LATTICE OF INTEGER FLOWS AND POSET OF STRONGLY CONNECTED ORIENTATIONS

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Abstract. — We show that the Voronoi cells of the lattice of integer flows of a finite connected graph G in the quadratic vector space of real valued flows have the following very precise combinatorics: the face poset of a Voronoi cell is isomorphic to the poset of strongly connected orientations of subgraphs of G. This confirms a recent conjecture of Caporaso and Viviani {*Torelli Theorem For Graphs and Tropical Curves*, Duke Math. J. **153**(1) (2010), 129-171}.

1. Introduction

1.1. The Lattice of Integer Flows. — Consider a finite graph $G = (V, E)$ with possible parallel edges and loops. All through the paper, we suppose that G is connected. Following the classical works (e.g.,[**3**], [**1**]), we replace the set of edges of G with the set of oriented edges $\mathbb{E}$, where each edge $e \in E$ is replaces with two oriented edges in opposite directions in $\mathbb{E}$. By the abuse of notation, we also denote by e an element of $\mathbb{E}$, and use $\bar{e}$ when referring to the same edge but with the inverse orientation.

For a ring A, let $C_0(G, A)$ be the free A-module generated by the set of vertices V and let $C_1(G, A)$ be the quotient of the free A-module generated by the set of oriented edges $\mathbb{E}$ by all the relations $(e) = -(\bar{e})$ for any oriented edge $e \in \mathbb{E}$. The basis elements of $C_0(G, A)$ are denoted by (v) for $v \in V$ and the basis elements of $C_1(G, A)$ are denoted by (e) for $e \in \mathbb{E}$. The chain complex $\mathcal{C}_*(G, A) : C_1(G, A) \xrightarrow{\partial} C_0(G, A)$ is the usual simplicial chain complex of G: the boundary map is given by $\partial(e) = (u) - (v)$ for u and v being respectively the head and the tail of the oriented edge e.

Similarly, one can define the cochain complex $\mathcal{C}^* : C^0(G, A) \xrightarrow{d} C^1(G, A)$ by letting $C^0(G, A)$ to be the A-module of all the functions $f : V \to A$ and $C^1(G, A)$ to be the A-module of all the functions $g : \mathbb{E} \to A$ verifying the property that $g(e) = -g(\bar{e})$ for any $e \in \mathbb{E}$. The differential d is defined as follows: $d(f)(e) = f(u) - f(v)$ where u and v are respectively the head and the tail of the oriented edge e. Note that the spaces $C_i(G, A)$ and $C^i(G, A)$ are canonically dual for $i = 0, 1$.

The (co)homology of $\mathcal{C}_*$ and $\mathcal{C}^*$ have a rather simple description:

– $H_1(\mathcal{C}_*(G, A)) \simeq A^g$, i.e., is a free A-module of rank g where g is the genus of the (unoriented) graph G. $H_0(\mathcal{C}_*(G, A)) \simeq A$.
– Similarly, $H^1(\mathcal{C}_*(G, A)) \simeq A^g$ and $H^0(\mathcal{C}^*(G, A)) \simeq A$.

The first homology group $H_1(\mathcal{C}_*(G, A))$ also coincides with the A-module of all the A-valued flows in G. Recall that a *flow* x in G with values in A is the data of $\{x_e \in A\}_{e \in \mathbb{E}}$ for any $e \in \mathbb{E}$ with the following properties:

– For any oriented edge $e \in \mathbb{E}$, $x_e = -x_{\bar{e}}$.
– For any vertex v, let $\mathbb{E}^{\mathrm{out}}(v)$ be the set of all the oriented edges emanating from v, i.e., with tail equal to v. Then $\sum_{e \in \mathbb{E}^{\mathrm{out}}(v)} x_e = 0$. (In other words: the amount of the flow coming out of u is equal to the amount of the flow entering u.)

Let F be the vector space of real-valued flows in G, i.e., $\mathrm{F} := H_1(\mathcal{C}_*(G, \mathbb{R}))$. And let Λ be the lattice of integer-valued flows in F, i.e., $\Lambda := H_1(\mathcal{C}_*(G, \mathbb{Z}))$, $\Lambda \subset \mathrm{F}$. Note that F is a vector subspace of dimension g in $C_1(G, \mathbb{R})$ and Λ has rank g.

1.2. The Positive Quadratic Form q. — There is a natural inner-product on $C_1(G, \mathbb{R})$ (and a natural inner product on $C_0(G, \mathbb{R})$) given by the simplicial structure of G. This is given on the basis as follows

$$\langle (e), (e') \rangle := \begin{cases} 1 & \text{if } e' = e, \text{ and} \\ -1 & \text{if } e' = \bar{e}; \\ 0 & \text{otherwise.} \end{cases}$$

(Similarly on $C_0(G, \mathbb{R})$, $\langle (u), (v) \rangle := \delta_{u,v}$ for vertices $u, v \in V$.)

It is straightforward to check that the above pairings naturally identify the dual spaces $C_i(G, \mathbb{R})$ and $C^i(G, \mathbb{R})$ and the adjoint d^* of d gets identified with ∂.

The above inner-product defines by restriction an inner-product on F, denoted again by $\langle , \rangle$; let q be the corresponding positive quadratic form on F given by $q(x) = \langle x, x \rangle$, for $x \in \mathrm{F}$.

1.3. Voronoi Diagram of Λ in H. — Consider a discrete subset S in the real vector space F. For a point λ in S, we define the *Voronoi cell* of λ with respect to q as

$$V_\lambda = \left\{ x \in \mathrm{F} \mid q(x - \lambda) \le q(x - \mu) \text{ for any other point } \mu \in S \right\}.$$

The *Voronoi diagram* $\mathrm{Vor}_q(\mathcal{S})$ is the decomposition of F induced by the cells V_λ, for $\lambda \in \mathcal{S}$. It is straightforward to see that each cell of the Voronoi diagram is a polytope or a polyhedron (in the case there are cells of infinite volume).

