MODULI OF HYBRID CURVES AND VARIATIONS OF CANONICAL MEASURES

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Abstract. We introduce the *moduli space of hybrid curves* as the hybrid compactification of the moduli space of curves thereby refining the one obtained by Deligne and Mumford. As the main theorem of this paper we then show that the universal family of *canonically measured* hybrid curves over this moduli space varies continuously.

On the way to achieve this, we present constructions and results which we hope could be of independent interest. In particular, we introduce *higher rank* variants of hybrid spaces which refine and combine both the ones considered by Berkovich, Boucksom and Jonsson, and metrized complexes of varieties studied by Baker and the first named author. Furthermore, we introduce canonical measures on hybrid curves which simultaneously generalize the Arakelov-Bergman measure on Riemann surfaces, Zhang measure on metric graphs, and Arakelov-Zhang measure on metrized curve complexes.

This paper is part of our attempt to understand the precise link between the non-Archimedean Zhang measure and variations of Arakelov-Bergman measures in families of Riemann surfaces, answering a question which has been open since the pioneering work of Zhang on admissible pairing in the nineties.

Contents

1. Introduction

In this paper we study canonical measures on Riemann surfaces and their *tropical* and *hybrid* limits.

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To Maryam Mirzakhani, with admiration for her sense of beauty in mathematics.

By canonical measure on a Riemann surface we mean the Arakelov-Bergman measure μ_{Ar} defined by the use of holomorphic one-forms. For a compact Riemann surface S of positive genus g, this is the positive density measure of total mass g on S given by

$$\mu_{\mathrm{Ar}} := \frac{i}{2}\sum_{j=1}^{g} \omega_j \wedge \bar{\omega}_j,$$

where $\omega_1, \dots, \omega_g$ form an orthonormal basis for the space of holomorphic one-forms on S, with respect to the hermitian inner product

$$\langle \eta_1, \eta_2 \rangle := \frac{i}{2}\int_S \eta_1 \wedge \bar{\eta}_2$$

for pairs of holomorphic one-forms η_1, η_2 on S.

Let $\mathscr{M}_g$ be the moduli space of curves of genus g and denote by $\overline{\mathscr{M}}_g$ its Deligne-Mumford compactification, consisting of stable curves of genus g. The question we address in this paper is the following:

Question 1.1. *Consider a sequence of smooth compact Riemann surfaces $S_1, S_2, \dots$ of genus g such that the corresponding points $s_1, s_2, \dots$ of $\mathscr{M}_g$ converge to a point in $\overline{\mathscr{M}}_g$. For each j, let μ_j^{can} be the canonical measure of S_j.*

What is the limit of the sequence of measures $\mu_1^{can}, \mu_2^{can}, \dots$?

Note that the measures $\mu_1^{can}, \mu_2^{can}, \dots$ do not live on the same space, so the first problem to handle consists in giving a precise mathematical meaning to the question. Moreover, once this has been taken care off, the answer to the question appears to be sensitive to the *speed* and *direction* of the convergence of the sequence.

Formalizing these points lead to the definition of hybrid curves and their moduli spaces studied in this paper.

1.1. **Hybrid curves.** Suppose s_∞ is the limit of the sequence of points $s_1, s_2, \dots$, and let S_∞ be the stable Riemann surface of genus g which corresponds to s_∞ in $\overline{\mathscr{M}}_g$. Denote by $G = (V, E)$ the dual graph of S_∞. Recall that the vertices of G are in bijection with the irreducible components of S_∞ and the edges are in bijection with the nodes (singular points) of S_∞. These bijections are also compatible, meaning that for a node lying on two components of S_∞, the vertices of the corresponding edge in G correspond to these two components. Moreover, if we keep the information of the genera of components of S_∞, we get a *genus function* $\mathfrak{g} : V \to \mathbb{N} \cup \{0\}$, and the pair $(G, \mathfrak{g})$ is called a *stable graph.*

With the help of the stable dual graph, it will be possible to capture additional information about the speed and direction of convergence of the sequence, in a way relevant to the question of interest to us.

We can view the speed of convergence as a way to distinguish the *relative order of appearance of the singular points* in the limit when the sequence $S_1, S_2, \dots$ approach the stable Riemann surface S_∞. This leads to an ordered partition $\pi = (\pi_1, \dots, \pi_r)$ of the edge set E, where π_1 are the edges corresponding to the *fastest appearing nodes*, π_2 are the ones which are *fastest among the remaining nodes*, and so on. We will call the ordered partition π a *layering* of the graph, the elements of π the *layers*, and the graph G endowed with the layering a *layered graph.*

Given the layering, capturing the direction will now correspond to the choice of edge length functions $\ell_1 : \pi_1 \to \mathbb{R}_+, \dots, \ell_r : \pi_r \to \mathbb{R}_+$, each of them well-defined up to multiplication by a positive scalar. Here and everywhere else, we denote by $\mathbb{R}_+$ the set of strictly positive real numbers.

We have now arrived naturally at the data underlying the definition of a hybrid curve:

- a stable curve S with stable dual graph $G = (V, E, \mathfrak{g})$ and
- an ordered partition $\pi = (\pi_1, \dots, \pi_r)$ of E, for some $r \in \mathbb{N}$, and
- an edge length function $\ell : E \to \mathbb{R}_+$. Moreover, ℓ is well defined up to multiplying its restrictions $\ell^j = \ell|_{\pi_j}$ by arbitrary positive real numbers.
  (Imposing $\sum_{e \in \pi_j} \ell^j(e) = 1$ for each $j = 1, \dots, r$ leads to a unique choice of ℓ.)

Given the triple (S, ℓ, π), we then define the associated *hybrid curve* $\mathscr{C}^{hyb}$ as the metrized complex $\mathcal{M}\mathscr{C}$ from [AB15], obtained by the *metric realization* of (S, ℓ), enriched with the data of the layering π on the edge set of $\mathcal{M}\mathscr{C}$.

We recall that $\mathcal{M}\mathscr{C}$ is obtained by taking first the normalization $\widetilde{S}$ of S, which is by definition the disjoint union of the (normalization of the) irreducible components of S (these are smooth compact Riemann surfaces), then taking an interval $\mathcal{I}_e$ of length ℓ_e for each edge $e \in E$, and finally gluing for each e the two extremities of $\mathcal{I}_e$ to the two points in $\tilde{S}$ corresponding to the node in S associated with e.

It might be helpful to remark that we can view each metrized complex $\mathcal{M}\mathscr{C}$ as a hybrid curve $\mathscr{C}^{hyb}$ endowed with the *trivial layering*. Here we mean the trivial ordered partition consisting of a unique element $\pi = (E)$. In this way, we see that hybrid curves provide an enrichment of the category of metrized curve complexes.

1.2. **Moduli space $\mathscr{M}_g^{hyb}$ of hybrid curves of genus g.** In order to give a meaning to Question 1.1, we will construct the *moduli space $\mathscr{M}_g^{hyb}$ of hybrid curves of genus g*, which provides a compactification of $\mathscr{M}_g$ naturally lying over the Deligne-Mumford compactification $\overline{\mathscr{M}}_g$. In a sense, $\mathscr{M}_g^{hyb}$ provides a *tropical refinement* of $\overline{\mathscr{M}}_g$ and allows to study problems involving both Archimedean and tropical non-Archimedean aspects of Riemann surfaces and their families. The precise meaning of this will be explained in our future work. The points in $\mathscr{M}_g^{hyb}$ correspond to *hybrid limits of Riemann surfaces of genus g* and the topology reflects aspects which are both reminiscent of Euclidean and Zariski topologies in $\overline{\mathscr{M}}_g$. In some sense, in our construction we replace each point of $\overline{\mathscr{M}}_g$ corresponding to a non-smooth stable Riemann surface S by infinitely many hybrid points corresponding to the hybrid curves with S as their underlying stable Riemann surface. We refer to Section 4.4 for more information on the construction.

The hybrid moduli space $\mathscr{M}_g^{hyb}$ naturally comes with the *universal family of hybrid curves of genus g*

$$\mathscr{C}_g^{hyb} \to \mathscr{M}_g^{hyb}$$

whose fiber $\mathscr{C}_{\mathbf{t}}^{hyb}$ over any point $\mathbf{t} \in \mathscr{M}_g^{hyb}$ corresponds to the hybrid curve represented by $\mathbf{t}$ in the moduli space. The universal family $\mathscr{C}_g^{hyb}$ inherits as well a natural topology which makes the projection map $\mathscr{C}_g^{hyb} \to \mathscr{M}_g^{hyb}$ continuous. Moreover, we get the following commutative diagram of continuous maps

$$\begin{tikzcd}
\mathscr{C}_g^{hyb} \arrow[r] \arrow[d] & \mathscr{M}_g^{hyb} \arrow[d] \\
\mathscr{C}_g \arrow[r] & \overline{\mathscr{M}}_g
\end{tikzcd}$$

which shows the constructions are compatible with the classical picture.

1.3. **Hybrid canonical measures.** In order to identify all the possible limits in Question 1.1, we will define a *canonical measure* on hybrid curves. These measures will combine and generalize both Archimedean canonical measures associated to Riemann surfaces and non-Archimedean measures on metric graphs, which arise naturally in non-Archimedean Arakelov geometry, when working with analytification of curves over non-Archimedean fields and their skeleta.

The canonical measure on a metric graph is the one introduced by Zhang in his work on admissible pairing [Zha93], using harmonic analysis on metric graphs. Zhang's measure μ_{Zh} is a positively weighted combination of Lebesgue measures on edges of the metric graph, with weights carrying interesting information about the combinatorics of the graph and its spanning trees, see Section 5 for the precise definition and several equivalent formulations. Moreover, when it is viewed on the analytified curve, it carries essential arithmetic geometric information [Zha93, Zha10, Cin11, Ami14, dJ19].

Consider now a hybrid curve $\mathscr{C}^{hyb}$ defined as the metric realization of the pair (S, ℓ), of a stable curve S and an edge-length function ℓ, endowed with an ordered partition $\pi = (\pi_1, \dots, \pi_r)$ on the edges of the dual graph G of S. We define the *canonical measure* μ^{can} on $\mathscr{C}^{hyb}$ as the sum

$$\mu^{can} := \mu_{\mathrm{Ar}} + \mu_{\mathrm{Zh}}^1 + \cdots + \mu_{\mathrm{Zh}}^r,$$

where

(1) the measure μ_{Ar}, *the Archimedean part of the canonical measure*, restricts to the Arakelov-Bergman measure on each positive-genus component of S, regarded inside $\mathscr{C}^{hyb}$, and has trivial support elsewhere;

(2) the measure μ_{Zh}^j, *the j^{th} graded non-Archimedean part of the canonical measure*, has support in the union of the intervals $\mathcal{I}_e$ in $\mathscr{C}^{hyb}$ for edges $e \in \pi_j$, and on each such interval $\mathcal{I}_e$, it has the same restriction as the Zhang's measure of the metric graph obtained by removing all the edges which appear in lower layers $\pi_1, \dots, \pi_{j-1}$ and then contracting all the edges which appear in upper layers $\pi_{j+1}, \dots, \pi_r$. (We call these graphs *graded minors* of the hybrid curve, appealing to Robertson-Seymour theory of graph minors.)

Note that when $\mathscr{C}^{hyb}$ corresponds to a smooth curve, then $\mu^{can} = \mu_{\mathrm{Ar}}$. Moreover, in the case that $\mathscr{C}^{hyb}$ is a metrized curve complex $\mathcal{M}\mathscr{C}$ endowed with the trivial ordered partition $\pi = (E)$, the measure $\mu^{can} = \mu_{\mathrm{Ar}} + \mu_{\mathrm{Zh}}$ is the corresponding Arakelov-Zhang measure on the metrized complex $\mathcal{M}\mathscr{C}$, considered in [Ami14] as the measure underlying the equidistribution of Weierstrass points on $\mathcal{M}\mathscr{C}$.

1.4. **Continuity of the canonical measures over the hybrid moduli space.** The main theorem of this paper can be stated as follows.

Theorem 1.2. *The universal family of canonically measured hybrid curves* $(\mathscr{C}_g^{hyb}, \mu^{can})$ *forms a continuous family of measured spaces over the hybrid moduli space* $\mathscr{M}_g^{hyb}$.

Continuity in the above statement is defined in a distributional sense. More precisely, we mean that for any continuous function $f : \mathscr{C}_g^{hyb} \to \mathbb{R}$, the function $F : \mathscr{M}_g^{hyb} \to \mathbb{R}$ defined by *integration along fibers*

$$F(\mathbf{t}) := \int_{\mathscr{C}_{\mathbf{t}}^{hyb}} f|_{\mathscr{C}_{\mathbf{t}}^{hyb}} \, d\mu_{\mathbf{t}}^{can}, \qquad \mathbf{t} \in \mathscr{M}_g^{hyb}$$

is continuous on $\mathscr{M}_g^{hyb}$. Here $\mu_{\mathbf{t}}^{can}$ denotes the canonical measure on the hybrid curve $\mathscr{C}_{\mathbf{t}}^{hyb}$, the fiber of $\mathscr{C}_g^{hyb}$ over the point $\mathbf{t} \in \mathscr{M}_g^{hyb}$.

The universal hybrid curve plays thus the role of an *englobing space*, making it possible to give a mathematical meaning to questions of the type stated in Question 1.1.

In particular, the above theorem provides a complete answer to Question 1.1. If the sequence $s_1, s_2, \dots$ converges in $\mathscr{M}_g^{hyb}$ to a point $\mathbf{t}$ corresponding to the hybrid curve $\mathscr{C}_{\mathbf{t}}^{hyb}$, then the canonical measures μ_j^{can} of S_j converge to the canonical measure of $\mathscr{C}_{\mathbf{t}}^{hyb}$. The converse is also true, and it follows from the fact that the hybrid moduli space $\mathscr{M}_g^{hyb}$ is a compact Hausdorff space.

Later on, in Sections 3 and 4, we will construct a tower of hybrid spaces $\mathscr{M}_g^{hyb(k)}$ interpolating between $\overline{\mathscr{M}}_g$ and $\mathscr{M}_g^{hyb}$. Instead of all hybrid curves of a fixed genus g, we restrict only to those having *depth at most* k meaning that their underlying ordered partition $\pi = (\pi_1, \dots, \pi_r)$ has at most k parts, i.e., $r \leqslant k$. The k-th hybrid space $\mathscr{M}_g^{hyb(k)}$ compactifies the original moduli space $\mathscr{M}_g$ by adding these hybrid curves as its boundary part. To ensure compactness, however, the notion of hybrid curves is slightly relaxed to allow some vanishing edge lengths, see Sections 3 and 4 for more details. The first hybrid space $\mathscr{M}_g^{hyb(1)}$ can be interpreted as a compactification of $\mathscr{M}_g$ by means of metrized complexes. Altogether, this leads to the following tower of hybrid compactifications

$$\overline{\mathscr{M}}_g \longleftarrow \mathscr{M}_g^{hyb(1)} \longleftarrow \mathscr{M}_g^{hyb(2)} \longleftarrow \dots \longleftarrow \mathscr{M}_g^{hyb(N)} = \mathscr{M}_g^{hyb} \tag{1.1}$$

for $N = 3g - 3$.

We refer to $\mathscr{M}_g^{hyb(k)}$ as *the rank k hybrid compactification of* $\mathscr{M}_g$, and by an abuse of notation (because of the presence of length zero edges in the intermediate rank), we also call it the *moduli space of hybrid curves of depth bounded by* k.

Again, Theorem 1.2 shows that neither of the intermediate spaces in the tower verify the continuity property stated in the theorem.

From the above theorem, we can deduce similar continuity results for other families of Riemann surfaces. Consider a family of stable Riemann surfaces $\pi : \mathcal{S} \to X$ whose *discriminant locus* (i.e., the locus of points in X whose fiber in the family is not smooth) forms a simple normal crossing (SNC) divisor D in X. Moreover, assume that π comes from a *toroidal map* $f : X \to \overline{\mathscr{M}}_g$, meaning that we have an isomorphism $\mathcal{S} \simeq f^*(\mathscr{C}_g)$, where $\mathscr{C}_g$ is the universal curve of genus g over $\overline{\mathscr{M}}_g$. To the pair (X, D) we associate the corresponding hybrid space $X^{^{hyb}}$, the hybrid compactification of $X^* := X \backslash D$ relative to the SNC divisor D (see Section 3 for the construction). Moreover, from the toroidal assumption we get a map $f^{^{hyb}} : X^{^{hyb}} \to \mathscr{M}_g^{^{hyb}}$. The space $X^{^{hyb}}$ lies above X and comes with the corresponding family of hybrid curves $\mathcal{S}^{^{hyb}} = f^{^{hyb}*}(\mathscr{C}_g^{^{hyb}})$, which is therefore obtained by replacing the fibers over the points of the discriminant locus by appropriate hybrid curves.

