph. A colour class contains all the edges whose list contains a given fixed colour. The random matching is with respect to a hardcore distribution. This way, at each step, a subset of edges can be coloured. The lists are then modified according to the attributed colours. A sufficient number of iterations of this naive colouring procedure results in a graph, consisting of all the uncoloured edges at that step, such that this graph has maximum degree T, for some integer T, and that the list sizes are at least $2T$., i.e., each uncoloured edge is in at least $2T$ of the H^I_α's. At this stage it is easy to finish the procedure by a simple greedy algorithm.

9.4. Cliques in squares of graphs

A trivial lower bound for the (list) chromatic number of a graph G is the *clique number* $\omega(G)$, the maximum size of a clique in G. For graphs with a Σ-system, we can define the following related concept. A Σ-*clique* is a subset $C \subseteq V$ such that every two different vertices in C are adjacent or appear together in some $\Sigma(v)$. Denote by $\omega(G;\Sigma)$ the maximum size of a Σ-clique in G. Then we trivially have $ch(G;\Sigma) \geq \omega(G;\Sigma)$, and so Theorem 135 means that for a graph G embeddable in some fixed surface S, we have $\omega(G;\Sigma) \leq \frac{3}{2}\,\Delta(G;\Sigma)+o(\Delta(G;\Sigma))$.

But in fact, the structural result we use to prove Theorem 135 is strong enough to give $\omega(G;\Sigma) \leq \frac{3}{2}\,\Delta(G;\Sigma) + O(1)$.

Theorem 144 ([**AEH**]).
For every surface S, there exist constants β_S and γ_S such that the following holds for all $\beta \geq \beta_S$. If G is a graph embeddable in S, with a Σ-system of width at most β, then every Σ-clique in G has size at most $\frac{3}{2}\,\beta + \gamma_S$.

In particular, Let S be a fixed surface. Then the square of every graph G embeddable in S and of maximum degree Δ has clique number at most $\frac{3}{2}\,\Delta + O(1)$.

From the proof of Theorem 144, it can be deduced that the square of a planar graph with maximum degree $\Delta \geq 11616$ has clique number at most $\frac{3}{2}\,\Delta + 76$.

Note that this is in contrast with the clique size of squares of general graphs. Indeed, Moore graphs of diameter two provide examples with the clique size of order $\Omega(\Delta^2)$ [**189**].

Personal bibliography

[AAST] L. Addario-Berry, O. Amini, J-S. Sereni, S. Thomassé, *Guarding art galleries: the extra cost for sculptures is linear* LNCS 5124, proceedings of SWAT (2008), 41-52.

[Ami1] O. Amini, *Equidistribution of Weierstrass points on curves over non-Archimedean fields*, preprint.

[Ami2] O. Amini, *The combinatorial Chow ring of products of graphs*, preprint.

[Ami3] O. Amini, *Logarithmic tree factorials*, preprint.

[Ami4] O. Amini, *The exchange graph and variation of the ratio of the two Symanzik polynomials*, Annales de l'Institut Henri Poincaré D, to appear.

[Ami5] O. Amini, *Reduced divisors and embeddings of tropical curves*, Trans. Amer. Math. Soc. 365 (2013), 4851-4880.

[Ami6] O. Amini, *Lattice of integer flows and poset of strongly connected orientations*, preprint.

[Ami7] O. Amini, *Geometry of graphs and applications in arithmetic and algebraic geometry*, survey notes written for the Simons Symposia 2015, `https://www.math.ens.fr/~amini/Publications/Survey.pdf`.

[Ami8] O. Amini, *Limit linear series: combinatorial theory*, preprint.

[AmiBa] O. Amini, M. Baker, *Linear series on metrized complexes of algebraic curves*, Mathematische Annalen 362 (2015), 55-106.

[ABBR1] O. Amini, M. Baker, E. Brugallé, J. Rabinoff, *Lifting harmonic morphisms I: metrized complexes and Berkovich skeleta*, Research in the Mathematical Sciences 2:7 (2015), 67 p.

[ABBR2] O. Amini, M. Baker, E. Brugallé, J. Rabinoff, *Lifting harmonic morphisms II: tropical curves and metrized complexes*, Algebra and Number Theory 9 (2015), no.2, 267-315.

[ABBF] O. Amini, S. Bloch, J. Burgos, J. Fresan, *Feynman amplitudes and limits of heights*, Izvestiya: Mathematics 80 (2016), no. 5, 813-848, special volume in honor of J-P. Serre.

[ABM] O. Amini, J-D. Boissonnat, and P. Memari, *Geometric tomography with topological guarantees*, Discrete and Computational Geometry 50 (2013), no. 4, 821-856.

[AmiCa] O. Amini, L. Caporaso, *Riemann-Roch for weighted graphs and tropical curves*, Advances in Mathematics 240 (2013), 1-23.

[AmiCo] O. Amini, D. Cohen-Steiner, *A transfer principle and applications to eigenvalue estimates for graphs,* Commentarii Mathematici Helvetici 93 (2018), 203-223.

[ACN] O. Amini, D. Coudert, N. Nisse, *Non-deterministic graph searching in trees*, Theoretical Computer Science 580 (2015), 101-121

[ADGO1] O. Amini, Devroye, S. Griffiths, N. Olver, *On explosion in heavy-tailed branching random walks*, Annals of Probability 41 (2013), 1864-1899.

[ADGO2] O. Amini, L. Devroye, S. Griffiths, N. Olver, *Explosion and linear transit time in infintie trees*, Probability Theory and Related Fields 167 (2017), no.1, 325-347.

[AEH] O. Amini, L. Esperet, and J. van den Heuvel, *A unified approach to distance two colourings of graphs on surfaces*, Combinatorica 33 (2013), Issue 3, 253-296.

[AmiEst1] O. Amini and E. Esteves, *Stacky limit linear series I: Voronoi tilings and degenerations of line bundles*, preprint.

[AmiEst2] O. Amini and E. Esteves, *Stacky limit linear series II: Toric tilings and the space of embedded sheaves*, in preparation.

[AmiEst3] O. Amini and E. Esteves, *Stacky limit linear series III: moduli spaces*, in preparation.