Consider the discrete subset Λ of F, which is of full rank. By translation-invariance of the distances defined by q, the cells of the Voronoi diagram are all simply translations of each other, i.e., for a point λ in Λ, $V_\lambda = V_O + \lambda$, where O denotes the origin. Thus, the Voronoi cell decomposition of F is completely understood by the Voronoi cell V_O, which is easily seen to be a polytope.

Consider now the set of all faces of the polytope V_0 ordered by inclusion. They form a finite poset that we denote by $\mathcal{FP}$. Our aim in this paper is to give a precise description of this poset in terms of the original graph G, that we now explain.

1.4. Poset of Strongly Connected Orientations of Subgraphs. — Let D be an orientation of a connected graph G. D is called *strongly connected* if any pair of vertices u and v in D are connected by an oriented path from u to v and an oriented path from v to u. An orientation of a (non necessarily connected) graph G is called strongly connected if the orientation induced on each of the connected components is strongly connected.

Let G be a given graph (with possibly multiple edges and loops). Define the following poset $\mathcal{SC}$, that we call the poset of strongly connected orientations of subgraphs of G: the elements of $\mathcal{SC}(G)$ are all the pairs (H, D_H) where H is a subgraph of G and D_H is a strongly connected orientation of the edges of H. (Note that the vertices and the edges of G are labeled, so parallel edges are distinguished in dealing with subgraphs.) A partial order $\preceq$ is defined on $\mathcal{SC}$ as follows: given two elements (H, D_H) and $(H', D_{H'})$ in $\mathcal{SC}$, we have $(H, D_H) \preceq (H', D_{H'})$ if and only if $H' \subseteq H$ and the orientation $D_{H'}$ is the orientation induced by D_H on H'. It is quite straightforward to see that $(\mathcal{SC}, \preceq)$ is a graded poset, and the grading is given by g minus the genus of the underlying subgraph. Note that $(\emptyset, \emptyset)$ is the maximum element of $(\mathcal{SC}, \preceq)$.

The main result of this paper is the following theorem.

Theorem 1. — *The two posets $\mathcal{FP}$ and $\mathcal{SC}$ are isomorphic.*

This is precisely Conjecture 5.2.8 of [**2**] which answers a question asked by Bacher *et al.* in [**1**]. The two posets $\mathcal{FP}$ and $\mathcal{SC}$ are the posets $\mathrm{Faces}(\mathrm{Vor}_\Gamma)$ and $\mathcal{OP}_\Gamma$ in the notations of [**2**] where $G = \Gamma$ is the metric graph with all lengths equal to one. The proof of our theorem can be extended in a straightforward way to more general metric graph, thus, Conjecture 5.2.8 of [**2**] holds. We have presented the proofs in the unweighted case to simplify the presentation. Note that Conjecture 5.2.8 of [**2**] has a second part (explained in Section 3): we will establish a precise bijection between the elements of the two posets, from this bijection the second part of the conjecture easily follows.

2. Proof of Theorem 1

A (non-necessarily connected) graph is Eulerian if every vertex has an even degree. An orientation of an Eulerian graph is called Eulerian if each vertex has the same in- and out-degrees. It is clear that every Eulerian orientation of a graph is necessarily strongly connected (in the sense we describes above). Also note that an Eulerian orientation D of a graph H has a canonical flow denoted by x^D. This is defined by setting $x_e^D = 1$ if $e \in D$ (thus, $x_e^D = -1$ if $\bar{e} \in D$) and $x_e^D = 0$ if neither e nor $\bar{e}$ is in D. For simplicity, we say a flow x in a graph G is Eulerian if it is of the form x^D for some D an Eulerian orientation of a subgraph H of G. The genus of an Eulerian flow $\mu = x^D$, for D an Eulerian orientation of an Eulerian graph H, is by definition the genus of H. In the case H is disconnected, this latter is the sum of the genus of the connected components of H.

Given a flow x, by the support of x, denoted by $\mathrm{supp}(x)$, we mean the set of all the oriented edges $e \in \mathbb{E}$ with $x_e > 0$. Clearly the oriented graph defined by $\mathrm{supp}(x)$ is strongly connected. (Different flows can however have the same support.) Note also that for an Eulerian flow x, $\mathrm{supp}(x)$ is the Eulerian orientation with $x = x^{\mathrm{supp}(x)}$.

Lemma 2 (Intersecting Voronoi Cells). — *A Voronoi cell V_λ, for $\lambda \in \Lambda$, intersects the Voronoi cell V_O of the origin if and only if λ is Eulerian. For two points λ and μ in Λ, $V_\lambda \cap V_\mu \neq \emptyset$ if and only if $\lambda - \mu$ is Eulerian.*

Proof. — We first show that if $V_\lambda \cap V_O \neq \emptyset$, then $\lambda/2 \in V_\lambda \cap V_O$. This will then easily implies the lemma.

The claim is a particular property of the Voronoi diagrams defined by the lattices. Suppose that $\lambda/2 \notin V_\lambda \cap V_O$. This means there exists a point $\mu \in \Lambda$ such that $q(\mu - \lambda/2) < q(\lambda/2)$, in other words, $\langle \lambda, \mu \rangle > q(\mu)$ or equivalently, $\langle \lambda - \mu, \mu \rangle > 0$. Consider now the parallelogram defined by O, λ, μ and $\lambda - \mu$ (which is in Λ). We show that a point $x \in V_O \cap V_\lambda$ is certainly closer in distance to one of μ or $\lambda - \mu$, and this will be a contradiction. By assumption $q(x) = q(x - \lambda)$, which gives $2\langle x, \lambda \rangle = q(\lambda)$. Since $q(x) \leq q(x - \mu)$, we have $2\langle x, \mu \rangle \leq q(\mu)$. Combining the two (in-)equalities, we have

$$2\langle x, \lambda - \mu \rangle \geq q(\lambda) - q(\mu) = q(\lambda - \mu) + 2\langle \lambda - \mu, \mu \rangle > q(\lambda - \mu).$$

This shows that $q(x) > q(x - \lambda + \mu)$, which is a contradiction to the assumption $x \in V_O$. And the claim follows.