Theorem 1.3. *The family of measured spaces $(\mathcal{S}_{\mathbf{t}}^{^{hyb}}, \mu_{\mathbf{t}}^{^{can}})_{\mathbf{t} \in X^{hyb}}$ is continuous.*

One specific case of the theorem concerns families which vary over a one-dimensional base, i.e., when $\dim(X) = 1$. The divisor D in this situation corresponds to a finite set of points $t_0, \dots, t_n$ in X, and our hybrid construction simply replaces the stable curves $\mathcal{S}_t$, for $t \in D$, with the corresponding metrized complex $\mathcal{MC}_t$. Denote by $\mathscr{G}_t$ the underlying stable metric graph of $\mathcal{MC}_t$. We have a projection map $\mathcal{MC}_t \to \mathscr{G}_t$. By the definition of the Arakleov-Zhang measure on $\mathcal{MC}_t$, the push-out of the canonical measure on $\mathcal{MC}_t$ coincides with the Zhang measure on the metric graph $\mathscr{G}_t$.

In this specific situation, we get the following theorem.

Theorem 1.4. *Let $\pi : \mathcal{S} \to X$ be a stable one-paramater family of complex curves of genus g with smooth generic fiber. Let t_0 be a point of X with singular fiber $\mathcal{S}_{t_0}$ and let $\mathcal{MC}_{t_0}$ be the corresponding metrized complex. Then the measured spaces $(\mathcal{S}_t, \mu_t^{^{can}})$ converge weakly to $(\mathcal{MC}_{t_0}, \mu_{t_0}^{^{can}})$ when t tends to t_0.*

In particular, the Zhang measure $\mu_{_{Zh}}$ on the metric graph $\mathscr{G}_{t_0}$ is the limit of the canonical measures $\mu_t^{^{can}}$ when t tends to t_0.

Theorem 1.4 is independently proved by Sanal Shivaprasad in a work parallel to ours with different methods based on non-Archimedean analysis [Shi20]. Moreover, he obtains results in one parameter families with non-reduced fibers, while we do not consider such families in our paper. A special consequence of this theorem, *the total mass of the limit measure on a singular point of* $\mathcal{S}_{t_0}$, was obtained by de Jong [dJ19]. The case where $\mathcal{S}_{t_0}$ is a curve of compact type or a rational curve with nodes is treated in a recent paper by Ng and Yeung [NY20]. Wentworth [Wen91] treated previously the case where the degenerate stable Riemann surface has a single node, i.e., the corresponding dual graph is a graph with a single edge and showed the convergence of canonical measures along a one-parameter family to the corresponding Zhang measure.

1.5. **Further directions.** In our upcoming work [AN21a], we show that hybrid curves introduced in this paper have a geometry parallel to that of smooth compact Riemann surfaces, by extending classical theorems such as Riemann-Roch and Abel-Jacobi theorem to the boundary of the hybrid moduli space.

In another direction, we introduce and study in [AN21b] a hybrid analogue of the canonical Green function and height pairing which will allow to obtain refined description of the asymptotic of these geometric quantities along a sequence of Riemann surfaces approaching the boundary of the hybrid moduli space.

The hybrid spaces introduced in this paper combine and refine the previous constructions of hybrid spaces [MS84, Ber09, AB15, BJ17, Fav18], and go beyond by producing on one side compactifications which are themselves hybrid on their boundaries, and on the other side by adding a depth to the level of degenerations providing higher rank versions of all these constructions. A geometric study of higher rank valuations is undertaken in [AI21], and the connection to the results in this paper should appear elsewhere.

1.6. **Notes and references to related work.** The work presented in this paper should be regarded as fitting into the current research in mathematics whose aim is to explain large scale limits of classical (e.g., algebraic and complex) geometry. In the past decade, this has given rise in particular to the development of tropical geometry, and has spread into diverse fields of mathematics, ranging from algebraic and complex geometry to mathematical physics and combinatorics, passing through computer science and combinatorial optimization.

The question of understanding non-Archimedean and hybrid limits of complex manifolds, in particular, has been an active area of research with connections to several other branches of mathematics. Pioneering work on the subject goes back already to the work of Bergman [Ber71], Bieri-Groves [BG84], Gelfand-Kapranov-Zelvinsky [GKZ94], Einsiedler-Kapranov-Lindand [EKL06], Passare and Rullgård [PR04, Rul01], Mikhalkin [Mik04], Jonsson [Jon16], and others. Recently, it has emerged in the work of Kontsevich and Soibelman [KS06], in relation with mirror symmetry, in a conjectural formulation of the non-Archiemdean Calabi metric as the *tropical limit* of maximal degenerations of Calabi-Yau varieties. This has resulted in a series of works linking non-Archimedean and tropical geometry to metric limits of complex geometry, see e.g. [KS06, GW00, GTZ13, GTZ16, BFJ15, Sus18, OO18, BJ18, Li20].

The first introduction of hybrid spaces probably goes back to the work of Morgan and Shalen [MS84], in the study of degenerations of hyperbolic structures on surfaces to real trees, and more recently, in the work of Berkovich [Ber09] in providing a non-Archimedean interpretation of the weight zero part of the limit mixed Hodge structures, in degenerating families of complex algebraic varieties. Recent work of Boucksom and Jonsson [BJ17] in connection with the conjecture of Kontsevich and Soibleman mentioned above, has resulted in an increase of interest in the development and use of hybrid spaces as a convenient bridge between the two worlds of Archimedean and non-Archimedean geometry. They in particular provide a proof of a measure theoretic version of the conjecture of Kontsevich and Soibelman. Hybrid spaces studied by Boucksom and Jonsson have been used in diverse direction, in particular in dynamics, in the work of Favre [Fav18], De Marco-Krieger-Ye [DKY20] and Dujardin-Favre [DF19], in the work of Pille-Schneider [PS19] on degenerations of Kähler-Einstein measures, and in the work of Shivaprasad on degenerations of log-Calabi-Yau varieties [Shi19].

In another direction, Baker and the first named author introduced in [AB15] specific classes of hybrid spaces called metrized complexes, as a tool to study degenerations of linear series, in connection with Berkovich non-Archimedean geometry. The point of view has found applications in the theory of limit linear series and in the study of moduli space of curves [ABBR15, BJ16, JP16, FJP20]. In particular, in connection with the results presented in this paper, the first named author has proved an equidistribution theorem for limits of Weierstrass points on families of complex curves. The equidistribution is taking place according to the Zhang measure on the level of metric graphs, and according to the Zhang-Arakelov measure on the level of metrized curve complexes [Ami14]. The Archimedean version of this result is a theorem of Neeman and Mumford [Nee84] and the measure of equidistribution in that setting is the Arakelov-Bergman measure on the Riemann surface. Recently, Richman [Ric18] has obtained a purely tropical equidistibution theorem for tropical Weierstrass points. These results and the recent works of Faltings [Fal20] and de Jong [dJ19] on non-Archimedean limits of Faltings δ-invariants, that of [ABBF16] on non-Archimedean limits of height pairing, and Shokrieh-Wu [SW19] on canonical and hyperbolic measures on finite and infinite metric graphs, in view of the previously mentioned works on measure-theoretic non-Archimedean limits of Archimedean geometry, suggested a deeper link between the Zhang and Arakelov measures, the leitmotive of the study undertaken in this paper.

In another direction related to the results of this paper, tropicalization of moduli spaces of curves has been studied by Abramovich-Caporaso-Payne [ACP15], and in the subsequent work of Chan-Galatius-Payne in the study of the cohomology of the moduli space of curves [CGP16, CGP18]. Tropical moduli spaces have been used in [OO18, Oda19], in connection with the constructions of Boucksom and Jonsson, to construct compactifications of moduli spaces of curves and $K3$ surfaces. The hybrid moduli space $\mathscr{M}_g^{hyb(1)}$ which appears through the constructions given in Sections 3.4 and 4.7 provides in a sense a refinement of these compactifications, replacing the dual complex with the dual metrized complex, making a more precise link between [AB15] and [BJ17].

Finally, we mention that degenerations of holomorphic differential forms and integrals to non-Archimedean differential forms and integrals have been studied in [CLT10, Lag12, CLD12, DHL19, BGJK20], and connections to asymptotic Hodge theory is studied in [AP20a, AP20b, GS06, GS10, IKMZ19, Rud10, Rud20].

1.7. **Sketch of the proof of the main theorem.** In this section, we briefly explain the big lines of the proof of Theorem 1.2

(1) By the definition of the hybrid moduli space, we will be able to proceed locally around each point of $\overline{\mathscr{M}}_g$. Let S_0 be a stable curve of genus g with the stable dual graph $G = (V, E, \mathfrak{g})$.

We consider the analytic family of stable curves $\mathcal{S} \to B$ over a polydisc B of dimension $N = 3g-3$ obtained from the versal deformation space of S_0. This means we can decompose $B = \Delta^{3g-3}$ for a small disk Δ around 0 in $\mathbb{C}$, and we have the collection of divisors $D_e \subset B$ for $e \in E$ given each by the equation $z_e = 0$ for coordinate functions $\underline{z}$ over the polydisk. Moreover, D_e is the locus of all points $t \in B$ such that in the family $\mathfrak{p} : \mathcal{S} \to B$, the fiber $\mathfrak{p}^{-1}(t)$ has a singular point corresponding to e.

By the general constructions in Section 3, we get the hybrid space B^{hyb} associated to the base $(B, \cup_{e\in E} D_e)$. Note that our construction refines the work of Boucksom and Jonsson [BJ17], but our hybrid space is different from the one studied in [BJ17]. In fact, we have a continuous forgetful map from our hybrid space to the one constructed by Boucksom and Jonsson.

Moreover, in Section 4.3 we construct a corresponding hybrid versal deformation curve $\mathcal{S}^{hyb}$ over B^{hyb}. It will be enough to prove the following local theorem.

Theorem 1.5 (Continuity: local case)**.** *Notations as above, the family of canonically measured hybrid spaces* $(\mathscr{C}_{\mathbf{t}}^{hyb}, \mu_{\mathbf{t}}^{can})_{\mathbf{t}\in B^{hyb}}$ *is continuous.*

The proof will be based on the use of asymptotic Hodge theory to reduce the problem to some specific problems either in matricial analysis or in combinatorics, concerning a tropical version of the theorem and spanning trees.

(2) The starting observation which opens the door to Hodge theory is to find a formulation of the Zhang measure as a determinantal measure, in the sense of [Lyo03] and reminiscent of the form of the Arakelov-Bergman measure. This will be done in Section 5, and is closely related to the work of the first named author with Bloch-Burgos-Fresàn [ABBF16] and the recent work of Shokrieh and Wu [SW19] on extension of canonical measures to infinite metric graphs. For Riemann surfaces, this expression involves the inverse of the imaginary part of the period matrix.

(3) Pursuing the Hodge theoretic approach, we use the nilpotent orbit theorem in asymptotic Hodge theory to describe the limiting behavior of the period matrices Ω_t of the curves $\mathcal{S}_t$ for t in the open part $B^* = B \backslash \bigcup_{e \in E} D_e$. This allows to describe the asymptotic of the matrices $\mathrm{Im}(\Omega_t)^{-1}$, and eventually leads to Theorem 1.5.

(4) More precisely, we first treat the *generic continuity*, i.e., the continuity of measured spaces through the open part B^*, at a given point of $B^{^{hyb}}$ with underlying stable curve S_0, cf. Section 9. In order to this, we decompose the canonical measures μ_t^{can} over the points $t \in B^*$ as a sum of measures $\mu_{i,j,t}$ for $1 \leqslant i, j \leqslant g$. Each measure $\mu_{i,j,t}$ is of the form $\mathrm{Im}(\Omega_t)^{-1}_{i,j} \omega_i \wedge \overline{\omega}_j$ for holomorphic one-forms ω_i, ω_j on the family chosen appropriately in relation with an *admissible symplectic basis* for the local system $H_1(\mathcal{S}_t, \mathbb{Z})$, cf. Section 7.4.

(5) We then prove that each family of complex-valued measures $\mu_{i,j,t}$ continuously extends over $\mathcal{S}^{^{hyb}}$. We need to consider three different regimes in which the measures $\mu_{i,j,t}$ extend to different type of measures on the hybrid curves in the hybrid family. The crucial lemma is the Inverse Lemma 9.3, which allows to control the asymptotic behavior of the inverse of a block matrix (in this case, the imaginary part of the period matrix) in terms of the block by block asymptotics of the original matrix.

This will allow to finish the proof of the continuity through the open part B^*.

(6) We then need to consider the continuity through any other stratum of the hybrid space $B^{^{hyb}}$. This is done in the final Section 10.

The proof of the continuity through another stratum can be reduced itself into two other continuity results.

- The generic continuity as in step (4-5-6) for smaller genus Riemann surfaces with marked points. As the analysis carried on through the previous parts is not affected by the presence of the marked points, this can be done by steps (4-5-6).
- The second part concerns a purely combinatorial continuity result about the continuity of the canonical measures on tropical curves over the *moduli space of tropical curves of given combinatorial type*, introduced in Secton 4. Our definition of a tropical curve encodes the data of a layering, so the canonical measure is obtained as the sum of Zhang measures associated to the graded minors. We prove this continuity phenomena using the formulation of the Zhang measures in terms of spanning trees, relying on a link between the spanning trees of graded minors and the spanning trees of the original graph. This is done in Section 6.

This finally finishes the proof of our main theorem.

1.8. **Basic notations.** A curve in this paper means a complex projective algebraic curve. The analytification of a complex curve is a compact Riemann surface. We deliberately use the terminology curve and Riemann surface to switch back and forth between the algebraic and analytic setting.

For a non-negative integer n, the symbol $[n]$ means the set $\{1, \dots, n\}$. *Almost all* in this paper means for all but a finite number of exceptions.

For a subset $F \subseteq E$, we denote by F^c the complement of F in E.

The letter G is used for graphs. We use the symbols $\mathscr{G}$, $\mathscr{C}^{\mathrm{trop}}$, $\mathcal{M}\mathscr{C}$ and $\mathscr{C}^{hyb}$ for metric graphs, tropical curves, metrized complexes and hybrid curves, respectively. Moreover, when working with the family of curves over a versal deformation space, we use the notation $\mathcal{S}^{hyb}$ for the corresponding family of hybrid curves over the hybrid deformation space.

In context with asymptotics, we will also make use of the standard Landau symbols. Let X be a topological space, x a point in X and $g\colon U\backslash\{x\} \to \mathbb{R}_+$ a function on a punctured neighborhood of x. Recall that the expressions $O(g)$ and $o(g)$ are used to abbreviate functions f such that the quotient $|f(y)|/g(y)$ remains bounded and goes to zero, respectively, as y goes to x in X (here, f is defined on a punctured neighborhood of x as well).

2. Preliminaries

In this preliminary section we introduce the combinatorial objects which will play a central role in the upcoming sections. These are graphs, ordered partitions, layered graphs and their graded minors, spanning trees, and augmented graphs.
We then provide a metric version of these notions, introducing metric graphs and tropical curves. We should clarify already here that our use of the notion of a tropical curve is a refinement of the current usage in the literature of this terminology. The one we use encodes in addition the choice of a *layering*, a concept introduced below, and has normalized edge lengths on each layer.
This naturally leads to the introduction of hybrid curves, as metrized complexes endowed with the choice of a layering on their underlying graphs and with normalized edge lengths on each layer.

2.1. **Graphs.** Part of the terminology and notation concerning graphs we use in this paper is either standard in graph theory or in algebraic geometry. The aim of this section is to recall some of these notions. We refer to classical text books on graph theory such as [Bol98, BM08, Die00] and on algebraic geometry such as [ACGH84] for more information. All our graphs are finite. We allow parallel edges and loops.

The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. If G is clear from the context, we simply use V and E and write $G = (V, E)$ to indicate the vertex and edge sets.

Two vertices u and v in G are called *adjacent* if there is an edge with extremities u and v. In this case we write $u \sim v$. An edge e and a vertex v are *incident* and we write $e \sim v$ if v is an extremity of e.