[AFS] O. Amini, F. Fomin, S. Saurabh, *Counting subgraphs via homomorphisms*, SIAM J. Discrete Math., 26 (2012), 695-717.

[AFS2] O. Amini, F. Fomin, S. Saurabh, *Implicite branching and parametrized partial cover problems*, J. Comput. System Sci. 77 (2011): 1159-1171.

[AGHP] O. Amini, F. Giroire, F. Huc, S. Pérennes, *Minimal selectors and fault tolerant networks*, Networks 55 (2010), 326-340.

[AGH] O. Amini, S. Griffiths, F. Huc, *Subgraphs of weakly quasi-random oriented graphs*, SIAM J. Discrete Math. 25 (2011), no. 1, 234-259.

[AHHT] O. Amini, F. Havet, F. Huc, S. Thomassé, *WDM and directed start arboricity*, Combinatorics Probability and Computing 19: 161-182, Cambridge U. Press, 2010.

[AHP] O. Amini, F. Huc, S. Pérennes, *On the path-width of planar graphs*, SIAM J. Discrete Math. 23 (2009), 1311-1316.

[AmiKool] O. Amini, J. Kool, *A spectral lower bound for the divisorial gonality of metric graphs*, International Mathematics Research Notices 8 (2016), 2423-2450.

[AmiMan] O. Amini, M. Manjunath, *Riemann-Roch for sub-lattices of the root lattice A_n*, Electron. J. Combin. 17 (2010), #R124, 50p.

[AMNT] O. Amini, F. Mazoit, N. Nisse, and S. Thomassé, *Submodular Partition Functions*, Discrete Mathematics 309 (2009), 6001-6008.

[APPSS] O. Amini, D. Peleg, S. Pérennes, I. Sau, S. Saurabh, *On the Approximability of Some Degree-constrained Subgraph Problems*, Discrete Applied Mathematics 160 (2012), 1661-1679.

[APS] O. Amini, S. Pérennes, I. Sau, *Hardness and approximation of traffic grooming*, Theoret. Comput. Sci. 410 (2009), 3751-3760.

[AR] O. Amini, B. Reed, *List colouring constants of triangle free graphs*, extended abstract in Electr. Notes Discrete Math. 30 (2008), 135-140.

[ASS] O. Amini, I. Sau, S. Saurabh, *Parameterized complexity of finding small degree-constrained subgraphs*, J. Discrete Algorithms 10 (2012), 70-83.

References

[1] D. Abramovich, *A linear lower bound on the gonality of modular curves*, Internat. Math. Res. Notices 20 (1996), 1005-1011.

[2] D. Abramovich, L. Caporaso, and S. Payne, *The tropicalization of the moduli space of curves*, Ann. Sci. c. Norm. Supr. 48 (2015), 765-809.

[3] L. Addario-Berry and B. Reed, *Minima in branching random walks*, *Ann. Appl. Probab.* 37 (2009), 1044-1079.

[4] K. Adiprasito, J. Huh, and E. Katz, *Hodge Theory for Combinatorial Geometries*, `http://arxiv.org/abs/1511.02888`.

[5] Aïdekon, E. (2011). Convergence in law of the minimum of a branching random walk, preprint. `http://arxiv.org/abs/1101.1810`.

[6] Aïdekon, E., Shi, Z. (2011). The Seneta-Heyde scaling for the branching random walk, preprint. `http://arxiv.org/abs/1102.0217`.

[7] G. Agnarsson and M. M. Halldórsson, *Coloring powers of planar graphs.* SIAM J. Discrete Math. 16 (2003), 651-662.

[8] D. J. Aldous, *The continuum random tree I.* Ann. Probab. 19 (1991), 1-28.

[9] D. J. Aldous, *Asymptotics in the random assignment problem*, Probab. Th. Rel. Fields 93 (1992), 507-534.

[10] D. J. Aldous and J. M. Steele, The objective method: Probabilistic combinatorial optimization and local weak convergence. In *Probability on Discrete Structures*, Encyclopedia of Mathematical Sciences (H. Kesten, ed.) 110 (2003) 1-72. Springer, Berlin.

[11] L. Allermann and J. Rau, *First Steps in Tropical Intersection Theory*, Mathematische Zeitschrift 264 (2010), Issue 3, 633–670.

[12] N. Alon and V.D. Milman, λ_1, *Isoperimetric inequalities for graphs, and superconcentrators*, Journal of Combinatorial Theory, Series B 38 (1985), 73-88.

[13] Y. Amice, *Interpolation p-adique*, Bull. Soc. Math. France 92 (1964), 117-180.

[14] Y. An, M. Baker, G. Kuperberg, and F. Shokrieh, *Canonical representatives for divisor classes on tropical curves and the Matrix-Tree Theorem*, Forum of Mathematics, Sigma. Cambridge University Press, 2014.

[15] I. V. Artamkin, *The discrete Torelli theorem*, (Russian) Mat. Sb. 197 (2006), 3-16, English translation in Sb. Math. 197 (2006), 1109-1120.

[16] Athreya, K. B. and Ney, P. E. (1972). *Branching Processes.* Die Grundlehren der mathematischen Wissenschaften 196. Springer-Verlag, Berlin.

[17] R. Bacher, P. de la Harpe, and T. Nagnibeda, *The lattice of integral flows and the lattice of integral cuts on a finite graph*, Bull. Soc. Math. Fr. 125 (1997), issue 2, 167-198.

[18] M. Bachmann, *Limit theorems for the minimal position in a branching random walk with independent logconcave displacements*, Adv. in Appl. Probab. 32 (2000), 159-176.

[19] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, M. Möller, *Compactification of strata of abelian differentials*, arXiv:1604.08834.

[20] M. Baker, *Specialization of linear systems from curves to graphs*, Algebra & Number Theory 2 (2008), 613-653.

[21] M. Baker and X. Faber, *Metric properties of the tropical Abel-Jacobi map*, J. Algebraic Combin. 33 (2011), no. 3, 349-381.

[22] M. Baker and D. Jensen, *Degeneration of linear series from the tropical point of view and applications*, in Nonarchimedean and Tropical Geometry, Simons Symposia, Springer-Verlag, 2016.