Suppose now $\lambda \in \Lambda$ a lattice point such that $\lambda/2 \in V_O \cap V_\lambda$. This latter condition is equivalent to the following:

$$\text{For all } \mu \in \Lambda, \qquad \langle \lambda, \mu \rangle \leq q(\mu).$$

Let $D = \mathrm{supp}(\lambda)$. We observed that D is a strongly connected orientation. For any oriented cycle C in D, consider the flow x^C in G. We must have $\langle \lambda, x^C \rangle = \sum_{e \in C} \lambda_e \geq q(C)$. By combining these two inequalities, we infer that the equality holds, and so, $\lambda_e = 1$ for all $e \in C$. Since every oriented edge in $\mathrm{supp}(\lambda)$ is contained in an oriented cycle $C \subset \mathrm{supp}(\lambda)$, we have $\mu_e = 1$ for all $e \in \mathrm{supp}(\lambda)$. This shows that λ is Eulerian and the first claim of the lemma follows. The second claim follows by translation invariance. $\qquad\square$

2.1. Codimension One Faces of $\mathcal{FP}$: Description of the Delaunay Edges. — We provide now a description of the faces of codimension one in $\mathcal{FP}$; this clearly provides a description of the dual Delaunay edges. (Recall that the one-sequelleton of the Delaunay dual is the graph obtained on Λ by joining two points λ and μ if $V_\lambda \cap V_\mu$ is a face of codimension one in both V_λ and V_μ.)

Let $F \in \mathcal{FP}$ be a face of codimension one and let x denote a generic point of F (an arbitrary point in the relative interior of F). Since F is of codimension one, there is a Voronoi cell V_λ such that $F \subseteq V_O \cap V_\lambda$. By Lemma 2, λ is Eulerian.

Lemma 3. — *The intersection $V_O \cap V_\lambda$ lies in an affine plane of codimension equal to the genus of $\mathrm{supp}(\mu)$. In particular, if V_O and V_λ intersect in a face of codimension one, then $\mathrm{supp}(\mu)$ is a circuit C (an orientation of a cycle) and $\mu = x^C$.*

Proof. — A point x in the intersection $V_O \cap V_\lambda$ has the following properties:

I. $q(x) = q(x - \lambda)$ or equivalently $2\langle x, \lambda \rangle = q(\lambda)$, and

II. for all $\mu \in \Lambda$, $2\langle x, \mu \rangle \leq q(\mu)$.

By Lemma 2, λ is Eulerian; $\mathrm{supp}(\lambda)$ is thus an Eulerian oriented graph and can be decomposed into a disjoint union of oriented cycles. In other words, there exist Eulerian $\lambda_1, \ldots, \lambda_k \in \Lambda$, for some $k \in \mathbb{N}$, such that

(i) for $1 \leq i \leq k$, $\mathrm{supp}(\lambda_i)$ is a circuit;

(ii) $\lambda = \lambda_1 + \cdots + \lambda_k$; and

(iii) for all $1 \leq i \neq j \leq k$, $\langle \lambda_i, \lambda_j \rangle = 0$.

In addition, for λ_i one may choose any oriented circuit C of $\mathrm{supp}(\lambda)$ and define $\lambda_1 = x^C$. (To see this, note that one way to obtain the decomposition is to proceed greedily, and in each step, choose an arbitrary circuit in the remaining oriented graph, delete it and proceed.)

Applying Property II above to each λ_i and using (ii) and (iii) above, one sees that

$$ 2\langle x, \lambda_i \rangle = q(\lambda_i), \qquad \text{for all } i \in \{1, \ldots, k\}. $$

Since there exists a decomposition such that $\lambda_1 = x^C$ for an arbitrary circuit of $\mathrm{supp}(\lambda)$, we infer that

$$ \textit{For every circuit } C \subset \mathrm{supp}(\lambda), \qquad 2\langle x, x^C \rangle = q(x^C). \tag{$*$} $$

Let $C_1, \ldots, C_{g_\lambda}$ be a basis for the cycle space of $\mathrm{supp}(\mu)$ and let P_λ be the affine plane defined by the g_λ equations $(*)$ for x^{C_i}, for $i = 1, \ldots, g_\lambda$. Here g_λ is the genus of $\mathrm{supp}(\lambda)$. Since the elements x^{C_i} are independent, and the inner product $\langle , \rangle$ is non-degenerate, P_λ has codimension g_λ and it contains $V_O \cap V_\lambda$. The lemma follows. $\qquad\square$

Proposition 4. — *Let Ξ_1 be the set of all Eulerian elements of Λ of genus 1. Ξ_1 forms a system of generates of Λ. The one-sequeletton of the Delaunay dual of Vor_q is isomorphic to $\mathrm{Cay}(\Lambda, \Xi_1)$.*

Proof. — Let $\mu \in \Lambda$. We show the existence of $\lambda_1, \ldots, \lambda_k$ such that each λ_i is of the form x^{C_i} for a circuit C_i and in addition $\mu = \sum_i \lambda_i$ (and in addition $C_i \subset \mathrm{supp}(\lambda)$). This clearly proves the proposition.

Let D be the Eulerian oriented graph obtained by replacing each arc e in $\mathrm{supp}(\mu)$ with λ_e different parallel arcs. Since D is Eulerian, it admits a decomposition into k circuits, for some $k \in \mathbb{N}$. Looking at the circuits in the original oriented graph $\mathrm{supp}(\mu)$, one obtains k circuits $C_1, \ldots, C_k \subset \mathrm{supp}(\mu)$, and it is clear that $\mu = \sum_1^k \lambda_i$ for $\lambda_i = x^{C_i}$. $\qquad\square$

Hyperplane Arrangement Defined by Circuits. — Let Ξ_1 be the set of all the Eulerian elements of Λ of genus one, and let λ be an element of Ξ_1. Let F_λ be the hyperplane in F defined by the equation

$$ \mathrm{F}_\lambda := \left\{ x \in \mathrm{F} \mid 2\langle x, x^\lambda \rangle = q(\lambda) \right\}. $$

More generally, for an arbitrary lattice point $\mu \in \Lambda$ and an Eulerian $\lambda \in \Lambda$ of genus one, let $\mathrm{F}_{\mu,\lambda}$ be the affine hyperplane $\mu + \mathrm{F}_\lambda$. By the results of the previous section, the Voronoi diagram Vor_q is the hyperplane arrangement of all the hyperplanes $\mathrm{F}_{\mu,\lambda}$ for $\mu \in \Lambda$ and $\lambda \in \Xi_1$. In addition, the Voronoi cell V_O (and more generally V_μ) is the cell containing O (resp. μ) in the hyperplane arrangement of the hyperplanes F_λ (resp. $\mathrm{F}_{\mu,\lambda}$) for $\lambda \in \Xi_1$.