The *degree* of a vertex v is defined by

$$\deg(v) := \Big|\big\{e \in E \,\big|\, e \sim v\big\}\Big| + \#\text{ loops based at } v.$$

In other words, $\deg(v)$ counts the number of half-edges incident to v.

A subgraph of $G = (V, E)$ is a graph $H = (U, F)$ with $U \subseteq V$ and $F \subseteq E$. We say a subgraph $H = (U, F)$ of G is *spanning* if $U = V$, i.e., if it contains all the vertices of G. For a collection of spanning subgraphs $H_1 = (V, E_1), \dots, H_r = (V, E_r)$ of G, we denote by $H_1 \cup \dots \cup H_r$ the spanning subgraph of G with edge set $E_1 \cup \dots \cup E_r$.

For a subset of edges $F \subseteq E$, we denote by $G[F]$ the spanning subgraph of G with edge set F, i.e., $G[F] = (V, F)$. This should not be confused with the same notation used for subsets of vertices: for a subset of vertices $U \subseteq V$, we denote by $G[U]$ the *induced subgraph of G on U* which has vertex set U and edge set consisting of all the edges of E whose extremities are both in U.

By a *spanning tree* of a graph $G = (V, E)$ we mean a spanning subgraph T of G without any cycle and with the same number of connected components as G. This is equivalent to requiring T to have the maximum possible number of edges without containing any cycle. We denote by $\mathcal{T}(G)$ the set of all spanning trees of G. Spanning trees are the central combinatorial structures underlying the definition of the canonical measure (see Section 5.1). They will be used later in order to study continuity properties of the Zhang measure above the moduli space of tropical curves of given combinatorial type (see Section 6).

There are two simple operations that one can define on graphs, *deletion* and *contraction*, which lead to the definition of minors of a graph. Despite their simplicity, these operations deeply reflect the *global properties of graphs*, when seen as a *whole*, for example within Robertson-Seymour graph minor theory. A special kind of minors called *graded minors*, that we will introduce in a moment, play a central role in this paper.

Given a graph $G = (V, E)$ and an edge $e \in E$, we denote by $G - e$ the spanning subgraph H of G with edge set $E \backslash \{e\}$. In this case we say H is obtained from G by *deleting* e. Moreover, we denote by G/e the graph obtained by *contracting* the edge e: it has as vertex set the set U obtained by identifying the two extremities of e, and as edge set, the set $E \backslash \{e\}$ viewed in U via the projection map $V \to U$. More generally, for a subset $F \subseteq E$, we denote by $G - F$ and G/F the graphs obtained by deletion and contraction of F in G, respectively: $G - F$ is the spanning subgraph of G with edge set $E \backslash F$, and G/F is the graph obtained by contracting one by one all the edges in F. (One shows that the order does not matter.) For the purpose of clarification, we mention that later on we will introduce a variant of the construction of contractions for *augmented graphs*, which are graphs endowed with a genus function on vertices. In this case, the contraction remembers the *genera of connected components of the contracted part*, thus keeping the genus of the graph constant before and after contraction.

A *minor* of a graph G is a graph H which can be obtained by contracting a subset of edges in a subgraph $H' = (W, F)$ of G.

For a graph G, we denote by $H_1(G)$ the first homology group of G with $\mathbb{Z}$ coefficients. The rank of $H_1(G)$, which coincides with the first Betti number of G viewed as a topological space, is called *genus* of the graph. In this paper we use the letter h when referring to the genus of graphs, and reserve the letter g and $\mathfrak{g}$ for the genus of augmented graphs, algebraic curves, and their analytic and hybrid variants introduced later.

The genus h of a graph $G = (V, E)$ is equal to $|E| - |V| + c(G)$ where $c(G)$ denotes the number of connected components of G.

2.2. **Ordered partitions.** Let E be a finite set. An *ordered partition* of a (non-empty) subset $F \subseteq E$ is an ordered sequence $\pi = (\pi_i)_{i=1}^r$ of non-empty, pairwise disjoint subsets $\pi_i \subseteq F$ such that

$$F = \bigsqcup_{i=1}^{r} \pi_i. \tag{2.1}$$

The integer r is called the *depth* of π. If $F = \varnothing$, then we make a slight abuse of notation and allow $\pi_\varnothing = \varnothing$ as its only ordered partition. The set of ordered partitions of a subset $F \subseteq E$ is denoted by $\Pi(F)$ (in particular, $\Pi(\varnothing) = \{\pi_\varnothing\}$).

The ordered partitions of a subset F are in bijective correspondence with its *filtrations*: A *filtration* $\mathscr{F}$ on a (non-empty) subset $F \subseteq E$ of *depth* r is an increasing sequence of non-empty subsets $F_1 \subsetneq F_2 \subsetneq F_3 \subsetneq \cdots \subsetneq F_r = F$ for $r \in \mathbb{N}$. Clearly, each filtration $\mathscr{F}$ on a subset $F \subseteq E$ defines an ordered partition $\pi = (\pi_i)_{i=1}^r$ of F by setting

$$\pi_i = F_i \backslash F_{i-1}, \qquad i = 1, ..., r,$$

with $F_0 := \varnothing$ by convention, and this gives a bijection. Sometimes we use the notation $\mathscr{F}^\pi$ to emphasize that the filtration is the one associated to the ordered partition π. Again, for $F = \varnothing$ we allow $\mathscr{F}^\varnothing := \varnothing$ as its only filtration.

Given an ordered partition $\pi = (\pi_1, \ldots, \pi_r)$, we set $E_\pi := \pi_1 \cup \cdots \cup \pi_r$, the subset of E underlying the ordered partition. We denote the maximum element of the filtration $\mathscr{F}$ by $\max(\mathscr{F})$. With these notations, we have $E_\pi = \max(\mathscr{F}^\pi)$.

We now define a natural partial order on the set of ordered partitions (equivalently, filtrations) of subsets of a given set E.

A filtration $\mathscr{F}$ is a *refinement* of the filtration $\mathscr{F}'$ if as a set we have $\mathscr{F}' \subseteq \mathscr{F}$. We say that an ordered partition π is a *refinement* of an ordered partition π', and write $\pi' \preceq \pi$, if this holds true for the corresponding filtrations $\mathscr{F}$ and $\mathscr{F}'$. It is easy to see that $\pi' \preceq \pi$ exactly when π is obtained from π' performing the following two steps:

(i) Replacing each set π'_i in the ordered sequence $\pi' = (\pi'_i)_{i=1}^r$ by an ordered partition $\varrho^i = (\varrho^i_k)_{k=1}^{s_i}$ of π'_i.
(ii) Possibly adding an ordered partition $\varrho = (\varrho_k)_{k=1}^s$ of a subset $F^\infty \subseteq E \backslash E_{\pi'}$ at the end of the ordered sequence π'.

(The use of the symbol F^∞ should become clear later on when we link these constructions to geometry.)

We stress that in the definition of a refinement, the ordered partitions/filtrations are not necessarily defined over the same subset $F \subseteq E$ (see Example 2.1). For instance, $\pi_\varnothing \preceq \pi$ for all ordered partitions π. However, if π is a refinement of π' (or equivalently, $\mathscr{F}' \subseteq \mathscr{F}$), then by definition this implies $E_{\pi'} \subseteq E_\pi$. It is also clear that "$\preceq$" defines a partial order on $\Pi := \bigcup_{F \subseteq E} \Pi(F)$, the set of all ordered partitions of subsets $F \subseteq E$.

Example 2.1. Assume that $E = \{e, f\}$ for two elements $e \neq f$. Then the ordered partitions of subsets $F \subseteq E$ are given by

$$\begin{aligned}\Pi(\varnothing) &= \big\{\pi_\varnothing\big\}\\ \Pi(\{e\}) &= \Big\{\big(\{e\}\big)\Big\}\\ \Pi(\{f\}) &= \Big\{\big(\{f\}\big)\Big\}\\ \Pi(E) &= \Big\{\big(\{e\},\{f\}\big), \big(\{f\},\{e\}\big), \big(\{e,f\}\big)\Big\}.\end{aligned}$$

The partial order "$\preceq$" on $\Pi = \bigcup_{F \subseteq E} \Pi(F)$ consists of the following relations

$$\begin{aligned}\big(\{e\}\big) &\preceq \big(\{e\},\{f\}\big)\\ \big(\{f\}\big) &\preceq \big(\{f\},\{e\}\big)\\ \big(\{e,f\}\big) &\preceq \big(\{e\},\{f\}\big)\\ \big(\{e,f\}\big) &\preceq \big(\{f\},\{e\}\big),\end{aligned}$$

and the trivial relations that $\pi_\varnothing \preceq \pi$ and $\pi \preceq \pi$ for all $\pi \in \Pi$. ◇

2.3. **Layered graphs and their graded minors.** A *layered graph* is the data of a graph $G = (V, E)$ and an ordered partition $\pi = (\pi_1, \dots, \pi_r)$ on the set of edges E. The elements π_j are called *layers*. The integer r is called the *depth* of the layered graph G, which thus equals the depth of the underlying ordered partition. By an abuse of the notation, we use the same letter G to denote the layered graph (V, E, π). The genus of a layered graph is defined to be equal to that of its underlying graph.

Let now $G = (V, E, \pi)$ be a layered graph with the ordered partition $\pi = (\pi_1, \dots, \pi_r)$ of E. We define the *graded minors* of G as follows.

Consider the filtration

$$\mathscr{F}^\pi_\bullet : F_0 = \varnothing \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r = E$$

associated to π. For each j, we let

$$E^j_\pi := E \backslash F_{j-1} = \pi_j \cup \pi_{j+1} \cup \cdots \cup \pi_r$$

and naturally arrive at a decreasing filtration

$$\mathscr{E}^\bullet_\pi : E^1_\pi = E \supsetneq E^2_\pi \supsetneq \dots E^r_\pi \supsetneq E^{r+1}_\pi = \varnothing.$$

This leads to a decreasing sequence of spanning subgraphs of G

$$G =: G^1_\pi \supset G^2_\pi \supset G^3_\pi \supset \cdots \supset G^r_\pi \supset G^{r+1}_\pi = (V, \varnothing)$$

where for each j, $G^j_\pi := (V, E^j_\pi)$ has edge set E^j_π.

For each integer $j \in [r]$, the j^{th} *graded minor* of G, denoted by $\mathrm{gr}^j_\pi(G)$, is obtained by contracting all the edges of E^{j+1}_π in G^j_π, i.e.,

$$\mathrm{gr}^j_\pi(G) := G^j_\pi / E^{j+1}_\pi = G[F^c_{j-1}]/F^c_j = G[\pi_j \cup \cdots \cup \pi_r]/\pi_{j+1} \cup \cdots \cup \pi_r.$$

It follows that $\mathrm{gr}^j_\pi(G)$ has edge set equal to π_j. (More precisely, we should view the elements of π_j in the contracted graph, but we simplify and identify the two sets in the writing.) We denote by V^j_π the vertex set of $\mathrm{gr}^j_\pi(G)$. Sometimes we will drop π or G

and simply write $\mathrm{gr}^j(G)$, gr^j_π, V^j, *etc.* if π or G is understood from the context. We denote by $\mathscr{K}_j : G^j_\pi \to \mathrm{gr}^j_\pi(G)$ the contraction map.

2.4. **Genus formula for layered graphs.** Let $G = (V, E, \pi)$ be a layered graph. We denote by c^j_π the number of connected components and by h^j_π the genus of $\mathrm{gr}^j_\pi(G)$. These are related to each other by the equality $h^j_\pi = |\pi_j| - |V^j_\pi| + c^j_\pi$.

Proposition 2.2 (Genus formula)**.** *Let G be a layered graph of genus h with the underlying ordered partition $\pi = (\pi_1, \dots, \pi_r)$. We have the equality*

$$h = \sum_{j=1}^{r} h^j_\pi.$$

In the rest of this section, we explain the proof of this proposition.

Consider $\mathrm{gr}^1_\pi(G) = (V^1_\pi, \pi_1)$. For each vertex $u \in V^1_\pi$, let $V_u \subseteq V$ be the set of vertices in V which are contracted to u, i.e., $V_u = \mathscr{K}_1^{-1}(u)$. Denote by E_u all edges of E^2_π with both end-points in V_u. Define $G_u := (V_u, E_u)$.

Proposition 2.3. *The subgraph $G_u = (V_u, E_u)$ of G is connected.*

Proof. The contraction of the edges of G_u results in a graph with a single vertex u. This can only happen if the graph is connected, which proves the claim. □

The ordered partition π of E induces a layering of E_u. This is obtained by taking first the ordered partition

$$\pi_1 \cap E_u = \varnothing, \pi_2 \cap E_u,\ \pi_3 \cap E_u,\ \dots,\ \pi_r \cap E_u,$$

and then removing all the empty sets from the sequence. We denote this ordered partition by π_u. The pair (G_u, π_u) is a layered graph. Note that π_u has depth at most $r - 1$ since the first term $\pi_1 \cap E_u$ is empty.

Proof of Proposition 2.2. This can be obtained by induction on the depth of the ordered partition π. If the partition π consists of a single set $\pi_1 = E$, the proposition is obvious. Otherwise, let $G^2_\pi = (V, E^2_\pi)$ be the spanning subgraph of G with edge set E^2_π. The set E^2_π is a disjoint union of the sets E_u for $u \in V^1_\pi$. It follows that G^2_π is the disjoint union of graphs G_u for $u \in V^1_\pi$. In particular, by the previous proposition, the number of connected components of G^2_π is $|V^1_\pi|$, and its genus is given by

$$h(G^2_\pi) = |E^2_\pi| - |V| + |V^1_\pi| = \sum_{u \in V_1} h(G_u). \tag{2.2}$$

We infer that

$$h = h(G) = |E| - |V| + 1 = |\pi_1| - |V^1_\pi| + 1 + |E^2_\pi| - |V| + |V^1_\pi| = h^1_\pi + h(G^2_\pi).$$

The result now follows by induction hypothesis applied to the layered graphs G_u, using Equation (2.2). □

2.5. **Spanning trees of layered graphs.** Let $G = (V, E)$ and let $\pi = (\pi_1, \dots, \pi_r)$ be an ordered partition of E. As in the previous section, we consider the decreasing filtration

$$\mathscr{E}_\pi^\bullet : E_\pi^1 = E \supsetneq E_\pi^2 \supsetneq \dots E_\pi^r \supsetneq E_\pi^{r+1} = \varnothing$$

and its associated decreasing sequence of spanning subgraphs of G

$$G =: G_\pi^1 \supset G_\pi^2 \supset G_\pi^3 \supset \cdots \supset G_\pi^r \supset G_\pi^{r+1} = (V, \varnothing).$$

For each $j = 1, \dots, r$, let T_j be a spanning tree of $\mathrm{gr}_\pi^j(G)$ with edge set $A_j \subset \pi_j$, so $T_j = (V_\pi^j, A_j)$. Let $A = A_1 \cup \cdots \cup A_r$. The following proposition can be regarded as a refinement of the genus formula.

Proposition 2.4. *Notations as above, the subgraph $T = (V, A)$ of G is a spanning tree.*

Definition 2.5 (Spanning trees of layered graphs)**.** A *spanning tree of a layered graph* $G = (E, V, \pi)$ is a spanning tree $T = (V, A)$ of G which is obtained as above, i.e., $A = A_1 \cup \cdots \cup A_r$ for A_j the edge set of a spanning tree of gr_π^j. We denote by $\mathcal{T}_\pi(G)$ the set of all such spanning trees. $\diamond$

Remark 2.6. We have $\mathcal{T}_\pi(G) \subseteq \mathcal{T}(G)$ but the two sets can be different in general. For example, consider $G = K_3$ the complete graph with three vertices v_1, v_2, v_3 and edges $e_1 = \{v_2, v_3\}, e_2 = \{v_1, v_3\}, e_3 = \{v_1, v_2\}$. Let $\pi = (\pi_1, \pi_2)$ with $\pi_1 = \big\{e_1\big\}$ and $\pi_2 = \big\{e_2, e_3\big\}$. Then $\mathcal{T}_\pi$ is the set of all spanning trees of K_3 which contain the edge e_1. More precisely,

$$\mathcal{T}_\pi = \{T_2, T_3\} \quad \text{and} \quad \mathcal{T} = \{T_1, T_2, T_3\}$$

where T_i is the spanning tree with edge set $E(T_i) = \{e_1, e_2, e_3\} \backslash \{e_i\}$. $\diamond$

Proof of Proposition 2.4. The sets $A_1, \dots, A_r$ are disjoint. Moreover, we have $|\pi_j \backslash A_j| = h_\pi^j$. By the genus formula, we see that $\pi_j \backslash A$ has size h. So we only need to prove that T does not contain any cycle. For the sake of a contradiction, suppose T has a cycle $C \subset G$, and let j be the smallest index with $E(C) \cap \pi_j \neq \varnothing$. But then, the set of edges $E(C) \cap \pi_j$ form a cycle in $\mathrm{gr}_\pi^j(G)$. This means T_j contains a cycle, which is a contradiction. This proves the proposition. $\square$

2.6. **Marked graphs.** Let n be a non-negative integer. A *graph with n (labeled) marked points* is a graph $G = (V, E)$ endowed with the data of labels $1, \dots, n$ placed on its vertices. More formally, the marking is encoded in the function $\mathfrak{m} : [n] \to V$ which to each element $j \in [n]$ associates the vertex $\mathfrak{m}(j) \in V$.