[23] M. Baker and L. DeMarco, *Preperiodic points and unlikely intersections*, Duke Math Journal 159, no. 1 (2011), 1-29.

[24] M. Baker and S. Norine, *Riemann-Roch and Abel-Jacobi theory on a finite graph*, Advances in Mathematics 215 (2007), 766-788.

[25] M. Baker and R. Rumley, *Harmonic analysis on metrized graphs*, Canadian Journal of Mathematics 59 (2007), 225-275.

[26] M. Baker and R. Rumely, *Potential Theorey and Dynamics on the Berkovich Projective Line,* Math. Surveys Monogr. 159, Amer. Math. Soc., Providence, 2010.

[27] M. Baker, S. Payne, J. Rabinoff, *Non-Archimedean geometry, tropicalization, and metrics on curves*, Preprint available at http://arxiv.org/abs/1104.0320.

[28] M. Baker and F. Shokrieh, *Chip-firing games, potential theory on graphs, and spanning trees*, J. Combin. Theory Ser. A, 120 (2013), issue 1, 164-182.

[29] A. A. Beilinson and Y. I. Manin, *The Mumford form and the Polyakov measure in string theory*, Communications in Mathematical Physics, 107(3), 359-376.

[30] A. A. Belavin and V. G. Knizhnik, *Complex geometry and the theory of quantum strings*, Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki (ISSN 0044-4510), vol. 91, Aug. 1986, p. 364-390. In Russian.

[31] A. A. Belavin and V. G. Knizhnik, *Algebraic geometry and the geometry of quantum strings*, Phys. Lett. B168, 201 (1986).

[32] P. Belkale and P. Brosnan, *Matroids, motives and a conjecture on Kontsevich*, Duke Math. Journal, Vol. 116 (2003) 147-188

[33] G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990.

[34] V. G. Berkovich, *Smooth p-adic analytic spaces are locally contractible*. Invent. Math. 137 (1999), no. 1, 1-84.

[35] V. G. Berkovich, *Smooth p-adic analytic spaces are locally contractible II.* Geometric aspects of Dwork theory, Walter de Gruyter and Co. KG, Berlin, 2004, 293-370.

[36] M. Bhargava, *P-orderings and polynomial functions on arbitrary subsets of Dedekind rings*, J. Reine Angew. Math. 490 (1997), 101-127.

[37] M. Bhargava, *On P-orderings, rings of integer-valued polynomials, and ultrametric analysis.* J. Amer. Math. Soc. 22 (2009), issue 4, 963-993.

[38] M. Bhargava, *The factorial function and generalizations*, Amer. Math. Monthly 107 (2000), 783-799.

[39] J.D. Biggins, *The central limit theorem for the supercritical branching random walk, and related results*. Stochastic Process. Appl. 34 (1990), 255-274.

[40] J. D. Biggins and A. E. Kyprianou*Seneta-Heyde norming in the branching random walk*, Ann. Probab. 25 (1997), 337-360.

[41] S. Billey and R. Vakil, *Intersections of Schubert varieties and other permutation array schemes*. In Algorithms in algebraic geometry. Springer New York, 208, p. 21–54.

[42] S. Bloch, H. Gillet, and C. Soulé, *Non-archimedean Arakelov theory*, Journal of Algebraic Geometry 4 (1995), 427-485.

[43] J.A. Bondy and U.S.R. Murty, *Graph Theory*. Grad. Texts in Math. 244, Springer-Verlag, New York, 2008.

[44] J. Borcea, P. Brändén, and T. M. Liggett, *Negative dependence and the geometry of polynomials*, J. Amer. Math. Soc. 22 (2009), 521-567.

[45] C. Bordenave, *A new proof of Friedman's second eigenvalue Theorem and its extension to random lifts*, https://arxiv.org/abs/1502.04482.

[46] O.V. Borodin, *Solution of the Ringel problem on vertex-face coloring of planar graphs and coloring of 1-planar graphs* (in Russian). Metody Diskret. Analyz. 41 (1984), 12-26.

[47] O.V. Borodin, H.J. Broersma, A. Glebov, and J. van den Heuvel, *Minimal degrees and chromatic numbers of squares of planar graphs* (in Russian). Diskretn. Anal. Issled. Oper. Ser. 1 8, no. 4 (2001), 9-33.

[48] O.V. Borodin, D.P. Sanders, and Y. Zhao, *On cyclic colorings and their generalizations*. Discrete Math. 203 (1999), 23-40.