2.2. Definition Of The Map ϕ From $\mathcal{FP}$ To $\mathcal{SC}$. — Each face F in $\mathcal{FP}$ is obtained as an intersection of some of the hyperplanes F_λ for $\lambda \in \Xi_1$.

Lemma 5. — *Let $F \in \mathcal{FP}$ be a face of V_O, and let λ and μ be two different elements of Ξ_1 such that $F \subset \mathrm{F}_\lambda \cap \mathrm{F}_\mu$. Then $\mathrm{supp}(\lambda)$ and $\mathrm{supp}(\mu)$ are consistent in their orientations: i.e., there is no oriented edge $e \in \mathbb{E}$ such that $e \in \mathrm{supp}(\lambda)$ and $\bar{e} \in \mathrm{supp}(\mu)$.*

Proof. — Let $C_\lambda = \mathrm{supp}(\lambda)$ and $C_\mu = \mathrm{supp}(\mu)$, and for the sake of a contradiction, suppose there exists an oriented edge $e \in \mathbb{E}$ with $e \in C_\lambda$ and $\bar{e} \in C_\mu$. We show that for every point x in $V_O \cap \mathrm{F}_\lambda \cap \mathrm{F}_\mu$, there exists an element $\gamma \in \Xi_1$ which is strictly closer to x. This will show that the intersection $V_O \cap \mathrm{F}_\lambda \cap \mathrm{F}_\mu$ is empty, which will be a contradiction, and thus, the lemma follows.

Let $x \in V_O \cap \mathrm{F}_\lambda \cap H_\mu$. We have

$$2\langle x, \lambda \rangle = q(\lambda) = \ell(C_\lambda) \qquad \text{and} \qquad 2\langle x, \mu \rangle = q(\mu) = \ell(C_\mu).$$

Thus,

$$\tag{1} 2\langle x, \lambda + \mu \rangle = \ell(C_\lambda) + \ell(C_\mu).$$

Let $D = \mathrm{supp}(\lambda + \mu)$. Observe that neither e nor $\bar{e}$ belong to D and so the sum $\sum_{e \in D} \lambda_e + \mu_e$ is strictly less than $\ell(C_\lambda) + \ell(C_\mu)$. By (the proof of) Proposition 4, $\lambda + \mu$ can be written as a sum $\gamma_1 + \cdots + \gamma_k$ such that $\gamma_i \in \Xi_1$ and $\mathrm{supp}(\gamma_i) \subset D$. This shows that

$$\tag{2} \sum_i \ell(\mathrm{supp}(\gamma_i)) = \sum_{e \in D} \lambda_e + \mu_e < \ell(C_\lambda) + \ell(C_\mu).$$

If for all i, $2\langle x, \gamma_i \rangle \leq q(\gamma_i) = \ell(\mathrm{supp}(\gamma_i))$, then by summing up all these inequalities and applying Inequality 2, we should have

$$\tag{3} 2\langle x, \lambda + \mu \rangle = \sum_i 2\langle x, \gamma_i \rangle \leq \sum_i \ell(\mathrm{supp}(\gamma_i)) < \ell(C_\lambda) + \ell(C_\mu),$$

which is impossible by Equation 1. Thus, there exists an i such that $2\langle x, \gamma_i \rangle > q(\gamma_i)$. And clearly, in this case $q(x) > q(x - \gamma_i)$, contradicting the assumption that $x \in V_O$. $\qquad \square$

Definition of the injective map ϕ from $\mathcal{FP}$ to $\mathcal{SC}$.— Applying Lemma 5, we now construct a map $\phi : \mathcal{FP} \to \mathcal{SC}$ as follows. Let $F \in \mathcal{FP}$, and consider the family $\mathcal{U}_F = \{\lambda \in \Xi_1 \mid F \subset \mathrm{F}_\lambda\}$. Define $\phi(F) = \bigcup_{\lambda \in \mathcal{U}_F} \mathrm{supp}(\lambda)$. By Lemma 5, two different λ and μ are consistent in their orientations, so the union in the definition of $\phi(F)$ defines a well-defined orientation of a subgraph of G. In addition, it is quite straightforward to check that $\phi(F)$ is a strongly connected orientation of that subgraph, i.e., $\phi(F)$ lies in $\mathcal{SC}$.

Lemma 6. — *Let F be a face in $\mathcal{FP}$, and $\mathcal{U}_F$ and $\phi(F)$ be defined as above. For any $\lambda \in \Xi_1$ with $\mathrm{supp}(\lambda) \subset \phi(F)$, we have $\lambda \in \mathcal{U}_F$, i.e., $F \subset \mathrm{F}_\lambda$. More precisely, $\lambda \in \mathcal{U}_F$ if and only if $\mathrm{supp}(\lambda) \subset \phi(F)$.*

Proof. — Define the flow $\mu = \sum_{\gamma \in \mathcal{U}_F} \gamma$. By applying Lemma 5 and the definition of $\mathcal{U}_F$, we have

$$\tag{4} \forall x \in F \qquad 2\langle x, \mu \rangle = \sum_{\gamma \in \mathcal{U}_F} \ell(\mathrm{supp}(\gamma)),$$

and since $F \subset V_O$, we have

$$\tag{5} \forall x \in F, \gamma \in \Xi_1, \qquad 2\langle x, \gamma \rangle \leq q(\gamma) = \ell(\mathrm{supp}(\gamma)).$$