To a graph with n marked points, we associate the *counting function* $\mathfrak{n} : V \to \mathbb{N} \cup \{0\}$ which at vertex v takes a value $\mathfrak{n}(v)$ equal to the number of labels placed at v, i.e., $\mathfrak{n}(v)$ is the number of elements $j \in [n]$ with $\mathfrak{m}(j) = v$.

2.7. **Augmented and stable graphs.** We now consider graphs endowed with a *genus function* on the set of vertices. Such objects appear with different names. For example in [AC13] they are called *weighted graphs*, while in [ABBR15], they are called *augmented graphs*. We use this latter terminology.

An *augmented graph* is a graph $G = (V, E)$ endowed with the data of an integer-valued *genus function* $\mathfrak{g} : V \to \mathbb{N} \cup \{0\}$ on the set of vertices. The integer $\mathfrak{g}(v)$ is called the *genus* of the vertex $v \in V$. The genus of an augmented graph $G = (V, E, \mathfrak{g})$ is by definition the integer

$$g := h + \sum_{v \in V} \mathfrak{g}(v)$$

where $h = |E| - |V| + c(G)$ is the genus of the underlying graph (V, E).

An *augmented graph with n marked points* is an augmented graph $(V, E, \mathfrak{g})$ with a marking $\mathfrak{m} : [n] \to V$.

In this paper, we mostly consider augmented graphs or augmented marked graphs which verify the *stability condition* below. In this case, they are called stable graphs. So by definition, a stable graph comes with a genus function on vertices.

A *stable graph with n marked points* is a quadruple $G = (V, E, \mathfrak{g}, \mathfrak{m})$ with marking and genus functions $\mathfrak{m} : [n] \to V$ and $\mathfrak{g} : V \to \mathbb{N} \cup \{0\}$, respectively, subject to the following *stability condition*:

- for every vertex v of genus zero, we have $\deg(v) + \mathfrak{n}(v) \geqslant 3$, and
- for every vertex v of genus one, we have $\deg(v) + \mathfrak{n}(v) \geqslant 1$.

A *stable graph* is a triple $(V, E, \mathfrak{g})$ which verifies the two above conditions for the counting function $\mathfrak{n} \equiv 0$.

By an abuse of the notation, we use G both for the stable (marked) graph and its underlying graph (V, E), forgetting the genus function (and the marking).

We extend the definition of deletion and contraction to stable marked graphs as follows. Let $G = (V, E, \mathfrak{g}, \mathfrak{m})$ be a stable marked graph of genus g with n marked points.

For an edge $e \in E$, we define the stable marked graph $G/e =: G' = (V', E', \mathfrak{g}', \mathfrak{m}')$ as follows. First, the underlying graph G' of G/e is the graph obtained by contracting e in G. Let v_e be the vertex of G' which corresponds to the identification of the two extremities u and v of e. The genus function $\mathfrak{g}'$ of G/e is then defined by

$$\mathfrak{g}'(w) = \begin{cases} \mathfrak{g}(w) & \text{if } w \neq v_e \\ \mathfrak{g}(u) + \mathfrak{g}(v) & \text{if } w = v_e \text{ and } e \text{ is not a loop, i.e., } u \neq v \\ \mathfrak{g}(u) + 1 & \text{if } w = v_e \text{ and } e \text{ is a loop, i.e., } u = v. \end{cases}$$

In the presence of a marking function $\mathfrak{m} : [n] \to V$, we define the marking function $\mathfrak{m}'$ of G' as $\mathfrak{m}' := \mathscr{K} \circ \mathfrak{m} : [n] \to V'$, where $\mathscr{K} : G \to G'$ denotes the contraction map.

The following is straightforward.

Proposition 2.7. *Notations as above, the stable graph G/e is of genus g.*

Finally, we define layered stable (marked) graphs as stable (marked) graphs $G = (V, E, \mathfrak{g}, \mathfrak{m})$ with the choice of an ordered partition $\pi = (\pi_1, \dots, \pi_r)$ on the edge set E.

2.8. **Partial order on stable (marked) graphs of given genus.** Let n be a non-negative integer. We define a partial order $\leqslant$ on the set of stable marked graphs of genus g with n marked points as follows. For two stable marked graphs G and H, we say $G \leqslant H$ if G can be obtained by a sequence of edge-contractions from H, in the sense of the previous paragraph.

2.9. **Metric graphs.** We briefly recall the definition of a *metric graph* and the way it arises as a *metric realization* of a finite graph with *lengths* associated to edges.

Suppose $G = (V, E)$ is a finite graph and $\ell : E \to \mathbb{R}_{>0}$ is a length function (i.e., it assigns a positive real number ℓ_e to every edge $e \in E$). To such a pair (G, ℓ), we associate a metric space $\mathscr{G}$ as follows: by assigning each edge a direction and calling one of its vertices the initial vertex v_i and the other one the terminal vertex v_t, every edge $e \in E$ can be identified with a copy of the interval $\mathcal{I}_e = [0, \ell_e]$ (the left and right endpoint correspond to v_i and v_t). The topological space $\mathscr{G}$ is obtained by further identifying the ends of edges corresponding to the same vertex v (in the sense of a topological quotient).

It is then clear that the topology on $\mathscr{G}$ is metrizable by the so-called *path metric*: the distance between two points $x, y \in \mathscr{G}$ is defined as the arc length of the shortest path connecting them.

By construction $\mathscr{G}$ is a compact metric space. Each point $x \in \mathscr{G}$ has a neighborhood isometric to a star-shaped set $S(\deg(x), r_x)$ of degree $\deg(x) \in \mathbb{Z}_{\geqslant 1}$,

$$S(\deg(x), r_x) := \big\{z = re^{2\pi ik/\deg(x)} \,|\, r \in [0, r_x),\ k = 1, \dots, \deg(x)\big\} \subset \mathbb{C}. \tag{2.3}$$

Notice that if x belongs to the vertex set V of G, then $\deg(x)$ in (2.3) coincides with the combinatorial degree. In particular, this justifies our use of the same notation $\deg(\cdot)$ for the combinatorial degree and the integer in (2.3). It is also clear that $\deg(x) = 2$ for every non-vertex point x of $\mathscr{G}$.

A *metric graph* is a compact metric space arising from the above construction for some pair (G, ℓ) of a graph G and length function ℓ. The metric graph $\mathscr{G}$ is then called the *metric realization* of the pair (G, ℓ). On the other hand, a pair (G, ℓ) such that the metric graph $\mathscr{G}$ is isometric to the metric realization of the pair (G, ℓ) is called a *finite graph model* of $\mathscr{G}$. Clearly, any metric graph has infinitely many finite graph models (e.g., they can be constructed by subdividing edges). However, as we explain now, it is possible to find a *minimal model* if the metric graph is *stable*.

An *augmented metric graph* is a metric graph with a genus function $\mathfrak{g} : \mathscr{G} \to \mathbb{N} \cup \{0\}$ which is *almost everywhere* zero, i.e., it is zero outside a finite set of points in $\mathscr{G}$. A *metric graph with n marked point* is a metric graph $\mathscr{G}$ with a choice of a marking function $\mathfrak{m} : [n] \to \mathscr{G}$. We define the counting function $\mathfrak{n} : \mathscr{G} \to [n]$ similar to the case of graphs.

A *finite graph model of an augmented metric graph with n marked points* is a finite graph model $(G = (V, E), \ell)$ of the underlying metric graph such that V contains all the points of positive genus and all the marked points.

An augmented metric graph $\mathscr{G}$ with n marked points is called *stable* if every point x of $\mathscr{G}$ of degree one verifies $\deg(x) + \mathfrak{g}(x) + \mathfrak{n}(x) \geqslant 3$, and any connected component of $\mathscr{G}$ entirely consisting of points of degree two, i.e., homeomorphic to circle, has either a point of positive genus or a marked point.

Proposition 2.8. *Any stable metric graph has a minimal finite graph model.*

Proof. The vertices of the minimal finite model consist of all the

- points x with $\deg(x) \neq 2$;
- points x with $\mathfrak{g}(x) > 0$; and
- points x with $\mathfrak{n}(x) > 0$.

Any finite graph model of $\mathscr{G}$ contains all these points. Under the stability condition, one verifies that this set is the vertex set of a finite graph model. $\square$

2.10. **Layered metric graphs and tropical curves.** A *layered augmented metric graph* (*with marking*) is a pair $(\mathscr{G}, \pi)$ consisting of an augmented metric graph $\mathscr{G}$ (with marking) together with a finite graph model (G, ℓ) and an ordered partition π on the edge set $E(G)$ of G. In this situation, (G, π) is called the combinatorial type of the augmented metric graph $(\mathscr{G}, \pi)$.

We define *the conformal equivalence relation* on layered augmented metric graphs as follows. Two layered augmented metric graphs (with marking) $\mathscr{G}$ and $\mathscr{G}'$ are called conformally equivalent if they have the same combinatorial type (G, π), the same genus function on the vertex set $V(G)$ and for each layer π_j of π, there is a positive number $\lambda_j > 0$ such that

$$\ell_{|\pi_j} = \lambda_j \, \ell'_{|\pi_j}$$

for the corresponding edge length functions $\ell, \ell' \colon E(G) \to \mathbb{R}_{>0}$. In other words, the layered augmented metric graphs $(\mathscr{G}', \pi')$ equivalent to some fixed $(\mathscr{G}, \pi)$ are simply obtained by multiplying the edge lengths for every layer π_j by some constant $\lambda_j > 0$.

A *tropical curve* (*with marking*) is a conformal equivalence class of layered metric graphs (with markings). Equivalently, a tropical curve (with marking) is a pair $\mathscr{C}^{\mathrm{trop}} = (\mathscr{G}, \pi)$ consisting of an augmented metric graph $\mathscr{G}$ (with marking) with an ordered partition $\pi = (\pi_1, \dots, \pi_r)$ on the edge set E of a finite graph model (G, ℓ) of $\mathscr{G}$ such that in addition, the normalization equality

$$\sum_{e \in \pi_j} \ell_e = 1, \quad j = 1, \dots, r$$

holds true. A layered metric graph (with marking) is called *stable* if its underlying metric graph (with marking) is stable. A tropical curve (with marking) is called *stable* if any, and so all, of its layered metric graph (with marking) representatives is stable.

For a stable tropical curve (with marking) in this paper we *always* assume that the layering is defined on the minimal model of the underlying metric graph.

Layered metric graphs and layered metrized complexes, that we define in the next section, naturally appear in the picture in our forthcoming work [AN21b] when we study asymptotics of geometric quantities along a sequence of Riemann surfaces approaching the boundary of the hybrid moduli space. For the purpose of the questions studied in this paper, we always work with conformal equivalence classes, by putting the normalization property on the edge lengths in each layer.

2.11. **Layered metrized complexes and hybrid curves.** We first recall the definition of metrized complexes, see [AB15] for more details.

A *metrized curve complex* $\mathcal{MC}$ consists of the following data:

- A finite graph $G = (V, E)$.
- A metric graph $\mathscr{G}$ with a model (G, ℓ) for a length function $\ell : E \to \mathbb{R}_+$.
- For each vertex $v \in V$, a smooth projective complex curve C_v.
- For each vertex $v \in V$, a bijection $e \mapsto p_v^e$ between the edges of G incident to v (with loop edges counted twice) and a subset $\mathcal{A}_v = \{p_v^e\}_{e \ni v}$ of $C_v(\mathbb{C})$.

By an abuse of the notation, we use the same letter $\mathcal{MC}$ for the metrized complex and its *geometric realization* defined as follows. Let $\mathcal{I}_e$ be an interval of length ℓ_e for each edge $e \in E$, as in the definition of the metric graph geometric realization. For each vertex, by an abuse of the notation, denote by C_v the analytification of C_v which is a compact Riemann surface. For each vertex v of an edge e, we identify the corresponding extremity of $\mathcal{I}_e$ with the marked point p_v^e. This identifies $\mathcal{MC}$ as the disjoint union of the Riemann surfaces C_v and intervals $\mathcal{I}_e$, for $v \in V$ and $e \in E$, quotiented by these identifications. Moreover, $\mathcal{MC}$ is naturally endowed with the quotient topology.

To any metrized complex, one naturally associates the underlying graph endowed with the genus function $\mathfrak{g} : V \to \mathbb{N} \cup \{0\}$ which to the vertex v associates $\mathfrak{g}(v)$ defined as the genus of C_v. The metric realization of this augmented graph with the edge length function ℓ is called the underlying augmented metric graph of $\mathcal{MC}$.

A *metrized complex with n marked points* is a metrized complex $\mathcal{MC}$ with a marking $\mathfrak{m} : [n] \to \mathcal{MC}$ such that for each $j \in [n]$, $\mathfrak{m}(j)$ lies outside the union of the intervals $\mathcal{I}_e$ for all the edges e of the underlying graph.

In this paper, unless otherwise stated, we only consider *stable metrized complexes* (*with marking*) which are those whose underlying augmented (marked) graph is stable.

A *layered metrized complex* is a pair $(\mathcal{MC}, \pi)$ consisting of a metrized complex $\mathcal{MC}$ and a layering π on the set of edges of $\mathcal{MC}$. Any layered metrized complex naturally gives rise to a layered augmented metric graph: we take the underlying augmented metric graph $\mathscr{G}$ of $\mathcal{MC}$ together with the ordered partition π of the edge set E of the graph G, which is a finite graph model of $\mathscr{G}$.

Again, we will only consider stable layered metrized complexes which are those whose underlying metrized complex is stable.

The definitions above naturally extend to the situation where we have markings.

We define *the conformal equivalence relation* on layered metrized complexes (with marking) by declaring two layered metrized complexes (with marking) $(\mathcal{MC}, \pi)$ and $(\mathcal{MC}', \pi')$ to be conformally equivalent if they have the same underlying graph G, the projective curves C_v with marked points A_v (and the marking) are the same, the ordered partitions π and π' coincide, and moreover, writing $\pi = (\pi_1, \dots, \pi_r)$, we can find positive real numbers $\lambda_1, \dots, \lambda_r > 0$ such that

$$\ell_{|\pi_j} = \lambda_j \, \ell'_{|\pi_j}, \quad j = 1, \dots, r$$

for the edge length functions ℓ and ℓ' of the two metrized complexes.

Finally, a *hybrid curve* (*with marking*) denoted by $\mathscr{C}^{hyb}$ is a conformal equivalence class of layered metrized complexes (with marking). Equivalently, we can define a hybrid curve (with marking) $\mathscr{C}^{hyb}$ as a layered metrized complex $(\mathcal{MC}, \pi = (\pi_1, \dots, \pi_r))$ whose underlying edge length function verifies the following normalization conditions

$$\sum_{e \in \pi_j} \ell_e = 1, \quad j = 1, \dots, r.$$

Again, we will only consider stable hybrid curves which are those whose underlying metrized complex is stable.

Any hybrid curve $\mathscr{C}^{hyb}$ naturally gives rise to a tropical curve $\mathscr{C}^{\mathrm{trop}}$: we take the underlying augmented metric graph $\mathscr{G}$ of $\mathcal{MC}$ with normalized edge lengths, together with the ordered partition π of the edge set E of the graph G, which is a finite graph model of $\mathscr{G}$.