[49] S. Bosch and W. Lütkebohmert, *Stable reduction and uniformization of abelian varieties I,* Math. Ann. 270 (1985), 349-379.
[50] J.-B. Bost, *Fibrés déterminants, déterminants régularisés et mesures sur les espaces de modules des courbes complexes.* Séminaire Bourbaki 29, 113-149.
[51] J.-B. Bost, *Green's currents and height pairing on complex tori*, Duke Math. J. 61 (1990), no. 3, 899-912.
[52] S. Boucksom, *Corps d'Okounkov, d'après Okounkov, Lazarsfeld-Mustata, Kaveh-Khovanski.* Séminaire Bourbaki, 65ème année, 2012-2013, n. 1059.
[53] S. Boucksom and M. Jonsson, *Tropical and non-Archimedean limits of degenerating families of volume forms*, Journal de l'École Polytechnique - Mathmatiques 4 (2017), 87-139.
[54] M. D. Bramson *Minimal displacement of branching random walks*, Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 45 (1978), 89-108.
[55] M. D. Bramson and O. Zeitouni, *Tightness for the minimal displacement of branching random walk*, J. Stat. Mech. Theory Exp. 7 (2007). P07010 (electronic).
[56] M. D. Bramson and O. Zeitouni, *Tightness for a family of recursive equations*, Ann. Probab. 37 (2009), 615-653.
[57] P. Brändén, *Polynomials with the half-plane property and matroid theory*, Adv. Math. 216 (2007), no. 1, 302-320.
[58] U. Brezner and M. Temkin, *Lifting problem for minimally wild covers of Berkovich curves*, preprint https://arxiv.org/abs/1709.10416.
[59] R. Brooks, *The spectral geometry of a tower of coverings*, Journal of Differential Geometry 23 (1986), 97-107.
[60] P. Brosnan and G. Pearlstein, *Jumps in the Archimedean Height*, https://arxiv.org/abs/1701.05527.
[61] M. Burger, *Petites valeurs propres du Laplacien et topologie de Fell*, thèse de doctorat (1986), Econom Druck AG (Basel).
[62] J. I. Burgos Gil, R. de Jong, D. Holmes, *Singularity of the biextension metric for families of abelian varieties*, preprint arxiv:1604.00686.
[63] L. Caporaso, *Geometry of tropical moduli spaces and linkage of graphs*, Journal of Combinatorial Theory, Series A, 119 (2012), 579-598.
[64] L. Caporaso and F. Viviani, *Torelli theorem for graphs and tropical curves*, Duke Math. J. 153:1 (2010), 129-171.
[65] S. Chaiken, *A combinatorial proof of the all minors matrix tree theorem*, SIAM J. Algebraic Discrete Methods 3 (1982), no. 3, 319-329.
[66] A. Chambert-Loir and A. Ducros, *Formes différentielles réelles et courants sur les espaces de Berkovich*, arXiv:1204.6277 (2012).
[67] M. Chan, *Lectures on tropical curves and their moduli spaces*, Preprint available at arXiv preprint arXiv:1606.02778.
[68] L.S. Chandran and C.R. Subramanian, *A spectral lower bound for the treewidth of a graph and its consequences,* Inf. Proc. Let. 87 (2003), 195-200.
[69] B. Chazelle, *The discrepancy method*, Cambridge University Press,
[70] J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, In R. C. Gunning, editor, Problems in Analysis, Pages 195-199, Princeton University Press, 1970.
[71] T. Chinburg and R. Rumely, *The capacity pairing*, J. reine angew. Math, 434 (1993), 1-44.
[72] Z. Cinkir, *Zhang's conjecture and the effective Bogomolov conjecture over function fields*. Inventiones mathematicae, 183 (2011), issue 3, 517–562.
[73] J. Clancy, N. Kaplan, T. Leake, S. Payne, and M. Wood, *On a Cohen-Lenstra heuristic for Jacobians of random graphs*, Journal of Algebraic Combinatorics 42 (2015), issue 3, 701-723.
[74] P.L. Clark, *On the Hasse principle for Shimura curves*, Israel J. Math. 171 (2009), 349-365.
[75] B. Colbois and A. El Soufi, *Extremal eigenvalues of the Lplacian in a conformal class of metrics : the "conformal spectrum"*, Annals of Global Analysis and Geometry 24 (2003), 337-349.
[76] B. Colbois and Y. Colin de Verdière, *Sur la multiplicité de la première valeur propre d'une surface de Riemann à courbure constante*, Commentarii Mathematici Helvetici 63 (1988), 194-208.
[77] B. Colbois and D. Maerten, *Eigenvalue estimates for the Neumann problem of a bounded domain*, J. Geom. Anal 18 (2008), 1022-1032.

[78] Y. Colin de Verdière, *Spectres de variétés Riemanniennes et spectres de graphes*, Proceedings of the International Congress of Mathematicians. Vol. 1. 1986.

[79] R. F. Coleman, *Stable maps of curves*, Documenta Math. Extra volume Kato (2003), 217-225.

[80] F. Cools, J. Draisma, S. Payne, and E. Robeva, *A tropical proof of the Brill-Noether Theorem.* Adv. Math. 230 (2012), no. 2, 759-776.

[81] R. F. Coleman, *Effective Chabauty*, Duke Math. J. 52 (1985), no. 3, 765-770.

[82] M. Coppens, *Clifford's theorem for graphs*, Advances in Geometry 16 (2016), issue 3, 389-400.

[83] R. Cori and Y. Le Borgne, *The sand-pile model and Tutte polynomials*, Adv. in Appl. Math., 30 (2003), no. 1-2, 44-52.

[84] G. Cornelissen, F. Kato, and J. Kool, *A combinatorial Li-Yau inequality and rational points on curves*, Mathematische Annalen 361 (2016), 211-258.

[85] N. Curien and B. Haas, *Random trees constructed by aggregation*, Annales de l'Institut Fourier, to appear.

[86] G. Davidoff, P. Sarnak, and A. Valette, *Elementary Number Theory, Group Theory, and Ramanujan Graphs*, LMS 55.

[87] F. M. Dekking and B. Host, *Limit distributions for minimal displacement of branching random walks*, Probab. Theory Related Fields 90 (1991), 403-426.

[88] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus,* Publications Mathématiques de l'IHES 36 (1969), no. 1, 75-109.

[89] D. Dhar, *Self-organized critical state of sandpile automaton models*, Phys. Rev. Lett., 64 (1990), issue 14: 1613-1616.

[90] D. Dhar, *Theoretical studies of self-organized criticality*, Phys. A, 369 (2006), issue 1, 29-70.

[91] R. Diestel, Graph Theory. *Grad. Texts in Math.* 173, Springer-Verlag, Berlin, 2005.

[92] R. Diestel and S-I. Oum, *Unifying duality theorems for width parameters in graphs and matroids. I. Weak and strong duality*, preprint available at `http://arxiv.org/abs/1406.3797`.

[93] R. A. Doney, *A note on some results of Schuh*, Journal of Applied Probability 21 (1984), 192-196.

[94] A. Ducros, *Triangulations et cohomologie étale sur une courbe analytique p-adique*, J. Algebraic Geom. 17 (2008), 503-575.

[95] A. Ducros, La structure des courbes analytiques, *book available on author's web-page.*

[96] J. Edmonds, *Maximum matching and a polyhedron with* $0,1$*-vertices.* J. Res. Nat. Bur. Standards Sect. B 69B (1965), 125-130.

[97] J. Ellenberg, C. Hall, and E. Kowalski, *Expander graphs, gonality and variation of Galois representations*, Duke Math. J. 161 (2012), 1233-1275.

[98] D. Eisenbud and J. Harris, *Limit linear series: Basic theory.* Invent. Math. 85 (1986), 337-371.