As the prove of Proposition 4 and Lemma 3 shows, since $\mathrm{supp}(\lambda) \subset \mathrm{supp}(\mu) = \phi(F)$, there exists another decomposition of $\mu = \sum_{i=}^k \gamma_i'$ with $\gamma_1' = \lambda$ and such that $\mathrm{supp}(\gamma_i') \subset \mathrm{supp}(\mu)$. This shows that $\sum_{i=1}^k \ell(\mathrm{supp}(\gamma_i')) = \sum_{\gamma \in \mathcal{U}_F} \ell(\mathrm{supp}(\gamma))$. Equation 4 and Inequalities 5 for γ_i' then imply that for all $i \in \{1, \ldots, k\}$, $2\langle x, \gamma_i' \rangle = q(\gamma_i')$, i.e., for all i, $\gamma_i' \in \mathcal{U}_F$, and in particular $\lambda = \gamma_1' \in \mathcal{U}_F$. The second part of the lemma follows from the first part and the definition of $\mathcal{U}_F$. $\qquad \square$

Lemma 7. — *The map ϕ is injective and order-preserving.*

Proof. — Injectivity follows easily from the previous lemma: if for two faces F_1 and F_2 we have $\phi(F_1) = \phi(F_2)$, then $\mathcal{U}_{F_1} = \mathcal{U}_{F_2}$. Since V_O is the cell containing the origin in the arrangement of the hyperplanes F_λ for $\lambda \in \Xi_1$, this shows that $F_1 = F_2$.

To show that the map ϕ is order preserving, note that if $F_1 \subseteq F_2$ for two faces in $\mathcal{FP}$, then $\mathcal{U}_{F_2} \subseteq \mathcal{U}_{F_1}$, thus, $\phi(F_2) = \bigcup_{\lambda \in \mathcal{U}_{F_2}} \mathrm{supp}(\lambda) \subseteq \bigcup_{\lambda \in \mathcal{U}_{F_1}} \mathrm{supp}(\lambda) = \phi(F_1)$ and so by the definition of the partial order in $\mathcal{SC}$, we have $\phi(F_1) \preceq \phi(F_2)$. $\qquad\square$

2.3. Explicit Description of the Faces in $\mathcal{FP}$. — We now show that the map ϕ defined in the previous section is surjective.

Let D be a strongly connected orientation of a subgraph H of G. Let F_D be the affine plane defined as the intersection of all the affine hyperplanes F_λ for $\lambda \in \Xi_1$ and $\mathrm{supp}(\lambda) \subseteq D$. From the analysis we did before, we know that $\mathrm{codim}(\mathrm{F}_D) = g_D$, the genus of H. We show that there is a face F of $\mathcal{FP}$ of codimension g_D which is included in F_D and $\phi(\mathcal{FP}) = D$. This proves the surjectivity of ϕ and Theorem 1 follows.

In what follows, by the abuse of the notation, by a strongly connected orientation of the whole graph we simply mean a strongly connected orientation of the graph minus the set of bridges. (Thus, if G does not have any bridge, i.e., is 2-edge connected, the orientation is indeed defined on the whole graph.)

Lemma 8. — *If D is a strongly connected orientation of the whole graph G, then F_D consists of precisely a vertex of V_O, denoted by v^D. All the vertices of V_O are of this form.*

Proof. — We already know that F_D is of codimension $g_D = g$. In other words, $\mathrm{F}_D = \{v^D\}$ for a point $v^D \in \mathrm{F}$. We now show that v^D is a vertex of V_O. To this end, we have to show that for all $\lambda \in \Xi_1$, $q(v^D) \leq q(v^D - \lambda)$, or equivalently $2\langle v^D, \lambda\rangle \leq q(\lambda) = \ell(\mathrm{supp}(\lambda))$. Let $\lambda \in \Xi_1$. Since D is a strongly connected orientation of D, the circuits of D form a basis for Λ. There exists thus a set of coefficients a_μ for $\mu \in \Xi_1$ with $\mathrm{supp}(\mu) \subset D$ such that

$$\lambda = \sum_{\mu \in \Xi_1 : \mathrm{supp}(\mu) \subset D} a_\mu\, \mu.$$

We have

$$2\langle v^D, \lambda\rangle = \sum_{\mu \in \Xi_1 : \mathrm{supp}(\mu) \subset D} 2a_\mu \langle v^D, \mu\rangle$$

$$= \sum_{\mu \in \Xi_1 : \mathrm{supp}(\mu) \subset D} a_\mu \ell(\mathrm{supp}(\mu))$$

$$= \sum_{e \in D}\Big(\sum_{\mu \in \Xi_1 : e \in \mathrm{supp}(\mu)\,\&\,\mathrm{supp}(\mu) \subset D} a_\mu\Big) = \sum_{e \in D} \lambda_e$$

$$\leq \sum_{e \in D} |\lambda_e| = \ell(\mathrm{supp}(\lambda)) = q(\lambda).$$

In addition, the last inequality is strict if and only if $\mathrm{supp}(\lambda) \not\subset D$. We infer that $v^D \in V_O$ and it is a vertex of V_O.

To show that a vertex v of V_O is of this form, consider the image of the face $\{v\}$ by ϕ. This is a strongly connected orientation of a subgraph H of G, and since the intersection

$\cap_{\lambda \in \mathcal{U}_{\{v\}}} H_\lambda$ is of codimenion g, the genus of H should be equal to that of G, i.e., $H = G$ and thus, $v = v^{\phi(\{v\})}$ and the lemma follows. $\qquad\square$

Consider now the general case, D being a strongly connected orientation of a subgraph H of D. Let $\mathcal{S}_D$ be the family of all the extensions of the orientation of H to a strongly connected orientation of the whole graph G, i.e., the family of all the strongly connected orientations D' of G such that $D \subseteq D'$. For each element $D' \in \mathcal{S}_D$, let $v^{D'}$ the vertex of V_O given by the previous lemma. By the convexity of V_O and F_D, we have

$$\text{conv-hull}_{D' \in \mathcal{S}_D} \langle v^{D'} \rangle \subset V_O \cap \mathrm{F}_D.$$

Lemma 9. — *The convex-hull of $v^{D'}$ for $D' \in \mathcal{S}_D$ defines a face F of V_O and we have $\phi(F) = D$. Thus, the map ϕ is surjective.*