3. Hybrid spaces of higher rank

The aim of the following sections 3.1–3.3 is to present a construction of hybrid spaces of higher rank. Loosely speaking, given the data of a complex manifold B and a simple normal crossing divisor D, we associate to it a hybrid space $B^{^{hyb}}$ by equipping the points $x \in D$ with additional simplicial coordinates. Specializing to the case of a polydisc $B = \Delta^N$ with the SNC divisor given by the coordinate axes, we obtain a suitable base space for the hybrid versal deformation in Section 4.2.

However, in light of, e.g., [Ber09, BJ17, Fav18, DF19, DKY20], the construction seems to be of independent interest and hence we present it in a general framework.

3.1. **The setting.** In the following, let B be an N-dimensional complex manifold, for an integer N, and let $D = \bigcup_{e\in E} D_e$ be a simple normal crossing divisor. More precisely, we require that $(D_e)_{e\in E}$ is a finite family of smooth, connected and closed submanifolds of codimension one in B such that for any subset $F \subseteq E$, the intersection

$$D_F := \bigcap_{e\in F} D_e \tag{3.1}$$

is either empty or a smooth submanifold of codimension $|F|$ (with only finitely many connected components). The intersection D_F is called the *stratum* of the subset $F \subseteq E$. The *inner stratum* is the subset

$$D_F^\circ := D_F \backslash \bigcup_{e\notin F} D_e = \big\{ t \in B \,|\, E_t = F \big\} \tag{3.2}$$

where for $t \in B$, we set

$$E_t := \big\{ e \in E \,|\, t \in D_e \big\}. \tag{3.3}$$

Notice that for $F = \varnothing$, we simply recover

$$D_\varnothing^\circ = B \backslash D =: B^*.$$

The inner strata (3.2) form a partition of B, that is,

$$B = \bigsqcup_{F\subseteq E} D_F^\circ. \tag{3.4}$$

An *adapted coordinate neighborhood*, also called a *system of local parameters*, for D around a point $t \in B$ is a pair (U, z) where U is an open neighborhood of t in B and $z = (z_i)_{i=1}^N$ are local coordinates on U with

(i) $|z_i| < 1$ on U for all $i = 1, \dots, N$, and
(ii) $D_e \cap U = \varnothing$ for all $e \notin E_t$, and
(iii) $D_e \cap U = \big\{ s \in U \,|\, z_e(s) = 0 \big\}$ for all $e \in E_t$. More precisely, for each $e \in E_t$, there is an associated coordinate $z_e := z_{i(e)}$ on (U, z) such that this holds true.

We will also need the following elementary fact:

Proposition 3.1. *Assume that (U, z) and (U', z') are adapted coordinate neighborhoods for points $t, t' \in B$, respectively. Then for all $e \in E_t \cap E_{t'}$,*

$$z'_e = g_e z_e \qquad \text{on } U \cap U'$$

for a non-vanishing holomorphic function $g_e \colon U \cap U' \to \mathbb{C}$.

3.2. **Definition of the hybrid space.** Before giving the formal definition of the hybrid space B^{hyb}, we briefly explain the motivation behind the construction. The idea is to enrich the points $t \in D$ with additional *simplicial coordinates* in order to distinguish different ways of approaching them in the complex topology of B.

Namely, fix some $t \in B$ and consider a small neighborhood of t in B. Upon choosing local coordinates $(z_i)_i$, we can assume that the divisors D_e, $e \in E_t$, are given by $D_e = \{z_e = 0\}$. This trivially implies that

$$\lim_{s \to t} \log(|z_e(s)|) = -\infty, \qquad e \in E_t. \tag{3.5}$$

However, the asymptotic behavior of the normalized logarithmic coordinates,

$$\mathrm{Log}_e(s) := \frac{\log(|z_e(s)|)}{\sum_{\hat{e} \in E_t} \log(|z_{\hat{e}}(s)|)}, \qquad e \in E_t, \tag{3.6}$$

is known to carry crucial information (see, e.g., [BJ17]). This hints that sequences $(s_n)_n$ in B^* converging to t in B with different limit values

$$y_e := \lim_{n \to \infty} \mathrm{Log}_e(s_n) \in [0, 1] \tag{3.7}$$

should be further distinguished and hence converge to different points in B^{hyb}. Since $\sum_e \mathrm{Log}_e(s) = 1$, the limit $y := (y_e)_e$ lies in the simplex $\sigma_{E_t} = \{(y_e)_e \in \mathbb{R}^{E_t}_{\geqslant 0} \mid \sum y_e = 1\}$. On the other hand, there might be a subset $E_t^2 \subsetneq E_t$ such that

$$y_e = \lim_{n \to \infty} \mathrm{Log}_e(s_n) = 0, \qquad e \in E_t^2. \tag{3.8}$$

These logarithmic coordinates are dominated by the others and their limit $y_e = 0$ carries only little information. To analyze them further, we renormalize and consider

$$y_e^2 := \lim_{n \to \infty} \frac{\log(|z_e(s_n)|)}{\sum_{\hat{e} \in E_t^2} \log(|z_{\hat{e}}(s_n)|)}, \qquad e \in E_t^2. \tag{3.9}$$

Repeating the same steps, we end up with subsets $E_t =: E_t^1 \supsetneq E_t^2 \supsetneq \cdots \supsetneq E_t^r \supsetneq E_t^{r+1} = \varnothing$ and corresponding limit values $y =: y^1 \in \sigma_{E_t}$, $y^2 \in \sigma_{E_t^2}$, $\cdots$, $y^r \in \sigma_{E_t^r}$.

Loosely speaking, the idea behind the topology of B^{hyb} is as follows: sequences $(s_n)_n \subseteq B^*$ approaching t in B have different limit points in B^{hyb}, according to the asymptotic behavior given by the values $(y^1, y^2, \cdots, y^r)$.

After these considerations, we now proceed with the formal definition of the space B^{hyb}. Recall from Section 2.2 that an ordered partition of a (non-empty) subset $F \subseteq E$ is an ordered sequence $\pi = (\pi_i)_{i=1}^r$ of non-empty, pairwise disjoint subsets $\pi_i \subseteq F$ such that $F = \bigsqcup_{i=1}^r \pi_i$. Recall as well that if $F = \varnothing$, then we set $\pi_\varnothing = \varnothing$ as its only ordered partition. The set of ordered partitions of a subset $F \subseteq E$ is denoted by $\Pi(F)$ (in particular, $\Pi(\varnothing) = \{\pi_\varnothing\}$). We denote by Π the union of $\Pi(F)$ for $F \subseteq E$.

For any subset $F \subseteq E$, the standard simplex σ_F in $\mathbb{R}^F$ is denoted by

$$\sigma_F = \Big\{(x_e)_e \in \mathbb{R}^F_{\geqslant 0} \;\Big|\; \sum_{e \in F} x_e = 1\Big\}$$

and its relative interior σ_F° is given by

$$\sigma_F^\circ = \Big\{(x_e)_e \in \sigma_F \;\Big|\; x_e > 0 \text{ for all } e \in F\Big\}.$$

For an ordered partition $\pi = (\pi_1, \dots, \pi_r) \in \Pi(F)$ of a subset $F \subseteq E$, we define its *hybrid stratum* as

$$D_\pi^{hyb} := D_\pi^\circ \times \sigma_\pi^\circ := D_\pi^\circ \times \sigma_{\pi_1}^\circ \times \sigma_{\pi_2}^\circ \cdots \times \sigma_{\pi_r}^\circ, \tag{3.10}$$

where $D_\pi^\circ := D_F^\circ$. For $\pi = \pi_\varnothing$ (i.e., the empty ordered partition of $F = \varnothing$), we set

$$D_{\pi_\varnothing}^{hyb} := D_\varnothing^\circ = B^*.$$

As a set, we introduce the *hybrid space* B^{hyb} as the disjoint union

$$B^{hyb} := \bigsqcup_{F \subseteq E} \bigsqcup_{\pi \in \Pi(F)} D_\pi^{hyb} = B^* \sqcup \bigsqcup_{\varnothing \subsetneq F \subseteq E} \bigsqcup_{\pi \in \Pi(F)} D_\pi^{hyb}. \tag{3.11}$$

In the sequel, elements of B^{hyb} will be written as pairs $\mathbf{t} = (t, x)$, meaning that $t \in D_\pi^\circ$ and $x \in \sigma_\pi^\circ$ for some ordered partition $\pi \in \Pi$.

In the last section of this paper, we will also work with a specific type of subspaces of B^{hyb}. Namely, fix two ordered partitions π and π' in Π satisfying $\pi' \preceq \pi$ (w.r.t. to the partial order "$\preceq$" on Π introduced in Section 2.2). Then we denote by $B^{hyb}_{[\pi,\pi']}$ the subspace

$$B^{hyb}_{[\pi,\pi']} := \bigsqcup_{\substack{\varrho \in \Pi \\ \pi' \preceq \varrho \preceq \pi}} D_\varrho^{hyb} \subseteq B^{hyb} \tag{3.12}$$

corresponding to the closed interval $[\pi', \pi] = \{\varrho \colon \pi' \preceq \varrho \preceq \pi\}$ in Π.

3.3. **The hybrid topology.** It remains to equip B^{hyb} with a suitable topology. Motivated by (3.5) – (3.9), this will be done by using normalized logarithmic coordinates. Let (W, z) be an adapted coordinate neighborhood for some point $t \in B$ (see Section 3.1). For a subset $F \subseteq E_t$, introduce the following Log-map

$$\begin{array}{rccc} \mathrm{Log}_F \colon & W \backslash \bigcup_{e \in F} D_e & \longrightarrow & \sigma_F^\circ \\ & s & \mapsto & \left(\frac{\log|z_e(s)|}{\sum_{\hat{e} \in F} \log|z_{\hat{e}}(s)|}\right)_{e \in F}. \end{array} \tag{3.13}$$

We will define a topology on B^{hyb} by specifying for each point $\mathbf{t} \in B^{hyb}$ a system of neighborhoods $\mathcal{U}(\mathbf{t})$. It suffices to define a neighborhood base $\big(U^\varepsilon(\mathbf{t})\big)_{\varepsilon > 0}$ around each $\mathbf{t} \in B^{hyb}$. The total system of neighborhoods $\mathcal{U}(\mathbf{t})$ will then consist of all sets V with $V \supseteq U^\varepsilon(\mathbf{t})$ for some $\varepsilon > 0$.

Hence, assume that $\mathbf{t} \in B^{hyb}$ and $\varepsilon > 0$ are fixed. Suppose that $\mathbf{t} = (t, x) \in D_\pi^{hyb}$ for the ordered partition π and fix some adapted coordinate neighborhood (W, z) of t. Bearing in mind the decomposition (3.10), we will define $U^{\varepsilon}(\mathbf{t})$ by a similar decomposition. More precisely, introduce

$$U^{\varepsilon}(\mathbf{t}) := \bigcup_{\pi' \preceq \pi} U^{\varepsilon}_{\pi'},$$

where the subset $U^{\varepsilon}_{\pi'} \subseteq D^{hyb}_{\pi'}$ is defined as follows:
Assume that $\pi' = (\pi_i)_{i=1}^{r}$ and that $\pi' \preceq \pi$. Recall that this implies that $E_{\pi'} \subseteq E_{\pi}$ and that π has the following form (see Section 2.2)

$$\pi = \Big((\varrho^i)_{i=1}^{r}, \varrho \Big), \tag{3.14}$$

where $\varrho^i = (\varrho^i_k)_k$ is an ordered partition of π'_i for $i = 1, \dots, r$ and the last part $\varrho = (\varrho_k)_k$ is an ordered partition of $E_{\pi} \backslash E_{\pi'}$. We allow that $E_{\pi} = E_{\pi'}$ and in this case, $\varrho = \varnothing$ is not present.
A point $\mathbf{s} = (s, y) \in D^{hyb}_{\pi'}$ belongs to $U^{\varepsilon}_{\pi'}$ if the following set of conditions is satisfied:

(i) For each complex coordinate z_i on W, that is, for $i = 1, ..., N$,
$$|z_i(s) - z_i(t)| < \epsilon.$$

(ii) Assume that $E_{\pi'} \subsetneq E_{\pi}$, that is, $\varrho = (\varrho_k)_k$ is non-trivial. Then
$$\max_k \left\{ \frac{\sum_{e \in \varrho_{k+1}} \log |z_e(s)|}{\sum_{e \in \varrho_k} \log |z_e(s)|} \right\} < \varepsilon \quad \text{and} \quad \max_k \Big\{ \| \mathrm{Log}_{\varrho_k}(s) - x_{\varrho_k} \|_{\infty} \Big\} < \varepsilon, \tag{3.15}$$
where $x_{\varrho_k} = (x_e)_{e \in \varrho_k} \in \sigma^{\circ}_{\varrho_k}$.

(iii) For each $i \in \{1, ..., r\}$, consider the ordered partition $\varrho^i = (\varrho^i_k)_{k=1}^{r_i}$ of π'_i. Then
$$\max_k \left\{ \frac{\sum_{e \in \varrho^i_{k+1}} y_e}{\sum_{e \in \varrho^i_k} y_e} \right\} < \varepsilon \qquad \text{and} \qquad \max_k \max_{e \in \varrho^i_k} \left| \frac{y_e}{\sum_{\hat{e} \in \varrho^i_k} y_{\hat{e}}} - x_e \right| < \varepsilon. \tag{3.16}$$

Notice that by Proposition 3.1, the neighborhood system $\mathcal{U}(\mathbf{t})$ is independent of the choice of the adapted coordinate neighborhood (W, z) (although the $U^{\varepsilon}(\mathbf{t})$'s are not). It turns out that this indeed defines a topology on B^{hyb}:

Theorem 3.2. *There is a unique topology on B^{hyb} such that $\mathcal{U}(\mathbf{t})$ coincides with the system of neighborhoods for any $\mathbf{t} \in B^{hyb}$.*

Proof. It suffices to verify that the set systems $\mathcal{U}(\mathbf{t})$, $\mathbf{t} \in B^{hyb}$, satisfy the axioms of a neighborhood system and this is a straightforward computation. □

Although the definition of the topology on B^{hyb} is slightly involved, there is a simple description of the convergence of sequences.

Proposition 3.3. *Let $\mathbf{t} \in B^{hyb}$ and assume $\mathbf{t} = (t, x) \in D^{hyb}_{\pi}$ for the ordered partition $\pi \in \Pi$. Suppose that $(\mathbf{t}_n)_n = (t_n, x_n)_n$ is a sequence in B^{hyb}. Then:*

(a) If $\mathbf{t}_n$ converges to $\mathbf{t}$ in B^{hyb}, then almost all $\mathbf{t}_n$ belong to hybrid strata $D^{hyb}_{\pi'}$ of ordered partitions π' with $\pi' \preceq \pi$.

(b) Assume that $(\mathbf{t}_n)_n \subseteq D^{hyb}_{\pi'}$ for some fixed ordered partition $\pi' \preceq \pi$. In particular, π is of the form (3.14). Then $\mathbf{t}_n$ converges to $\mathbf{t}$ in B^{hyb} if and only if the following conditions hold:

(i) t_n *converges to* t *in* B.

(ii) *Assume that* $E_{\pi'} \subsetneq E_\pi$, *that is,* $\varrho = (\varrho_k)_k$ *is non-trivial (see (3.14)). Then*

$$\lim_{n\to\infty} \mathrm{Log}_{\varrho_k}(t_n) = x_{\varrho_k} \tag{3.17}$$

for all k, *where* $x_{\varrho_k} = (x_e)_{e\in\varrho_k} \in \mathbb{R}^{\varrho_k}$. *Moreover, for coordinates* $e \in \varrho_k$ *and* $e' \in \varrho_{k'}$ *with* $k < k'$,

$$\lim_{n\to\infty} \frac{\log|z_{e'}(t_n)|}{\log|z_e(t_n)|} = 0. \tag{3.18}$$

In the above statements, (W, z) *is a fixed (or equivalently, any) adapted coordinate neighborhood of* t.

(iii) *For every* $i = 1, \dots, r$, *let* $\varrho^i = (\varrho^i_k)_k$ *be the ordered partition of* π'_i *in (3.14). Then for all* k *and* $e \in \varrho^i_k$,

$$\lim_{n\to\infty} \frac{x_{n,e}}{\sum_{\hat e\in\varrho^i_k} x_{n,\hat e}} = x_e. \tag{3.19}$$

Moreover, if $e \in \varrho^i_k$ *and* $e' \in \varrho^i_{k'}$ *with* $k < k'$, *then*

$$\lim_{n\to\infty} \frac{x_{n,e'}}{x_{n,e}} = 0 \tag{3.20}$$

Proof. The claims are simple consequences of the definition of the topology on B^{hyb}. □

Remark 3.4. Proposition 3.3 confirms that the topology on B^{hyb} formalizes the ideas of (3.5)–(3.9). Indeed, (3.17) and (3.18) correspond exactly to the ordering of the singular coordinates z_e, $e \in E_t$ around a point t in groups (according to their logarithmic growth) and then renormalizing. The other conditions (3.19) and (3.20) represent the same idea for the simplicial coordinates. ◇

The following proposition summarizes basic properties of the hybrid topology.