[99] D. Eisenbud and J. Harris, *The Kodaira dimension of the moduli space of curves of genus* ≥ 23. Invent. Math. 90 (1987), no. 2, 359-387.

[100] D. Eisenbud and J. Harris, *Existence, decomposition, and limits of certain Weierstrass points.* Invent. Math. 87 (1987), no. 3, 495-515.

[101] D. Eisenbud and J. Harris, *The monodromy of Weierstrass points.* Invent. Math. 90 (1987), no.2, 333-341.

[102] K. Eriksson and S. Linusson, *A combinatorial theory of higher-dimensional permutation arrays*, Advances in Applied Mathematics, 25 (2000), n. 2, 194–211.

[103] K. Eriksson and S. Linusson, *A decomposition of* $Fl(n)^d$ *indexed by permutation arrays,* Advances in Applied Mathematics, 25 (2000), n.2, 212–227.

[104] E. Esteves and N. Medeiros, *Limit canonical systems on curves with two components*, Inventiones mathematicae 149 (2002), issue 2, 267-338.

[105] X. Faber, *Spectral convergence of the discrete Laplacian on models of a metrized graph*, New York J. Math.12 (2006), 97-121.

[106] E. M. Feichtner and S. Yuzvinsky, *Chow rings of toric varieties defined by atomic lattices*, Inventiones Mathematicae 155 (2004), no. 3, 515-536.

[107] M. Fekete, *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffiziente.* Mathematische Zeitschrift 17 (1923), 228-249.

[108] Feller, W. (1970). An introduction to probability theory and its applications, *Vol. 2. Wiley, New York.*

[109] T. Foster, D. Ranganathan, M. Talpo, and M. Ulirsch, *Logarithmic Picard groups, chip firing, and the combinatorial rank*, https://arxiv.org/pdf/1611.10233.pdf.

[110] G. Frey, *Curves with infinitely many points of fixed degree*, Israel J. Math. 85 (1994), 79-83.

[111] J. Friedman, *A proof of Alon's second eigenvalue conjecture*, Memoirs of the A. M. S., 2010.

[112] M. Fuhr, *Revêtements abéliens et graphes de Ramanujan*, master thesis 2015, available at `http://www.math.ens.fr/~amini/HdR/memoire-fuhr.pdf`.

[113] W. Fulton, Intersection Theory, *Springer-Verlag, Berlin* (1998).

[114] A. Gathmann and M. Kerber, *A Riemann-Roch theorem in tropical geometry*, Math. Z. 259(1) (2008):217-230.

[115] A. O. Gel'fond, *On a uniqueness theorem*, Mat. Zametki. 1 (1967), 321-324.

[116] L. Gerritzen, *Die Jacobi-Abbildung über dem Raum der Mumfordkurven*, Math. Ann. 261 (1982), no.1 , 81-100.

[117] H. Gillet and C. Soulé, *Intersection theory using Adams operations.* Inventiones mathematicae 90 (1987), no. 2, 243–277.

[118] D. R. Grey, *A note on explosiveness of Markov branching processes*, Advances in Applied Probability 21 (1989), 226-228.

[119] P. Griffiths and J. Harris, *On the variety of special linear systems on a general algebraic curve*, Duke Math. J. 47 (1980), no. 1, 233-272.

[120] A. Grigor'yan, Y. Netrusov, and S.T. Yau, *Eigenvalues of elliptic operators and geometric applications*, Surveys in Differential Geometry **IX** (2004), 147-217.

[121] B. Gross and C. Schoen, *The modified diagonal cycle on the triple product of a pointed curve.* Ann. Inst. Fourier 45 (1995), 649–679.

[122] A. Grothendieck, *Esquisse d'un programme* (1986), in *L. Schneps, The Grothendieck Theory of Dessins d'enfants, London Mathematical Society, Lecture Notes Series 200, Cambridge University Press, Cambridge 1994.*

[123] W. Gubler and K. Künnemann, *A tropical approach to non-Archimedean Arakelov theory,* arXiv preprint arXiv:1406.7637 (2014).

[124] R. Hain, *Biextensions and heights associated to curves of odd genus*, Duke Math. J. 61 (1990), no. 3, 859-898.

[125] C. Hall, D. Puder, and W. Sawin, *Ramanujan Coverings of Graphs*, https://arxiv.org/abs/1506.02335.

[126] J. M. Hammersley, *Postulates for subadditive processes,* Ann. Probab. 2 (1974), 652-680.

[127] T. E. Harris, The Theory of Branching Processes. *Die Grundlehren der Mathematischen Wissenschaften* 119, Springer, Berlin, 1963.

[128] A. Hassannezhad, *Conformal upper bounds for the eigenvalues of the Laplacian and Steklov problem*, Journal of Functional Analysis 261 (2011), 3419-3436.

[129] F. Havet, J. van den Heuvel, C. McDiarmid, and B. Reed, *List colouring squares of planar graphs* (2008). Preprint available at `www.cdam.lse.ac.uk/Reports/Files/cdam-2008-09.pdf`.

[130] T. Hayama and G. Pearlstein, *Asymptotics of degenerations of mixed Hodge structures*, Adv. Math. 273 (2015), 380-420.

[131] J. Hersch, *Quatre propriétés isopérimétriques de membranes sphériques homogènes*, C. R. Acad. Sci. Paris 270 (1970), 1645-1648.

[132] T.J. Hetherington and D.R. Woodall, *List-colouring the square of a K_4-minor-free graph.* Discrete Math. 308 (2008), 4037-4043.

[133] G. Hetyei, *On the Stanley ring of a cubical complex*, Discrete and Computational Geometry 15 (1995), 305-330.

[134] J. van den Heuvel and S. McGuinness, *Coloring the square of a planar graph.* J. Graph Theory 42 (2003), 110-124.

[135] J. Hladký, D. Kràl' and S. Norine, *Rank of divisors on tropical curves*, J. Combin. Theory Ser. A 120 (2013), no 7, 1521-1538.

[136] D. Holmes and R. de Jong, *Asymptotics of the Néron height pairing*, Math. Res. Lett. 22 (2015), no. 5, 1337-1372.