Proof. — Let $F = \text{conv-hull}_{D' \in \mathcal{S}_D} \langle v^{D'} \rangle$. We first show that F is a face of V_O, the second statement $\phi(F) = D$ easily follows. By the previous lemma, any vertex v of V_O which lies in F_D is of the form $v^{D'}$ for a strongly connected orientation D' of G. Since $\mathcal{U}_D \subset \mathcal{U}_{D'}$, and D and D' are strongly connected orientation, we infer that $D \subseteq D'$, i.e., $D' \in \mathcal{S}_D$, and so all the vertices of V_O which lie in F_D are of the form $v^{D'}$ for $D' \in \mathcal{S}_D$. It follows that F is a face of V_O. To show that $\phi(F) = D$, note that for any $\lambda \in \Xi_1$, by the proof of the previous lemma, we have if $\text{supp}(\lambda) \not\subset D$, then for every $D' \in \mathcal{S}_D$, $2\langle v^{D'}, \lambda \rangle < \ell(\text{supp}(\lambda))$, i.e., $\lambda \notin \mathcal{U}_F$. And clearly if $\text{supp}(\lambda) \subset D$, then for every $D' \in \mathcal{S}_D$, $2\langle v^{D'}, \lambda \rangle = \ell(\text{supp}(\lambda))$, and since $F = \text{conv-hull}_{D' \in \mathcal{S}_D} \langle v^{D'} \rangle$, we have $2\langle x, \lambda \rangle = \ell(\text{supp}(\lambda))$ for every $x \in F$, i.e., $\lambda \in \mathcal{U}_F$. We conclude that $\phi(F) = D$, and the lemma follows. $\qquad\square$

The proof of Theorem 1 is now complete. Note that, as the proof shows, the codimension of a face $F \in \mathcal{FP}$ is equal to the genus of $\phi(F)$.

3. Quotient Posets $\overline{\mathcal{FP}}$ and $\overline{\mathcal{SC}}$

The Voronoi diagram Vor_q is naturally equipped with the action of Λ obtained by translation. (More precisely this is defined as follows: for an element $\lambda \in \Lambda$, and a face F of a Voronoi cell V_μ, $\mu.F := \mu + F$ is the corresponding face of the Voronoi cell $V_{\lambda+\mu}$ under the natural isomorphism obtained by translation). The poset $\overline{\mathcal{FP}}$ is defined as the quotient of Vor_q under this action. Another way to define $\overline{\mathcal{FP}}$ is to start from $\mathcal{FP}$ and identify those faces of $\mathcal{FP}$ which differ by a translation by an element of Λ.

There is also a quotient poset $\overline{\mathcal{SC}}$ associated to $\mathcal{SC}$ obtained roughly by identifying those strongly connected orientations of subgraphs of G which differ by reversing the orientation of an Eulerian orientation of an Eulerian subgraph of G. More precisely, $\overline{\mathcal{SC}}$ is defined as follows.

Let $D \in \mathcal{SC}$ be a strongly connected orientation of a subgraph H of G. The outdegree function $d_D^+ : V \to \mathbb{N}$ is define by sending a vertex v of G to the number of oriented edges emanating from v in the orientated graph D, i.e., the number of arcs e in D whose tail is v. The equivalence relation $\sim$ is defined on $\mathcal{SC}$ as follows. For two strongly connected orientations D_1 of a subgraph H_1 and D_2 of a subgraph H_2 of G, we set $D_1 \sim D_2$ if an only if

  – $H_1 = H_2$, and
  – the two corresponding out-degree functions are equal, i.e., $d_{D_1}^+ = d_{D_2}^+$.

Lemma 10. — *Two strongly connected orientations D_1 and D_2 of a subgraph H of G are equivalent if and only there exists an Eulerian subgraph H_0 of H and an Eulerian orientation D_0 of H_0 such that $D_0 \subset D_1$ and D_2 is the orientation obtained by replacing each edge e in D_0 by $\bar{e}$, i.e., $\mathbb{E}(D_2) = \Big(\mathbb{E}(D_1) \setminus \mathbb{E}(D_0)\Big) \cup \{\bar{e} \,|\, e \in \mathbb{E}(D_0)\}$.*

Proof. — The proof is quite easy: consider the set of all oriented edges $e \in \mathbb{E}(D_1)$ such that $\bar{e} \in D_2$. This defines an Eulerian oriented graph D_0 and lemma easily follows. $\square$

The poset $\overline{\mathcal{SC}}$ is the quotient of $\mathcal{SC}$ obtained by identifying equivalent elements of $\mathcal{SC}$, in other words $\overline{S}C = \mathcal{SC}/\sim$. Note that this is indeed a poset: if $D_1 \preceq D_1'$, and $D_1' \sim D_2'$, then by the definition of the partial order $\preceq$ and the lemma above, there exists an Eulerian orientation D_0' of a subgraph of G such that $D_0' \subseteq D_1' \subseteq D_1$ and D_1' is obtained from D_1 by reversing the orientation of the oriented edges of D_0'. Now reversing the orientation of the oriented edges of D_0' in D_1, one obtained an element $D_2 \in \mathcal{SC}$ such that $D_2 \sim D_1$ and $D_2 \preceq D_2'$. So the quotient partial order $\preceq$ is well-defined.

Theorem 11. — *The two posets $\overline{\mathcal{SC}}$ and $\overline{\mathcal{FP}}$ are isomorphic.*

Proof. — The proof follows from the proof of our Theorem 1 and Lemma 2. Namely, consider the bijective map ϕ defined in the previous section. Let F_1 and F_2 be two faces in $\mathcal{FP}$ which are identified in $\mathcal{FP}$, i.e., there exists an element $\lambda \in \Lambda$ such that $F_2 = F_1 + \lambda$. Since F_2 is also a face of V_λ, we infer that $V_\lambda \cap V_O \neq \emptyset$, and by Lemma 2, λ is Eulerian. Let $D_0 = \text{supp}(\lambda)$, note that D is Eulerian. We show that $D_0 \subset \phi(F_2)$ and $\phi(F_1)$ is obtained by reversing the orientation of D_0 in $\phi(F_2)$, from this we obtain a well-defined quotient map $\overline{\phi}$, which becomes automatically bijective, and the theorem follows.