Proposition 3.5. B^{hyb} *has the following topological properties.*

(i) B^{hyb} *is a second countable, locally compact Hausdorff space and in particular,* B^{hyb} *is metrizable. The convergent sequences are described in Proposition 3.3.*

(ii) *The natural projection map* $\mathrm{pr}\colon B^{hyb} \to B$, $\mathbf{t} = (t, x) \mapsto t$ *is continuous. Moreover,* B *is homeomorphic to the quotient space* $B^{hyb}/\sim$, *where* $\mathbf{t} \sim \mathbf{s}$ *for* $\mathbf{t}, \mathbf{s} \in B^{hyb}$ *if and only if* $\mathrm{pr}(\mathbf{t}) = \mathrm{pr}(\mathbf{s})$.

(iii) *The subset* $B^* \subseteq B^{hyb}$ *is open in* B^{hyb}. *Moreover, the topology on* B^* *induced by* B^{hyb} *coincides with the one from* B.

(iv) *In general, the hybrid stratum* D^{hyb}_π *of an ordered partition* $\pi \in \Pi$ *is neither open nor closed in* B^{hyb}. *On each* D^{hyb}_π, *the topology induced by* B^{hyb} *coincides with the product topology induced from the product decomposition (3.10).*

3.4. **Relationship to former hybrid spaces.** Our construction presented above refines and combines the previous constructions of hybrid spaces [BJ17] and metrized complexes [AB15] in two aspects.

First, the simplicial boundary in [BJ17], which is the dual complex of the divisor D, is replaced with a space which is itself hybrid. Moreover, this hybrid boundary space can be regarded as a higher rank refinement of the metrized complex $\mathcal{M}\mathscr{C}(D)$ associated to the divisor D [AB15]. Recall that $\mathcal{M}\mathscr{C}(D)$ is obtained by taking the disjoint union $D_F \times \sigma_F$ for all non-empty subsets $F \subseteq E$, and then by identifying for each pair of non-empty subsets $F_1 \subseteq F_2$ the two cells $D_{F_1} \times \sigma_{F_1}$ and $D_{F_2} \times \sigma_{F_2}$ along the common subset $D_{F_2} \times \sigma_{F_1}$. The resulting space is then endowed with the quotient topology. In our construction of hybrid spaces, we have replaced this metrized complex boundary with an *unfolded variant*, defined in terms of ordered partitions.

Second, we include a refined version of *logarithmic convergence* by adding a notion of *depth* to the level of convergence, refining the limit further according to the relative speed of convergence of the sequence.

We briefly discuss a comparison of these constructions.

Denote by $B^{hyb(0)}$ the hybrid space from [BJ17]. Recall that $B^{hyb(0)} = B^* \sqcup \Sigma(D)$ where $\Sigma(D)$ is the simplicial complex which for every non-empty subset $F \subset E$ and each connected component $Z \subseteq D_F$, contains one copy σ_Z of the standard simplex σ_F.

We have a forgetful projection map $\mathfrak{q}_0 : B^{hyb} \to B^{hyb(0)}$ defined as follows. The map $\mathfrak{q}_0$ restricts to identity on the open subset $B^* \subset B^{hyb}$. For a non-empty subset $F \subseteq E$ and an ordered partition $\pi = (\pi_1, \dots, \pi_r)$, a point $\mathbf{t} = (t, x) \in D_F^{hyb} = D_F^\circ \times \sigma_\pi^\circ$ has image $\mathfrak{q}_0(\mathbf{t}) \in \Sigma(D)$ obtained as follows. Let Z be the connected component of D_F which contains t. Let x^1 be the projection of x to $\sigma_{\pi_1}^\circ$ via the product decomposition $\sigma_\pi^\circ = \sigma_{\pi_1}^\circ \times \dots \times \sigma_{\pi_r}^\circ$. Then, the projection sends $\mathbf{t}$ to $\mathfrak{q}_0(\mathbf{t}) := x^1 \in \sigma_{\pi_1}^\circ \hookrightarrow \sigma_Z \simeq \sigma_F$, seen as an element of the copy σ_Z of σ_F.

Proposition 3.6. *The map* $\mathfrak{q}_0 : B^{hyb} \to B^{hyb(0)}$ *is continuous.*

We omit a formal proof here.

We can actually go further and define a tower of hybrid spaces

$$B^{hyb(0)} \longleftarrow B^{hyb(1)} \longleftarrow B^{hyb(2)} \longleftarrow \dots \longleftarrow B^{hyb(|E|)-1} \longleftarrow B^{hyb(|E|)} = B^{hyb} \tag{3.21}$$

interpolating between B^{hyb} and $B^{hyb(0)}$.

To see this, fix an integer $r \in \mathbb{N}$. We define $B^{hyb(r)}$ as follows. Let Π_r be the set of all ordered partitions of subsets of E which have depth bounded by r. For each subset $F \subseteq E$ and ordered partition $\pi = (\pi_1, \dots, \pi_k)$ of F, with $k \leqslant r$, define

$$D_F^{hyb(r)} := \begin{cases} D_F^\circ \times \sigma_{\pi_1}^\circ \times \dots \times \sigma_{\pi_{r-1}}^\circ \times \sigma_{\pi_r} & \text{if } k = r \\ D_F^\circ \times \sigma_{\pi_1}^\circ \times \dots \times \sigma_{\pi_{k-1}}^\circ \times \sigma_{\pi_k}^\circ & \text{otherwise.} \end{cases}$$

Note that in the case $k = r$, for the last simplex σ_{π_r} in the above product, we allow coordinates to take value zero. For $\pi = \pi_\varnothing$, we set $D_{\pi_\varnothing}^{hyb(r)} = B^*$.

We define $B^{hyb(r)}$ as the disjoint union

$$B^{hyb(r)} := \bigsqcup_{F\subseteq E} \bigsqcup_{\pi\in\Pi_r(F)} D_\pi^{hyb(r)} = B^* \sqcup \bigsqcup_{\varnothing\subsetneq F\subseteq E} \bigsqcup_{\pi\in\Pi_r(F)} D_\pi^{hyb(r)}. \tag{3.22}$$

In particular we remark that the hybrid boundary $D^{hyb(1)} := B^{hyb(1)}\backslash B^* \subseteq B^{hyb(1)}$ coincides with the metrized complex associated to D. Moreover, by definition B^{hyb} is exactly the last space in this chain, that is, $B^{hyb} = B^{hyb(|E|)}$.

We define the hybrid topology on $B^{hyb(r)}$ by a construction similar to the one described for B^{hyb}.

For each $j > i$, we get moreover a forgetful map $\mathfrak{q}_{j>i} : B^{hyb(j)} \to B^{hyb(i)}$ defined as follows. The map $\mathfrak{q}_{j>i}$ restricts to identity on any stratum $D_\pi^{hyb(j)}$ with $\pi\in\Pi_i$. For an element $\pi = (\pi_1,\dots,\pi_k)\in\Pi_k$ with $k \geqslant i+1$, let $\pi' := (\pi_1,\dots,\pi_{i-1},\pi_i\cup\dots\cup\pi_k)$. Then the restriction of $\mathfrak{q}_{j>i}$ to $D_\pi^{hyb(j)}$ has image in the stratum $D_{\pi'}^{hyb(i)}$,

$$\mathfrak{q}_{j>i} : D_\pi^{hyb(j)} \to D_{\pi'}^{hyb(i)},$$

and it is given by sending a point $\mathbf{t} = (t, x = (x^1,\dots,x^k))$ of $D_\pi^{hyb(j)}$ to the point (t,y) with $y = (x^1,\dots,x^{i-1}, y^i = (x^i,0,\dots,0))$.
Similarly, we define the map $\mathfrak{q}_i : B^{hyb} \to B^{hyb(i)}$. This leads to the tower of hybrid spaces given in (3.21), which consist of continuous maps, as stated in the following general form of Proposition 3.6.

Proposition 3.7. *Notations as above, the maps $\mathfrak{q}_{j>i}$ and $\mathfrak{q}_i$ are continuous for all pairs of non-negative integers $j > i$.*

4. Hybrid deformation spaces and moduli of hybrid curves

The aim of this section is to construct the moduli spaces of hybrid curves $\mathscr{M}_g^{hyb}$ and $\mathscr{M}_{g,n}^{hyb}$ as well as the universal hybrid curves $\mathscr{C}_g^{hyb}$ and $\mathscr{C}_{g,n}^{hyb}$ which lie above them.

In order to simplify the presentation, we refrain to develop a theory of hybrid Deligne-Mumford stacks in this paper, and only consider coarse moduli spaces. In this setting, the terminology of universal curve is used by an abuse of the notation, since in the presence of automorphisms, the fibers of the family are the corresponding hybrid curves quotiented out by the action of their automorphism groups. In proving the continuity of the canonical measures in our main theorem, however, we reduce to showing the continuity over hybrid replacements of versal deformation spaces. These provide étale charts for the fine moduli space and thus the continuity theorem holds more strongly in the setting of hybrid Deligne-Mumford stacks. We will elaborate on the hybrid replacement of toroidal embeddings and Deligne-Mumford stacks in our future work.

We will thus proceed by applying first the construction of the previous section to the deformation space of a stable curve, which provide local étale charts in $\overline{\mathscr{M}}_{g,n}$ around points. These local hybrid charts will be used in order to define the hybrid topology.

In order to define $\mathscr{M}_g^{hyb}$, set theoretically, we will define hybrid strata associated to stable graphs and then take their disjoint union.

Finally, we define the universal hybrid curve and describe its topology.

We start by recalling basic facts about the deformation theory of stable curves. In order to perform an induction step in the last part of the proof of our main theorem, we also need to more generally consider deformations of stable curves with markings. So we will explain how the constructions extend in this more general setting.

4.1. **Deformations of stable curves.** This section recalls basic standard results about deformations and degenerations of complex stable curves, with or without markings, which will be used in the construction of the hybrid moduli spaces as well as in the study of the monodromy and period maps, later in Section 7. Our main reference is [ABBF16], which we follow closely, and refer to [ACG11, DM69, Hof84] for more details.

Let S_0 be a stable curve with dual graph $G = (V, E, \mathfrak{g})$. For $v \in V$, we denote by C_v the (normalization of the) corresponding irreducible component of S_0 and let $\mathfrak{g} : V \to \mathbb{Z}_{\geqslant 0}$ be the *genus function* which associates to each vertex v of G the integer $\mathfrak{g}(v)$ the genus of C_v. We denote by g the arithmetic genus of S_0, which, we recall (see below), is equal to

$$g = h + \sum_{v \in V} \mathfrak{g}(v).$$

Here h denotes the first Betti number of G, i.e., $h = \operatorname{rank} H_1(G)$. Each edge $e = uv$ corresponds to a point of intersection of C_u and C_v in S_0. These points are denoted by p_u^e and p_v^e in C_u and C_v, respectively. In this way, for each vertex v in G, we get a *marked Riemann surface of genus* $\mathfrak{g}(v)$ *with* $\deg(v)$ *marked points* p_v^e in correspondence with edges e of the graph incident to v. The collection of these marked points on C_v is denoted by $\mathcal{A}_v$.

4.1.1. *The genus formula.* Let $\mathcal{O}_{S_0}$, resp. ω_{S_0}, be the sheaf of holomorphic functions, resp. holomorphic one-forms, on S_0. Recall that the arithmetic genus of S_0 is by definition the dimension of $H^1(S_0, \mathcal{O}_{S_0})$. By Serre duality this coincides with the dimension of $H^0(S_0, \omega_{S_0})$. We have the following proposition, see e.g. [ABBF16].

Proposition 4.1. *The natural projection map* $\mathfrak{P} : \coprod_{v \in V} C_v \to S_0$ *induces a canonical isomorphism*

$$H^1(G, \mathbb{C}) \simeq \ker\Big(H^1(S_0, \mathcal{O}_{S_0}) \xrightarrow{\mathfrak{P}^*} \bigoplus_{v \in V} H^1(C_v, \mathcal{O}_{C_v}) \Big).$$

Moreover, we have

$$H^1(G, \mathbb{Z}) \simeq \ker\Big(H^1(S_0, \mathbb{Z}) \xrightarrow{\mathfrak{P}^*} \bigoplus_{v \in V} H^1(C_v, \mathbb{Z}) \Big).$$

It follows that the arithmetic genus of S_0 is equal to $h + \sum_{v \in V} \mathfrak{g}(v)$.

Proof. The proof of the first statement can be obtained by applying the long exact sequence of cohomology to the exact sequence of sheaves

$$0 \to \mathcal{O}_{S_0} \to \mathfrak{P}_* \mathcal{O}_{\coprod C_v} \xrightarrow{\varphi} \mathscr{S} \to 0, \tag{4.1}$$

where $\mathscr{S}$ is a skyscraper sheaf with stalk $\mathbb{C}$ over each singular point of S_0 and the map $\varphi = (\varphi_e)_{e \in E}$ is defined as follows. First choose an orientation of the edges of G. Now, if f is a local section of $\mathfrak{P}_* \mathcal{O}_{\coprod C_v}$ near the singular point p^e corresponding to the edge $e = \{w, v\}$ with its orientation $\vec{e} = \vec{wv}$, then set $\varphi_e(f) = f(p_v^e) - f(p_w^e)$.

The proof of the second assertion goes in the same way, replacing the exact sequence (4.1) by the analogous sequence of constructible sheaves calculating Betti cohomology. □

4.1.2. *Formal deformations.* Let S_0 be a stable curve of arithmetic genus g. Standard results in deformation theory provide a smooth formal scheme $\widehat{B} = \mathrm{Spf}\,\mathbb{C}[[t_1, \dots, t_N]]$ and a versal formal family of curves $\mathfrak{p}\colon \widehat{\mathcal{S}} \to \widehat{B}$ with fiber $\mathcal{S}_0$ over the point $0 \in \widehat{B}$ isomorphic to S_0.

The total space $\widehat{\mathcal{S}}$ is formally smooth over $\mathbb{C}$ and the tangent space T to $\widehat{B}$ at 0 can be identified with the Ext group $\mathrm{Ext}^1(\Omega^1_{S_0}, \mathcal{O}_{S_0})$ in the category of sheave over S_0. Locally for the étale topology at a singular point of S_0, which corresponds to an edge of the dual graph G, we have

$$S_0 \simeq \mathrm{Spec}\, R, \quad \text{for } R = \mathbb{C}[x, y]/(xy). \tag{4.2}$$

This implies that locally we have an isomorphism $\Omega^1_{S_0} \simeq Rdx \oplus Rdy/(xdy + ydx)$. Moreover, the element $xdy \in \Omega^1_{S_0}$ is killed by both x and y. This means that $\Omega^1_{S_0}$ has a non-trivial torsion subsheaf supported at the singular points of S_0.

By local-to-global Ext-spectral sequence, since all the higher terms in the spectral sequence are vanishing, we get the following short exact sequence

$$0 \to H^1(S_0, \underline{\mathcal{H}om}(\Omega^1_{S_0}, \mathcal{O}_{S_0})) \to T \to \Gamma(S_0, \underline{\mathcal{E}xt}^1(\Omega^1_{S_0}, \mathcal{O}_{S_0})) \to 0. \tag{4.3}$$

By local presentation (4.2) at the singular points, we infer that the $\mathcal{E}xt$ sheaf on the right hand side of the equation above is locally given by the extension

$$\begin{aligned} 0 \to R &\longrightarrow Rdx \oplus Rdy \to \Omega^1_R \to 0 \\ 1 &\longmapsto xdy + ydx. \end{aligned}$$

In other words, $\underline{\mathcal{E}xt}^1(\Omega^1_{S_0}, \mathcal{O}_{S_0})$ is the skyscraper sheaf which contains one copy of $\mathbb{C}$ supported at each singular point. This implies that the space of global sections of $\underline{\mathcal{E}xt}^1(\Omega^1_{S_0}, \mathcal{O}_{S_0})$ is given by

$$\Gamma\big(S_0, \underline{\mathcal{E}xt}^1(\Omega^1_{S_0}, \mathcal{O}_{S_0})\big) \simeq \mathbb{C}^E. \tag{4.4}$$

By the results of Deligne-Mumford [DM69], the global sections $\Gamma\big(S_0, \underline{\mathcal{E}xt}^1(\Omega^1_{S_0}, \mathcal{O}_{S_0})\big)$ correspond to smoothings of the singular points of S_0. Those deformations in which the dual graph remains the same (and so only the marked curves $(C_v, \mathcal{A}_v)$ deform) are moreover represented by the subspace $H^1(S_0, \underline{\mathcal{H}om}(\Omega^1_{S_0}, \mathcal{O}_{S_0})) \subset T$. (Recall that the set of marked points $\mathcal{A}_v$ is in bijection with the edges in G which are incident to v.)