[137] S. Hoory, N. Linial and A. Widgerson. *Expander Graphs and their Applications*, Bull. A. M. S., 43:439-561, 2006.

[138] Y. Hu and Z. Shi, *Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees*, Ann. Probab. 37 (2009), 742-789.

[139] I. Itenberg, L. Katzarkov, G. Mikhalkin, I. Zharkov, *Tropical homology*, http://arxiv.org/abs/1604.01838.

[140] C. Itzykson and J.-B. Zuber, Quantum Field Theory, *International Series in Pure and Applied Physics*, McGraw-Hill International, New York, 1980.

[141] T.R. Jensen and B. Toft, Graph Coloring Problems. *John-Wiley & Sons*, New York, 1995.

[142] T.K. Jonas, *Graph coloring analogues with a condition at distance two: $L(2,1)$-labelings and list λ-labelings*. Ph.D. Thesis, University of South Carolina, 1993.

[143] J. Kahn, *Asymptotics of the chromatic index for multigraphs*. J. Combin. Theory Ser. B 68 (1996), 233-254.

[144] J. Kahn, *Asymptotics of the list-chromatic index for multigraphs*. Random Structures Algorithms 17 (2000), 117-156.

[145] J. Kahn and P.M. Kayll, *On the stochastic independence properties of hard-core distributions*. Combinatorica 17 (1997), 369-391.

[146] K. Kato, C. Nakayama, and S. Usui, $SL(2)$-orbit theorem for degeneration of mixed Hodge structure, *J. Algebraic Geom.* 17 (2008), no. 3, 401-479.

[147] E. Katz and D. Zureick-Brown, *The Chabauty-Coleman bound at a prime of bad reduction and Clifford bounds for geometric rank functions*, Compositio Math. 149 (2013), no. 11, 1818-1838.

[148] E. Katz, J. Rabinoff and D. Zureick-Brown, *Uniform bounds for the number of rational points on curves of small Mordell–Weil rank*, Duke Math. J. 165 (2016), n. 16 , 3189-3240.

[149] K. Kaveh and A. G. Khovanskii, *Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory.* Annals of Mathematics 176 (2012), 925-978.

[150] J. Kelner, *Spectral partitioning, eigenvalue bounds, and circle packings for graphs of bounded genus*, SIAM Journal on Computing 35 (2006), 882-902.

[151] J.A. Kelner, J.R. Lee, G.N. Price, and S.H. Teng, *Metric uniformization and spectral bounds for graphs,* Geometric and Functional Analysis 21 (2011), 1117-1143.

[152] J. F. Kingman, *The first birth problem for an age-dependent branching process.* Ann. Probab. 3 (1975), 790-801.

[153] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird, *Annalen der Physik* 148 (1847), no. 12, 497-508.

[154] P. Klein, S.A. Plotkin, S. Rao, *Excluded minors, network decomposition, and multicommodity flow*, In Proceedings of the twenty-fifth annual ACM Symposium on Theory of Computing (STOC '93). ACM, New York, NY, USA, 682-690.

[155] F. Knudsen and D. Mumford, *The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div"*. Math. Scand. 39:1, 19-35 (1976)

[156] G. Kokarev, *Variational aspects of Laplace eigenvalues on Riemannian surfaces*, Advances in Mathematics 258 (2014), 191-239.

[157] J. Kolb, *A simplicial calculus for local intersection numbers at non-archimedean places on products of semi-stable curves.* Abhandlungen aus dem Mathematischen Seminar der Universitt Hamburg 86.1 (2016), 97–132.

[158] J. Kolb, *An analytic description of local intersection numbers on non-Archimedean places for products of semi-stable curves.* Ann. Fac. Sci. Toulouse Math. 25 (2016), issue 1, 19–63.

[159] ,V. F. Kolchin, Random Mappings, *Optimization Software Inc.*, New York, 1986.

[160] N. Korevaar, *Upper bounds for eigenvalues of conformal metrics*, Journal of Differential Geometry 37 (1993), 73-93.

[161] A.V. Kostochka and D.R. Woodall, *Choosability conjectures and multicircuits.* Discrete Math. 240 (2001), 123-143.

[162] D. Laksov, *Weierstrass points on curves.* Astérisque 87-88 (1981), 221-247.

[163] D. Laksov, *Wronskians and Plücker formulas for linear systems on curves*, Ann. Sci. Éc. Norm. Sup. 17 (1984), 45-66.

[164] S. Lang, Introduction to Arakelov Theory, *Springer-Verlag.*

[165] R. Lazarsfeld and M. Mustata, *Convex bodies associated to linear series.* Ann. Sci. Éc. Norm. Sup. 42 (2009), 783-835.