By the definition of ϕ, we have to show that for every $\mu \in \Xi_1$ with $\text{supp}(\mu)$ an oriented cycle in D_0, we have $\mu \in \mathcal{U}_{F_2}$. Let $\mu \in \Xi_1$ be such that $\text{supp}(\mu) \subset D_0$. Since F_1 is a face of V_O, for all $x \in F_1$, we have $-2\langle x, \mu\rangle = 2\langle x, -\mu\rangle \leq q(-\mu) = q(\mu) = \ell(\text{supp}(\mu))$. It follows that

$$2\langle x + \lambda, \mu\rangle = 2\langle x, \mu\rangle + 2\ell(\text{supp}(\mu)) \qquad \text{since supp}(\mu) \subseteq \text{supp}(\lambda)$$
$$\geq \ell(\text{supp}(\mu)) = q(\mu).$$

Since $x + \lambda \in F_2$, and F_2 is a face of V_O, the above inequality is indeed an equality for all $x \in F_1$, or equivalently for all $x + \lambda \in F_2$, i.e., for all $y \in F_2$, $2\langle y, \mu\rangle = q(\text{supp}(\mu))$. Thus, $\mu \in \mathcal{U}_{F_2}$. We infer that $D_0 = \text{supp}(\lambda) \subset \phi(F_2)$.

Let $D_1 = \phi(F_1)$, $D_2 = \phi(F_2)$ and D_1' the element of $\mathcal{SC}$ obtained from D_2 by reversing the orientation of every oriented edge in $D_0 \subset D_2$. We aim to show that $D_1' = D_1$.

In the proof of Theorem 1, we show that F_2 (resp. F_1) is the convex hull of all the vertices v^D of the Voronoi cell V_0 for $D \in \mathcal{S}_{D_2}$ (resp. $D \in \mathcal{S}_{D_1}$), i.e., for D being a strongly connected orientation of G which contains D_2 (resp. D_1) as an oriented subgraph. On the other hand, since $F_1 = F_2 + \lambda$, we have conv-hull$_{D \in \mathcal{S}_{D_2}}(v^D + \lambda) = F_1$. We now show that the set $\{v^D + \lambda\}$ coincides with $\mathcal{S}_{D_1'}$, i.e., with the set of all the strongly connected orientations of G which contains D_1' as an oriented subgraph. From this, and the proof of Theorem 1, it easily follows that $D_1' = D_1$.

Recall that v^D has the property that for every $\mu \in \Xi_1$ with $\text{supp}(\mu) \subset D$,

$$2\langle v^D, \mu\rangle = \ell(\text{supp}(\mu)).$$

Since D is in $\mathcal{SC}$, the set of μ's with $\mathrm{supp}(\mu) \subset D$ generate Λ. And since the above equations are all linear, we easily have for all $\mu \in \Xi_1$ (not only μ with $\mathrm{supp}(\mu) \subset D$),

$$2\langle v^D, \mu \rangle = \ell^+(\mathrm{supp}(\mu)) - \ell^-(\mathrm{supp}(\mu)),$$

where $\ell^+(\mathrm{supp}(\mu))$ (resp. $\ell^-(\mathrm{supp}(\mu))$) is the number of edges e of $\mathrm{supp}(\mu)$ with $e \in \mathbb{E}(D)$ (resp. with $\bar{e} \in \mathbb{E}(D)$). From this, we have for all $\mu \in \Xi_1$,

$$(6) \qquad 2\langle v^D + \lambda, \mu \rangle = \ell^+(\mathrm{supp}(\mu)) - \ell^-(\mathrm{supp}(\mu)) + 2\langle \lambda, \mu \rangle$$

We claim that $v^D + \lambda$ is equal to $v^{D'}$ for the orientation D' obtained by reversing the orientation of D_0 in D (note that $D_0 \subset \phi(F_2) = D_2 \subset D$). For this, we have to show that for all $\mu \in \Xi_1$ with $\mathrm{supp}(\mu) \subset D'$, we have $2\langle v^D + \lambda, \mu \rangle = \ell(\mathrm{supp}(\mu))$. But this easily follows from the Equation 6 above. Indeed, for such a μ we have

$$\begin{aligned}
2\langle v^D + \lambda, \mu \rangle &= \ell^+(\mathrm{supp}(\mu)) - \ell^-(\mathrm{supp}(\mu)) + 2\langle \lambda, \mu \rangle \\
&= \ell^+(\mathrm{supp}(\mu)) - \ell^-(\mathrm{supp}(\mu)) + 2\ell^-(\mathrm{supp}(\mu)) \\
&= \ell(\mathrm{supp}(\mu)).
\end{aligned}$$

The proof of the theorem is now complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4. Concluding Remarks

As we said, the results of the previous sections on the lattice of integer flows extend directly to the weighted case, i.e., to the case of metric graphs.

Covering Number and Well-(Un)Balanced Orientations. — Let $\mathcal{L}$ be a lattice of full dimension in a real vector space $\mathcal{E}$ equiped with a positive quadratic form q. The *covering number* of $\mathcal{L}$ denoted by $\mathrm{Cov}(\mathcal{L})$ is the smallest real number r such that the closed balls of radius r centered at the lattice points cover $\mathcal{E}$, i.e., such that for every point $x \in \mathcal{E}$, there is an element $m \in \mathcal{L}$ such that $q(x - m) \leq r$.

By the results of the previous section, it is possible to give an explicit description of the covering number of Λ in terms of certain orientations of the graph G (that we will call balanced) as follows.

Recall that all the vertices of the Voronoi cell of the origin, V_O, are of the form v^D for a strongly connected orientation D of the whole graph G. In addition, v^D is the flow uniquely determined by the equations

$$2\langle v^D, x^C \rangle = \ell(C),$$

for all circuits $C \subset D$.

Proposition 12. — *The covering number of Λ, $\mathrm{Cov}(\Lambda)$, is the maximum of $q(v^D)$ for D a strongly connected orientation of the whole graph G.*

Thus, it will be enough to find a strongly connected orientation D of G such that $q(v^D)$ is maximized.