For any pair of non-negative integers a and n, the moduli space $\mathscr{M}_{a,n}$ of Riemann surfaces of genus a with n marked points has dimension $3a - 3 + n$. It follows that for each vertex v of the graph, the pairs $(C_v, \mathcal{A}_v)$ have $3\mathfrak{g}(v) - 3 + \deg(v)$ moduli. This leads to the following equality of dimensions

$$\dim H^1(S_0, \underline{\mathcal{H}om}(\Omega^1_{S_0}, \mathcal{O}_{S_0})) = \sum_{v \in V} (3\mathfrak{g}(v) - 3 + \deg(v)), \qquad \text{and}$$

$$\dim T = \sum_{v \in V} (3\mathfrak{g}(v) + \deg(v) - 3) + |E| = 3g - 3.$$

The above analysis also shows that deformations of S_0 which preserve the singularity at p^e are given by a divisor $\widehat{D}_e \subset \widehat{B}$. Denoting by $z_e \in \mathcal{O}_{\widehat{B}}$ the local parameters around 0 for the divisors $\widehat{D}_e$, for $e \in E$, the projection map $T \to \mathbb{C}^E \xrightarrow{pr_e} \mathbb{C}$ is seen to be defined by dz_e, and that the surjective map $T \to \mathbb{C}^E$ in (4.3) is given precisely by the differentials of these local parameters. It thus follows that we have a collection of principal divisors $\widehat{D}_e \subset \widehat{B}$ indexed by the edges of G which meet transversally. Moreover, the intersection of these divisors in $\widehat{B}$ is precisely the locus of those deformations of S_0 which keep the dual graph fixed.

More generally, we get the following correspondence. For any subset $F \subseteq E$, the intersection of the divisors $\widehat{D}_e$ for $e \in E$ is the locus of those deformations whose dual graph contains all the edges $F \subseteq E$. This is naturally stratified into those deformations whose dual graph is obtained by contracting a subset of edges $A \subseteq E \backslash F$. This dual graph is denoted by G/A.

In addition to local parameters z_e for edges $e \in E$, we have $N - |E|$ more local parameters which correspond to the deformations which preserve the dual graph.

4.1.3. *Formal deformations with markings.* Let $(S_0, q_1, \dots, q_n)$ be a stable curve with n marked points. Let $G = (V, E, \mathfrak{g}, \mathfrak{m})$ be the corresponding stable dual graph with n marked points. Recall that $\mathfrak{m} : [n] \to V$ is the marking function which associates to any integer $i \in [n] = \{1, \dots, n\}$ a vertex of the graph (representing the component on which the ith marked point q_i lies). Let $\mathfrak{n} : V \to \mathbb{N} \cup \{0\}$ be the counting function which at each vertex $v \in V$, takes value $\mathfrak{n}(v)$ given by the number of elements $i \in [n]$ with $\mathfrak{m}(i) = v$. The stability condition means the inequality

$$\mathfrak{g}(v) + \deg(v) + \mathfrak{n}(v) \geqslant 0$$

for any vertex v of the dual graph. Moreover, in the case $g = 1$ and S_0 is smooth of genus one, the stability means $n \geqslant 1$.

Similarly as in the previous section, for a complex stable curve $(S_0, q_1, \dots, q_n)$ of arithmetic genus g with n marked points, there exists a formal disc $\widehat{B}$ of dimension $N = 3g - 3 + n$ and a versal formal deformation $\mathfrak{p} \colon \widehat{\mathcal{S}} \to \widehat{B}$ such that the tangent space to $0 \in \widehat{B}$ is identified with $\mathrm{Ext}^1(\Omega^1_{S_0}, \mathcal{O}_{S_0}(-\sum_{i=1}^n q_i))$. The fiber at 0 of the family is isomorphic to S_0, and in addition, the family comes with sections $\sigma_i \colon \widehat{B} \to \widehat{\mathcal{S}}$ such that $\sigma_i(0) = q_i$.

The local study of the previous section is carried out in a similar way, and leads to local parameters z_e for $e \in E$ leading to formal divisors $\widehat{D}_e \subset \widehat{B}$ indexed by the edges of G which meet transversally. In addition to local parameters z_e for edges $e \in E$, we have $N - |E| = 3g - 3 + n - |E|$ more local parameters, n of which correspond to the deformations of the n markings, and the rest of the parameters correspond to the deformations of the stable curve which preserve the dual graph.

As above, we get the following correspondence. For any subset $F \subseteq E$, the intersection of the divisors $\widehat{D}_e$ for $e \in E$ is the locus of those deformations whose dual graph contains all the edges $F \subseteq E$. This is naturally stratified into those deformations whose dual marked graph is obtained by contracting a subset of edges $A \subseteq E \backslash F$, whose dual stable graph with n markings is the one obtained by taking the stable dual graph with marking G/A, as defined in Section 2. Recall that the marking function in this graph is defined as the composition of the map $\mathfrak{m} : [n] \to V(G)$ with the projection map $\mathscr{K} : V(G) \to V(G/A)$. We denote this stable marked graph still with G/A, as it will be clear from the context whether markings will be involved or not.

The discussion of the previous section corresponds to the case $n = 0$.

4.1.4. *Analytification.* From the local theory above using formal schemes, we get analytic deformations $\mathcal{S} \to B$ of stable Riemann surfaces (with markings if $n \neq 0$) over a polydisc B of dimension $N = 3g-3+n$. This means that $B = \underbrace{\Delta \times \Delta \times \cdots \times \Delta}_{N \text{ times}}$ for Δ a small disk around 0 in $\mathbb{C}$. Moreover, the formal divisors above gives rise to analytic divisors $D_e \subset B$ which are defined by equations $\{z_e = 0\}$ for analytic local parameters z_e for edges $e \in E$. Once again, D_e is the locus of all points $t \in B$ such that in the family $\mathfrak{p} : \mathcal{S} \to B$, the fiber $\mathfrak{p}^{-1}(t)$ is a Riemann surface which has a singular point corresponding to e.

In the following, B will denote the base of the versal deformation of the stable curve S_0, or the stable marked curve $(S_0, q_1, \dots, q_n)$, and we set $B^* := B \setminus \bigcup_{e \in E} D_e$ and denote by $\mathcal{S}^* := \mathfrak{p}^{-1}(B^*)$ the locus of points whose fibers in the family π is a smooth Riemann surface.

We follow our convention in the introduction and deliberately use the terminology complex curve when referring to the Riemann surface in this family.

4.2. **Hybrid deformation space.** By the discussion which just preceded and that of Section 3, we assume now that $B = \Delta^N$ is a polydisc in $\mathbb{C}^N$ and $D = \bigcup_{e \in E} D_e$, where the divisors D_e, $e \in E$, are given by (w.l.o.g., we can assume that $E = \{1, ..., |E|\} \subseteq \{1, ...N\}$)

$$D_e = \{z \in \Delta^N | z_e = 0\}.$$

The corresponding hybrid space B^{hyb} will serve as the base space for the hybrid versal deformation. Notice that in this setting, adapted coordinate neighborhoods (U, z) can be defined in a particularly simple way: for each $t \in B$, we can choose $U = B \setminus \bigcup_{e : t_e \neq 0} D_e$ and take $z = (z_i)_i$ as the standard coordinates on $U \subseteq \Delta^N$.

4.3. **Versal curve over hybrid deformation space.** In order to simplify the presentation, in what follows we omit any mention of the markings, the presentation takes place in the more general setting of stable curves with n marked points.

In this section, we introduce the versal hybrid curve $\mathfrak{p}^{hyb} : \mathcal{S}^{hyb} \to B^{hyb}$ associated to a stable curve S_0. The idea is to replace non-smooth fibers $\mathcal{S}_t$ in the versal curve $\mathfrak{p} \colon \mathcal{S} \to B$ by hybrid curves. More precisely, for a hybrid point $\mathbf{t} = (t, x) \in B^{hyb}$, its fiber $\mathcal{S}_{\mathbf{t}}^{hyb}$ will be the hybrid curve obtained by replacing the singular points in $\mathcal{S}_t$ with intervals of lengths given by x. The corresponding ordered partition π of the edge set is the unique one such that $\mathbf{t} \in D_\pi^{hyb}$.

More precisely, let S_0 be a stable curve with stable dual graph $G = (V, E, \mathfrak{g})$ and let $\mathfrak{p} \colon \mathcal{S} \to B$ be its associated versal deformation family (see Section 4.1). The hybrid versal deformation family $\mathcal{S}^{hyb}$ will be defined over the hybrid deformation space B^{hyb} (see Section 4.2). In accordance with the decomposition

$$B^{hyb} = B^* \sqcup \bigsqcup_{\varnothing \subsetneq F \subseteq E} \bigsqcup_{\pi \in \Pi(F)} D_\pi^{hyb},$$

we will introduce $\mathcal{S}^{hyb}$ by a similar decomposition. Namely, we will define

$$\mathcal{S}^{hyb} := \mathcal{S}^* \sqcup \bigsqcup_{\varnothing \subsetneq F \subseteq E} \bigsqcup_{\pi \in \Pi(F)} \mathcal{S}_\pi^{hyb} \tag{4.5}$$

where each $\mathfrak{p}_\pi^{hyb} : \mathcal{S}_\pi^{hyb} \to D_\pi^{hyb}$ is a family of hybrid curves defined over D_π^{hyb}. Moreover, we will define a suitable topology on $\mathcal{S}^{hyb}$ which is again closely related to the decomposition (4.5). The construction of $\mathcal{S}_\pi^{hyb}$ and its topology will be be given below.

4.3.1. *Normalized families.* We first recall from Section 4.1 that the fiber $\mathcal{S}_t$ of each $t \in B$ is a stable Riemann surface with exactly $|E_t|$ singular points $p^e(t)$, $e \in E_t$. Its associated stable dual graph is the graph $G_t := G/(E\backslash E_t)$ obtained by contracting all the edges in $E\backslash E_t$. Moreover, the stable Riemann surface $\mathcal{S}_t$ has precisely $|V(G_t)|$ irreducible components C_v, $v \in V(G_t)$, and the sections $p^e \colon D_e \to \mathcal{S}$, $e \in E_t$, are continuous. In what follows, it will be convenient to denote the set of all singular points of fibers in the family $\mathcal{S}/B$ associated to an edge $e \in E$ as

$$P^e := p^e(D_e) = \Big\{ p^e(t) \,\Big|\, t \in D_e \Big\} \subseteq \mathcal{S} \tag{4.6}$$

and consider the complement

$$P^0 := \mathcal{S} \setminus \bigcup_{e \in E} P^e \subseteq \mathcal{S}. \tag{4.7}$$

In other words, P^0 consists of all the points in $\mathcal{S}$ which are also smooth points of their corresponding fiber. We will refer to such points as *normal smooth points*. However, let us stress that the total family $\mathcal{S}$ is a smooth complex manifold and hence all points $p \in \mathcal{S}$ are smooth points of $\mathcal{S}$. Notice that each P^e is a closed subset of $\mathcal{S}$ and in particular, $P^0 \subseteq \mathcal{S}$ is open.

For each point $p \in P^e$, there is a small neighborhood U of p in $\mathcal{S}$ and local coordinates

$$z = \Big((z_i)_{i \neq e}, z_u^e, z_v^e \Big) \tag{4.8}$$

with the following properties:

(i) All coordinates have absolute value smaller than one, that is, $|z_u^e|$, $|z_v^e|$ and $|z_i| < 1$ on U for all $i = 1, \dots N$, $i \neq e$, and

(ii) the coordinates are compatible with the base B in the sense that for all $q \in U$,
$$\mathfrak{p}(q)_i = z_i(q), \qquad i \neq e,$$
and $\mathfrak{p}(q)_e = z_u^e(q) z_v^e(q)$, and

(iii) for each base point $t \in \mathfrak{p}(U)$, we have
$$C_u \cap U = \Big\{ q \in U \;\Big|\; \mathfrak{p}(q) = t \text{ and } z_v^e(q) = 0 \Big\}$$
and the same holds true for C_v and z_u^e. Here, C_u and C_v are the smooth components of the fiber $\mathcal{S}_t$ corresponding to the vertices u and v of the dual graph G_t of $\mathcal{S}_t$ with $e = uv$. In particular, $P^e \cap U = \Big\{ q \in U \;\Big|\; z_u^e(q) = z_v^e(q) = 0 \Big\}$.

In the following, we call a pair (U, z) of a neighborhood U and local coordinates z on U with the above properties a *standard coordinate neighborhood* of $p \in P^e$ on $\mathcal{S}$.

As the first step in the definition of $\mathcal{S}_\pi^{hyb}$ (see (4.5)), we construct the *Riemann surface components* for the hybrid curves described above.

Let π be a fixed ordered partition of a non-empty subset $F \subseteq E$. From the preceding discussion, it is clear that for each point $t \in D_F^\circ$, the fiber $\mathcal{S}_t$ is a stable Riemann surface with the corresponding stable dual graph denoted $G_F := G/(E\backslash F)$. Moreover, $\mathcal{S}_t$ has $|V(G_F)|$ components C_v, $v \in V(G_F)$ and $|F|$ singular points $p^e(t)$, $e \in F$. The sections $p^e\colon D_F^\circ \to \mathcal{S}_F$ of the restricted family $\mathcal{S}_F := \mathcal{S}|_{D_F^\circ} \to D_F^\circ$ are moreover continuous. Note that by an abuse of the notation, we identify the set F with the set of edges of the contracted graph G_F.

Consider now the corresponding *normalized family*

$$\widetilde{\mathcal{S}}_F \to D_F^\circ.$$

More precisely, in each fiber $\mathcal{S}_t$, we consider the singular points $p^e(t)$, $e = uv \in F$, and disconnect the smooth components C_u and C_v. That is, we replace the point $p^e(t)$ by two different points $p_u^e(t)$ and $p_v^e(t)$, which lie on C_u and C_v, thereby making the components disjoint. This can be easily made precise using standard coordinate neighborhoods (it amounts to replacing points q with $z_u^e(q) = z_v^e(q) = 0$ by two new points, as in the normalization of a single fiber).
Altogether we get a holomorphic family $\widetilde{\mathcal{S}}_F/D_F^\circ$. Each fiber of $\widetilde{\mathcal{S}}_F$ is the disjoint union of the smooth Riemann surfaces C_v, $v \in V(G_F)$ (the smooth irreducible components of $\mathcal{S}_t$). For each edge $e = uv$ of F, there are corresponding sections $p_u^e, p_v^e\colon D_F^\circ \to \widetilde{\mathcal{S}}_F$ such that p_u^e and p_v^e lie in the components C_u and C_v, respectively. Moreover, $\mathcal{S}_F$ is obtained from $\widetilde{\mathcal{S}}_F$ by glueing the sections p_u^e and p_v^e for each edge $e \in F$ above any point $t \in D_F^\circ$.

Recalling that $D_\pi^{hyb} = D_F^\circ \times \sigma_\pi^\circ$, we can naturally extend $\widetilde{\mathcal{S}}_F \to D_F^\circ$ to

$$\widetilde{\mathcal{S}}_F \times \sigma_\pi^\circ \to D_\pi^{hyb} \tag{4.9}$$

and $p_u^e, p_v^e\colon D_F^\circ \to \widetilde{\mathcal{S}}_F$ to sections

$$\tilde{p}_u^e, \tilde{p}_v^e\colon D_\pi^{hyb} \to \widetilde{\mathcal{S}}_F \times \sigma_\pi^\circ \tag{4.10}$$

for each edge $e = uv$ in F.