[166] D. Lear, *Extensions of normal functions and asymptotics of the height pairing*, PhD thesis, University of Washington, 1990.
[167] C.W. Lee, *Some recent results on convex polytopes.* In: J.C. Lagarias and M.J. Todd, eds., *Mathematical Developments Arising from Linear Programming.* Contemp. Math. 114 (1990), 3-19.
[168] J.R. Lee, S. Oveis Gharan, L. Trevisan, *Multi-way spectral partitioning and higher-order Cheeger inequalities*, In Proceedings of the forty-fourth annual ACM Symposium on Theory of Computing (STOC '12). ACM, New York, NY, USA, 1117-1130.
[169] Y. Len, *Hyperelliptic graphs and metrized complexes*, Forum of Mathematics, Sigma, Vol. 5 (2017), Cambridge University Press.
[170] P. Li and S. T. Yau, *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Invent. math. 69 (1982), 269-291.
[171] K.-W. Lih, W. F. Wang and X. Zhu, *Coloring the square of a K_4-minor free graph.* Discrete Math. 269 (2003), 303-309.
[172] Q. Liu, *Stable reduction of finite covers of curves*, Compositio Math. 142 (2006), 101-118.
[173] Q. Liu and D. Lorenzini, *Models of curves and finite covers*, Compositio Math. 118 (1999), 61-102.
[174] D. Lorenzini, *Two-variable zeta-functions on graphs and Riemann-Roch theorems*, International Mathematics Research Notices, 2012 (22), 5100-5131.
[175] D. J. Lorenzini and T. J. Tucker, *Thue equations and the method of Chabauty-Coleman*, Invent. Math. 148 (2002), no. 1, 47-77.
[176] A. Louis, P. Raghavendra, P. Tetali, S. Vempala, *Many sparse cuts via higher eigenvalues*, In Proceedings of the forty-fourth annual ACM Symposium on Theory of Computing (STOC '12). ACM, New York, NY, USA, 1131-1140.
[177] A. Lubotzky, *Discrete Groups, Expanding Graphs and Invariant Measures*, vol. 125, Springer, 1994.
[178] Y. Luo, *Rank-determining sets of metric graphs*, J. Combin. Theory Ser. A 118 (2011):1775-1793.
[179] R. Lyons, *Random Walks and Percolation on Trees*, Ann. Probab. 18 (1990), no. 3, 931-958.
[180] T. J. Lyons, *A simple criterion for transience of a reversible Markov chain,* Ann. Probab. 11 (1983), 393-402.
[181] R. Lyons and Y. Peres, *Probability on Trees and Networks*, Cambridge Series in Statistical and Probabilistic Mathematics 2016.
[182] D. Maclagan and B. Sturmfels, Introduction to Tropical Geometry, *AMS Graduate Texts in Mathematics* 161, 2015.
[183] K. Mahler, *An interpolation series for a continuous function of a p-adic variable*, J. reine angew. Math. 199 (1958), 23-34.
[184] A. Marcus, D. Speilman, N. Srivastava, *Interlacing families I: bipartite Ramanujan graphs of all degrees*, Ann. Math. 182 (2015), 307-325.
[185] T. Mantuano, *Discretization of compact Riemannian manifolds applied to the spectrum of the Laplacian*, Annals of Global Analysis and Geometry 27 (2005), 33-46.
[186] W. McCallum and B. Poonen, *The method of Chabauty and Coleman*, June 14, 2010. Preprint, `http://www-math.mit.edu/~poonen/papers/chabauty.pdf`, to appear in Panoramas et Synthèses, Société Math. de France.
[187] C. McDiarmid, *Minimal positions in a branching random walk*, Ann. Appl. Probab. 5 (1995), 128-139.
[188] G. Mikhalkin and I. Zharkov, *Tropical curves, their Jacobians and theta functions*, Contemp. Math. 465 (2008), Amer. Math. Soc., Providence, RI, 203-230.
[189] M. Miller and J. Sirán, *Moore graphs and beyond: A survey of the degree/diameter problem*, Electronic Journal of Combinatorics (2005), #DS14.
[190] G. L. Miller, S. H. Teng, W. Thurston, S. A. Vavasis, *Geometric separators for finite element meshes*, SIAM J. Scientific Computing 19 (1998), 364-386.
[191] N. E. Mnëv, *The universality theorems on the classification problem of configuration varieties and convex polytopes varieties*, pp. 527?543, in "Topology and geometry: Rohlin Seminar." Edited by O. Ya. Viro. Lecture Notes in Mathematics, 1346. Springer-Verlag, Berlin, 1988.
[192] B. Mohar, *Some applications of Laplace eigenvalues of graphs.* In: Hahn G., Sabidussi G. (eds) Graph Symmetry. NATO ASI Series (Series C: Mathematical and Physical Sciences), vol 497. Springer, 1997.

[193] M. Molloy and B. Reed, Graph Colouring and the Probabilistic Method, *Algorithms Combin.* 23, Springer-Verlag, Berlin, 2002.

[194] M. Molloy and M.R. Salavatipour, *A bound on the chromatic number of the square of a planar graph.* J. Combin. Theory Ser. B 94 (2005), 189-213.

[195] D. Mumford, Curves and their Jacobians, *Univ. of Mich. Press*, Ann Arbor.

[196] D. Mumford, *Stability of projective varieties*, Enseign. Math. 23 (1977), 39-100.

[197] A. Neeman, *The distribution of Weierstrass points on a compact Riemann surface*, Ann. Math. 120 (1984), 317-328.

[198] A. Nilli, *On the second eigenvalue of a graph*, Discrete Math., 91(2):207-210, 1991.

[199] B. Olsen, *On higher order Weierstrass points.* Ann. of Math. 95 (1972), 357-364.

[200] O. Ore and M.D. Plummer, *Cyclic coloration of plane graphs.* In: *Recent Progress in Combinatorics; Proceedings of the Third Waterloo Conference on Combinatorics.* Academic Press, San Diego (1969) 287-293.

[201] B. Osserman, *Limit linear series for curves not of compact type*, Journal für die reine und angewandte Mathematik, to appear.

[202] B. Osserman, *Limit linear series and the Amini-Baker construction*, arXiv preprint arXiv:1707.03845.

[203] A. Ostrowski, *Uber ganzwertige Polynome in algebraischen Zahlkörpern*, J. reine angew. Math. 149 (1919) 117-124.

[204] J. G. Oxley, Matroid theory, *Oxford Science Publications*, The Clarendon Press, Oxford University Press, New York, 1992.

[205] G. J. Pearlstein, *Degenerations of Mixed Hodge Structures*, Duke Math. J. 110, no. 2, (2001), 217-251.

[206] G. J. Pearlstein, *SL_2-orbits and degenerations of mixed Hodge structures*, J. Diff. Geometry 74 (2006), 1-67.

[207] G. J. Pearlstein, *Variations of mixed Hodge structure, Higgs fields, and quantum cohomology,* Manuscripta Math. 102 (2000), no. 3, 269-310.

[208] R. Pemantle and Y. Peres, *Domination Between Trees and Application to an Explosion Problem*, Ann. Probab. 22 (1994), 180-194.

[209] M. Piquerez, *A generalization of Symanzik polynomials*, master thesis 2017, available at `https://www.eleves.ens.fr/home/piquerez/pdf/A_generalization_of_Symanzik_polynomials.pdf`.

[210] G. Pólya, *Über ganzwertige ganze Funktionen*, Rend. Circ. Mat. Palermo 40 (1915) 1-16.