Let Δ be the Laplcaian operator. We recall that for a function $f : V \to \mathbb{R}$, $\Delta(f)$ is the function on V taking on $y \in V$ the value $\Delta(f)(y) := \sum_{z:\, \{z,y\} \in E(G)} \big(f(y) - f(z)\big)$.

Let $\mathcal{S}$ be the family of all the strongly connected orientations of the whole graph. (Recall that if G contains bridges, by our convention as before, we mean all the strongly connected

orientation of G minus the set of bridges of G.) Let D be a fixed element of $\mathcal{S}$, and for a vertex $y \in V$, let d_y^{in} and d_y^{out} be the in- and the out-degree of y in D respectively. Define the function $\mu : \mathbb{E}(D) \to \mathbb{R}$ on the set of arcs of D as follows: for all $e \in \mathbb{E}(D)$, $\mu_e = 2v_e^D - 1$. Then we have

$$\text{For all circuit } C \subset D, \quad \langle \mu, x^C \rangle = 0.$$

(In other words, μ is a tension on D.)

It follows that there exists a function $f = f_D : V \to \mathbb{R}$ such that for all arc $e = (y, z) \in \mathbb{E}(D)$, $\mu_e = f_D(z) - f_D(y)$. (In other words, there exists a potential function for μ.)

In addition, since v^D is a flow, the function f should satisfy the following set of equations:

$$\forall y \in V, \quad \sum_{z:\,(z,y)\in\mathbb{E}(D)} \big(f(y) - f(z) + 1\big) \;=\; \sum_{z:\,(y,z)\in\mathbb{E}(D)} \big(f(z) - f(y) + 1\big).$$

Or equivalenty,

$$\forall y \in V, \ \Delta(f)(y) = d_y^{\text{out}} - d_y^{\text{in}}.$$

In other words, f is the solution to the Laplacian equation $\Delta(f) = d^{\text{out}} - d^{\text{in}}$. We now show how to describe $q(v^D)$ in terms of f. First we need the following lemma.

Lemma 13. — *We have $2\,q(v^D) = |v^D|_{\ell_1} = \sum_{e\in\mathbb{E}(D)} v_e^D$.*

Proof. — We already know that $\text{supp}(v^D) = D$. Write v^D as a positive combination of the flows x^C for $C \subset D$, $v^D = \sum_{C\subset D} \alpha_C x^C$. We have

$$2\,q(v^D) = \sum_{C\subset D} 2\alpha_C \langle v^D, x^C \rangle$$

$$= \sum_{C} \alpha_C \ell(C) = \sum_{e\in\mathbb{E}(D)} \sum_{C:\,e\in C} \alpha_C$$

$$= \sum_{e\in\mathbb{E}(D)} v_e^D = |v^D|_{\ell_1}.$$

$\square$

Since for $e = (y, z) \in \mathbb{E}(D)$, we have $v_e^D = f(z) - f(y) + 1$, it follows that

$$2q(v^D) = \sum_{(y,z)\in\mathbb{E}(D)} \big(f(z) - f(y) + 1\big)$$

$$= |\mathbb{E}(D)| + \sum_{y\in V} (d_y^{\text{in}} - d_y^{\text{out}})f(y)$$

$$= |\mathbb{E}(D)| - \sum_{y\in V} f(y)\Delta(f)(y) = |\mathbb{E}(D)| - \langle f, \Delta(f) \rangle.$$

In addition, $\mathbb{E}(D)$ is the number of the non-bridge edges of G. Thus, we have

Definition-Theorem 14. — A strongly connected orientation D of G for which the quantity $\langle f_D, \Delta(f_D) \rangle$ is minimized (for f_D satisfying the Laplacian equation $\Delta(f_D) = d^{\text{out}} - d^{\text{in}}$) is called well-balanced.

Let ϵ be the number of the non-bridge edges of G. Let D be a well-balanced orientation of G. The covering number of Λ is given by

$$\mathrm{Cov}(\Lambda) = \frac{\epsilon}{2} - \frac{1}{2}\langle f_D, \Delta(f_D)\rangle.$$

In particular if G is Eulerian, then $\mathrm{Cov}(\Lambda) = \frac{|E(G)|}{2}$.

Proof. — It only remains to prove the last claim: if G is Eulerian, then $\epsilon = |E(G)|$, and there exists an orientation D of G such that $d_y^{\mathrm{in}} = d_y^{\mathrm{out}}$ for all $y \in V$. Clearly D is strongly connected and we have $\Delta(f_D) = 0$, and so $\langle f_D, \Delta(f_D)\rangle = 0$, i.e., D is well-balanced! The claim follows. $\square$

Thus, we are left with the following optimization problem.

Find a strongly connected orientation D of G such that for f satisfying $\Delta(f) = d^{\mathrm{out}} - d^{\mathrm{in}}$, the quantity $\sum_{y \in V} f(y)(d_y^{\mathrm{out}} - d_y^{\mathrm{in}})$ is minimized.

We do not know the complexity of the above problem in general, however, we suspect it to be $\mathscr{NP}$−hard.

Remark 15. — Suppose for simplicity that G is two-edge connected. Let $c(y) := d_y^{\mathrm{in}} - d_y^{\mathrm{out}}$ for all $y \in V$. Then the following properties are satisfied

(i) $\sum_y c(y) = 0$;

(ii) for all $y \in Y$, $\quad c(y) \equiv \deg(y) \pmod 2$; and

(iii) for all non-empty $X \subsetneq V$, $c(X) = \sum_{y \in X} c(y) < \delta(X)$, where $\delta(X)$ is the number of edges with one end-point in X and one end-point in $V \setminus X$.

It is not hard to show that the converse is also true: a function $c : V \to \mathbb{Z}$ is of the form $d^{\mathrm{in}} - d^{\mathrm{out}}$ for a strongly connected orientation D of G if and only if it satisfies the properties $(i), (ii)$, and (iii) above (and in this case, it is possible to find the orientation in polynomial time). Thus, while calculating the covering number, we are left with the optimization problem of finding the minimum of $\sum_{y \in V} f(y)c(y)$ over all the functions c with the above three set of properties and for f satisfying the Laplacian equation $\Delta(f) = c$.

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