[211] G. Pólya, *Uber ganzwertige Polynome in algebraischen Zahlkörpern*, J. reine angew. Math. 149 (1919) 97-116.

[212] L. Pyber and E. Szabó, *Growth in finite simple groups of Lie type of bounded rank*, Preprint available at `http://arxiv.org/abs/1005.1858`.

[213] Y. Rabinovich, A. Sinclair and A. Wigderson, *Quadratic dynamical systems.* In: *Proceedings of the 33rd Annual Conference on Foundations of Computer Science (FOCS)*, (1992), 304-313.

[214] B.A. Reed, *Tree width and tangles: A new connectivity measure and some applications*, Surveys in Combinatorics 241 (1997), 87-162.

[215] N. Robertson and P. Seymour, *Graph Minors. I - XXIII*, J. Comb. Theory Series B. 1983-2013.

[216] M. Saïdi, *Revêtements modérés et groupe fondamental de graphe de groupes*, Compos. Math. 107 (1997), 319–338.

[217] D.P. Sanders and Y. Zhao, *A new bound on the cyclic chromatic number.* J. Combin. Theory Ser. B 83 (2001), 102-111.

[218] A. Schrijver, Combinatorial Optimization; Polyhedra and Efficiency. *Algorithms Combin.* 24, Springer-Verlag, Berlin, 2003.

[219] A. Schweizer, *On the uniform boundedness conjecture for Drinfeld modules*, Math. Z. 244 (2003), 601-614.

[220] H. J. Schuh, *Sums of i.i.d. random variables and an application to the explosion criterion for Markov branching processes,* J. Appl. Prob. 19 (1982), 29-38.

[221] A. Scott and A. Sokal, *Complete monotonicity for inverse powers of some combinatorially defined polynomials*, Acta Mathematica 213 (2014), Issue 2, 323?392.

[222] B. A. Sevast'yanov, *On the regularity of branching processes, Mat. Zametki.* 1 (1967), 53-62.

[223] B. A. Sevast'yanov, *A necessary condition for the regularity of branching processes*, *Mat. Zametki.* 7 (1970), 389-396.

[224] P. Seymour and R. Thomas, *Graph searching and a min-max theorem for tree-width,* J. Comb. Theory, Series B, 58 (1993), 22-33.

[225] J-P. Serre, Arbres, amalgames, SL2, *Astérique* 46, Soc. math. France, 1977.

[226] K. Shaw, *A tropical intersection product in matroidal fans*, SIAM J. Discrete Math. 27 (2013), issue 1, 459-491.

[227] S-J. Smit, *String theory and algebraic geometry of moduli spaces*, Communications in Mathematical Physics 114 (1988), 645-685.

[228] D. Spielman, S.H. Teng, *Spectral partitioning works: Planar graphs and finite element meshes*, Linear Algebra and its Applications 421 (2007), 284-305.

[229] M. Stoll, *Independence of rational points on twists of a given curve*, Compositio Math. 142 (2006), no. 5, 1201-1214.

[230] M. Stoll, *Uniform bounds for the number of rational points on hyperelliptic curves of small Mordell-Weil rank*, Journal of the European Mathematical Society, to appear.

[231] G. Szegö, *Inequalities for certain eigenvalues of a membrane of given area*, J. Rational Mech. Anal 3 (1954), 343-356.

[232] M. Temkin, *Introduction to Berkovich analytic spaces*, `http://arxiv.org/abs/1010.2235v1`.

[233] P. Tourkine, *Tropical amplitudes*, Annales Henri Poincaré, Volume 18, Issue 6, pp 2199-2249.

[234] A. Thuillier, *Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications à la théorie d'Arakelov.* Ph.D. thesis, University of Rennes, 2005, `http://tel.ccsd.cnrs.fr/documents/archives0/00/01/09/90/index.html`.

[235] R. Vakil, *Murphy's Law in algebraic geometry: Badly-behaved deformation spaces*, Invent. math. 164, 569-590 (2006).

[236] V. A. Vatutin, *A condition for the regularity of the Bellman-Harris branching process* (Russian) Dokl. Akad. Nauk SSSR 230 (1976), no. 1, 15-18.

[237] V. A. Vatutin, *Sufficient conditions for the regularity of Bellman-Harris branching processes*, Theory Probab. Appl. 31 (1987), 50-57.

[238] V. A. Vatutin and A. M. Zubkov, *Branching processes II*, Probability theory and mathematical statistics 1 (1993), J. Soviet Math. 67 3407-3485.

[239] A. A. Voronov, *A unified approach to string scattering amplitudes*, Comm. Math. Phys. 131 (1990), no. 1, 179-218.

[240] G. Wegner, *Graphs with given diameter and a coloring problem.* Technical Report, University of Dortmund, 1977.

[241] W. Wewers, *Deformation of tame admissible covers of curves*, Aspects of Galois theory (Gainesville, FL, 1996), London Math. Soc. Lecture Note Ser. 206 (1999), Cambridge Univ. Press, 239–282.

[242] H. Whitney, *Congruent Graphs and the Connectivity of Graphs*, American Journal of Mathematics 54 (1932), No. 1, 150-168.

[243] H. Whitney, *2-isomorphic graphs*, Amer. Journ. Math. 55 (1933), 245-254.

[244] S.A. Wong, Colouring graphs with respect to distance. M.Sc. Thesis, Department of Combinatorics and Optimization, University of Waterloo, 1996.

[245] M. Wood, *The distribution of sandpile groups of random graphs*, Journal of the American Mathematical Society, 30 (2017), issue 4, 915-958.

[246] S. Wolpert, *Asymptotics of the spectrum and the Selberg zeta function on the space of Riemann surfaces*, Communications in Mathematical Physics 112(1987), 283-315.

[247] P. Yang, S.T. Yau, *Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4 (1980), 55-63.

[248] S. Zhang, *Admissible pairing on a curve,* Invent. math. 112 (1993), 171-193.

[249] S.-W. Zhang, *Small points and adelic metrics.* J. Algebraic Geom. 4 (1995), issue 2, 281–300.

[250] S.-W. Zhang, *Gorss-Schoen cycles and dualising sheaves.* Inventiones mathematicae 170 (2010), 1–73.