For later reference, we also introduce the following family: for each edge $e \in E$, consider the normalized family

$$\breve{\mathcal{S}}_e \to D_e, \tag{4.11}$$

which is constructed in the same way as above by resolving the singular points $p^e(t)$, $t \in B$. We stress that $\breve{\mathcal{S}}_e$ is defined over the full stratum D_e (instead of the interior part). In particular, if F contains the edge e, then $D_F^\circ \subseteq D_e$, and it is also clear from the construction that $\breve{\mathcal{S}}_e$ and $\widetilde{\mathcal{S}}_F$ coincide on D_F°, except along the sections p^f (of $\breve{\mathcal{S}}_e$) which has been separated into two sections p_u^f and p_v^f in $\widetilde{\mathcal{S}}_F$, for any edge $f \neq e$ in F.

In the same way, we see that the construction of $\widetilde{\mathcal{S}}_F$ leaves normal smooth points $p \in P^0$ invariant (see (4.7) for the definition). More formally, we have

$$\Big\{q \in P^0 \,\Big|\, \mathfrak{p}(q) \in D_F^\circ\Big\} \subseteq \widetilde{\mathcal{S}}_F. \tag{4.12}$$

4.3.2. *Metric graph part of hybrid curves.* Next, we will construct the *versal family of intervals* for the hybrid curves. For each edge $e = uv \in F$, define $\mathcal{I}_\pi^e \subseteq \sigma_\pi^\circ \times \mathbb{R}^2_{\geqslant 0}$ by

$$\mathcal{I}_\pi^e = \Big\{(x, \lambda_u, \lambda_v) \in \sigma_\pi^\circ \times \mathbb{R}^2_{\geqslant 0} \;\Big|\; \lambda_u + \lambda_v = x_e\Big\}. \tag{4.13}$$

Clearly, the natural projection map to the first factor $\mathrm{pr}_1 \colon \mathcal{I}_\pi^e \to \sigma_\pi^\circ$ is continuous and the fiber of any point $x = (x_f)_{f\in F} \in \sigma_\pi^\circ$ can be identified with the *standard interval* $[0, x_e]$ of length $x_e > 0$. Moreover, $\mathcal{I}_\pi^e/\sigma_\pi^\circ$ is a fiber bundle, topologically isomorphic to the trivial bundle $(\sigma_\pi^\circ \times [0,1])/\sigma_\pi^\circ$.

It is also clear that the following sections

$$\begin{aligned} s_u^e \colon \sigma_\pi^\circ &\longrightarrow \mathcal{I}_\pi^e, \qquad (x_f)_f \mapsto (x, 0, x_e)\\ s_v^e \colon \sigma_\pi^\circ &\longrightarrow \mathcal{I}_\pi^e, \qquad (x_f)_f \mapsto (x, x_e, 0) \end{aligned}$$

are continuous. Informally, for each point $x = (x_f)_{f\in F}$ in σ_π°, we see its fiber $\mathcal{I}_\pi^e|_x$ as the edge e in the graph G_F (turned into a metric graph with edge lengths given by x). In this picture, $s_u^e(x)$ and $s_v^e(x)$ are the endpoints corresponding to the vertices u and v of e, respectively.

Finally, recalling that $D_\pi^{hyb} = D_F^\circ \times \sigma_\pi^\circ$, we can naturally extend $\mathcal{I}_\pi^e \to \sigma_\pi^\circ$ to

$$D_F^\circ \times \mathcal{I}_\pi^e \to D_\pi^{hyb} \tag{4.14}$$

and $s_u^e, s_v^e \colon \sigma_\pi^\circ \to \mathcal{I}_\pi^e$ to sections

$$\tilde{s}_u^e, \tilde{s}_v^e \colon D_\pi^{hyb} \to D_F^\circ \times \mathcal{I}_\pi^e \tag{4.15}$$

for each edge $e = uv$ in F.

4.3.3. *Construction of $\mathcal{S}^{hyb}$.* In the following, we combine the previous steps to define the family $\mathcal{S}_\pi^{hyb}$ over the hybrid stratum D_π^{hyb}. The idea is to connect the disjoint components C_v in the normalized family $\widetilde{\mathcal{S}}_F$ by putting the interval $\mathcal{I}_\pi^e$ between $\tilde{p}_u^e$ and $\tilde{p}_v^e$.

For every partition π of some non-empty subset $F \subseteq E$, from the preceding considerations we get the *normalized hybrid family*

$$\widetilde{\mathcal{S}}_\pi^{hyb} := \Big(\widetilde{\mathcal{S}}_F \times \sigma_\pi^\circ\Big) \sqcup \Big(\bigsqcup_{e\in F} D_F^\circ \times \mathcal{I}_\pi^e\Big),$$

obtained as the fibered sum over D_π^{hyb} of the families $\widetilde{\mathcal{S}}_F \times \sigma_\pi^\circ$ and $D_F^\circ \times \mathcal{I}_\pi^e$, for $e \in F$. The normalized hybrid family thus comes with a map (see (4.9) and (4.14))

$$\widetilde{\mathfrak{p}}_\pi^{hyb} : \widetilde{\mathcal{S}}_\pi^{hyb} \to D_\pi^{hyb}$$

and continuous sections $\tilde{p}_u^e, \tilde{p}_v^e, \tilde{s}_u^e, \tilde{s}_v^e \colon D_\pi^{hyb} \to \widetilde{\mathcal{S}}_\pi^{hyb}$ (see (4.10) and (4.15)).

Finally, we define $\mathcal{S}_\pi^{hyb}$ by glueing together the section $\tilde{p}_u^e$ with $\tilde{s}_v^e$ and the section $\tilde{s}_u^e$ with $\tilde{p}_v^e$. More precisely, $\mathcal{S}_\pi^{hyb}$ is the quotient under an equivalence relation,

$$\mathcal{S}_\pi^{hyb} := \widetilde{\mathcal{S}}_\pi^{hyb} / \sim, \tag{4.16}$$

where for each point $\mathbf{t} \in D_\pi^{hyb}$, the two points $\tilde{p}_u^e(\mathbf{t})$ and $\tilde{s}_u^e(\mathbf{t})$ are identified, and similarly for $\tilde{p}_v^e(\mathbf{t})$ and $\tilde{s}_v^e(\mathbf{t})$. The resulting points in the quotient space $\mathcal{S}_\pi^{hyb}$ are denoted by $\mathbf{p}_u^e(\mathbf{t})$ and $\mathbf{p}_v^e(\mathbf{t})$.

Equipping $\mathcal{S}_\pi^{hyb}$ with the quotient topology, we get a continuous map

$$\mathfrak{p}_\pi^{hyb} : \mathcal{S}_\pi^{hyb} \to D_\pi^{hyb} \tag{4.17}$$

By construction, the fiber $\mathcal{S}_\pi^{hyb}|_{\mathbf{t}}$ at some $\mathbf{t} = (t, x) \in D_\pi^{hyb}$ is the metrized complex $\mathcal{MC}(\mathcal{S}_t)$ defined by the fiber $\mathcal{S}_t$ and the edge lengths of x on its dual graph G_F. To turn $\mathcal{S}_\pi^{hyb}|_{\mathbf{t}}$ into a hybrid curve, we add the data of the ordered partition π (see Section 2.11). So altogether, $\mathcal{S}_\pi^{hyb}$ is a family of hybrid curves over D_π^{hyb}, which all have G_F as their underlying graph and π as their layering.

In addition, the sections $\mathbf{p}_u^e, \mathbf{p}_v^e \colon D_\pi^{hyb} \to \mathcal{S}_\pi^{hyb}$ are continuous and for each $\mathbf{t} \in D_\pi^{hyb}$, the points $\mathbf{p}_u^e(\mathbf{t})$ and $\mathbf{p}_v^e(\mathbf{t})$ are exactly the two attachment points for the interval representing the edge e on the hybrid curve.

Notice that $\mathcal{S}_\pi^{hyb}$ consists of three different types of points: it decomposes naturally as

$$\mathcal{S}_\pi^{hyb} = \mathbf{P}_\pi^0 \sqcup \mathbf{P}_\pi^1 \sqcup \mathbf{P}_\pi^2 \tag{4.18}$$

for the following three sets:

- $\mathbf{P}_\pi^0$ is the set of all points in $\mathcal{S}_\pi^{hyb}$, which are smooth points of some component C_v of their corresponding fiber. More formally, we can write this set as (see (4.12) and (4.16))

$$\mathbf{P}_\pi^0 := \bigsqcup_{\mathbf{t}=(t,x)\in D_\pi^{hyb}} (\mathcal{S}_t \cap P^0) \times \{x\} = (\mathcal{S}_F \cap P^0) \times \sigma_\pi^\circ \subseteq \mathcal{S}_\pi^{hyb}. \tag{4.19}$$

- $\mathbf{P}_\pi^1$ consists of all points of attachment of intervals in the fibers. More precisely,

$$\mathbf{P}_\pi^1 := \bigsqcup_{\mathbf{t}\in D_\pi^{hyb}} \bigsqcup_{e=uv\in F} \Big\{\mathbf{p}_u^e(\mathbf{t}), \mathbf{p}_v^e(\mathbf{t})\Big\}. \tag{4.20}$$

- $\mathbf{P}_\pi^2$ is the set of all points in $\mathcal{S}_\pi^{hyb}$, which lie in the strict interior of some interval of their corresponding fiber. That is,

$$\mathbf{P}_\pi^2 := \Big(\bigsqcup_{e\in F} D_F^\circ \times \mathcal{I}_\pi^e\Big) \backslash \mathbf{P}_\pi^1 \subseteq \mathcal{S}_\pi^{hyb}. \tag{4.21}$$

  (Here we see the points $\mathbf{p}_u^e(\mathbf{t})$ and $\mathbf{p}_v^e(\mathbf{t})$ as the endpoints of the intervals.)

Finally, as a set, we introduce the family $\mathcal{S}^{hyb}$ by (4.5), namely, by

$$\mathcal{S}^{hyb} := \mathcal{S}^* \sqcup \bigsqcup_{\varnothing\subsetneq F\subseteq E} \bigsqcup_{\pi\in\Pi(F)} \mathcal{S}_\pi^{hyb}. \tag{4.22}$$

Moreover, recalling the decomposition $D = \bigsqcup_\pi D_\pi^{hyb}$, we see that the projection maps $\mathfrak{p}_\pi^{hyb}$ given in (4.17), stratum by stratum, glue together to define a *hybrid projection map*

$$\mathfrak{p}^{hyb} : \mathcal{S}^{hyb} \to D^{hyb}. \tag{4.23}$$

Notice that over $\mathcal{S}^* \subseteq \mathcal{S}^{hyb}$, the hybrid projection $\mathfrak{p}^{hyb}$ coincides with the original projection $\mathfrak{p}\colon \mathcal{S}^* \to B^*$ for the original versal family $\mathcal{S}$. In the sequel, if there is no risk of confusion, we drop the mention of hybrid and use the same notation $\mathfrak{p}$ for both maps.

4.3.4. *The topology on $\mathcal{S}^{hyb}$.* It remains to introduce a suitable topology on $\mathcal{S}^{hyb}$. By the preceding constructions, we already have a topology on each of the subfamilies $\mathcal{S}_\pi^{hyb}$, $\pi \in \Pi$ in the decomposition (4.5). Hence it remains to clarify the topological relation between two subfamilies $\mathcal{S}_\pi^{hyb}$ and $\mathcal{S}_{\pi'}^{hyb}$ for different ordered partitions $\pi \neq \pi'$. Similar to the hybrid base B^{hyb}, we will rely on the use of logarithmic coordinates.

The precise construction of the topology on $\mathcal{S}^{hyb}$ is slightly involved and hence we begin with an informal outline. Suppose that $\mathbf{t} = (t, x) \in B^{hyb}$ is a hybrid base point and consider its fiber $\mathcal{S}_\mathbf{t}^{hyb}$. First of all, the topology on $\mathcal{S}^{hyb}$ should ensure the continuity of the (hybrid) projection map $\mathfrak{p}\colon \mathcal{S}^{hyb} \to D^{hyb}$ (see (4.23)). Hence it will suffice to clarify the topological relationship between $\mathcal{S}_\mathbf{t}^{hyb}$ and nearby fibers $\mathcal{S}_\mathbf{s}^{hyb}$, that is, those fibers whose hybrid base points $\mathbf{s} = (s, y) \in B^{hyb}$ are close to $\mathbf{t}$ in the hybrid base B^{hyb}.

By definition, any such nearby fiber $\mathcal{S}_\mathbf{s}^{hyb}$ is a hybrid curve: it is obtained by replacing the nodes $p^e(s)$, $e \in E_s$, of the stable Riemann surface $\mathcal{S}_s$ with intervals of lengths $(y_e)_{e \in E_s}$ and by remembering the layering on the edge set E_s. Since $\mathbf{s}$ is close to $\mathbf{t}$ in B^{hyb}, by definition of the topology on B^{hyb}, we have $E_s \subseteq E_t$ or, in other words, the stable Riemann surface $\mathcal{S}_t$ has more nodes than $\mathcal{S}_s$. Moreover, for any additional node $e \in E_t \backslash E_s$, we can find a part of $\mathcal{S}_\mathbf{s}$ which is "almost degenerate". In order to make this rigorous, we fix a small neighborhood U^e of the singular point $p^e(t)$ in $\mathcal{S}$. The intersection region $U_s^e = U^e \cap \mathcal{S}_s$ can then be regarded as close to being singular (at least, when seen as part of the whole family $\mathcal{S}$).

Recall that our aim was to define a good topological relation between the fibers $\mathcal{S}_\mathbf{s}^{hyb}$ and $\mathcal{S}_\mathbf{t}^{hyb}$. The key idea of the construction is to relate the singular parts U_s^e, $e \in E_t \backslash E_s$ to the additional intervals $\mathcal{I}_e$ on $\mathcal{S}_\mathbf{t}^{hyb}$. This will be done by using logarithmic coordinates. Namely, after choosing standard coordinates $z = ((z_i)_{i \neq e}, z_u^e, z_v^e)$ on U (see (4.8)), we perform the following change

$$z_u^e \leftrightarrow \Big(\arg(z_u^e), |\log(|z_u^e|)|\Big)$$

to logarithmic coordinates. In these new coordinates, the region U_s^e in the fibre $\mathcal{S}_\mathbf{s}^{hyb}$ essentially becomes a cylinder of length $\sim |\log(|z_e(s)|)|$. Forgetting about the angle and rescaling appropriately, it can be naturally related to the interval corresponding to e on the hybrid curve $\mathcal{S}_\mathbf{t}^{hyb}$.

Remark 4.2. The use of logarithmic coordinates is crucial in the context of canonical measures (see Section 9.4.1) since the one-forms involved have precisely logarithmic singularities. $\diamond$

To formalize the above idea, we will use the Log-maps defined as follows. Fix some $e \in E$ and assume that p is a point in the set $P^e \subseteq \mathcal{S}$ (that is, it is a node corresponding to the divisor D_e, see (4.6)). Suppose further that (U, z) is a standard coordinate neighborhood for p in $\mathcal{S}$ (see (4.8)). Then we define the corresponding Log-map as

$$\begin{array}{rccc} \mathrm{Log}_e\colon & U\backslash \mathfrak{p}^{-1}(D_e) & \longrightarrow & [0,1] \\ & q & \mapsto & \frac{\log|z_u^e(q)|}{\log|z_e(q)|}, \end{array} \tag{4.24}$$

where $z_e(q) = \mathfrak{p}(q)_e = z_u^e(q)z_v^e(q)$ and $\mathfrak{p}\colon \mathcal{S} \to B$ is the projection map.

Before introducing the topology on $\mathcal{S}^{hyb}$ in detail, we would like to illustrate its properties by another result. The following proposition characterizes the convergence of points p in the non-hybrid part $\mathcal{S}^* \subseteq \mathcal{S}^{hyb}$ to points $\mathbf{p}$ in the hybrid family $\mathcal{S}^{hyb}$.

Proposition 4.3. *Let $\mathbf{p}$ be a point in $\mathcal{S}^{hyb}$ and $\mathbf{t} = (t, x) := \mathfrak{p}(\mathbf{p})$ its hybrid base point. Suppose further that $(p_n)_n$ is a sequence in $\mathcal{S}^* \subseteq \mathcal{S}^{hyb}$ with base points $t_n := \mathfrak{p}(p_n)$, $n \in \mathbb{N}$ in $B^* \subseteq B^{hyb}$. If $\lim_{n\to\infty} p_n = \mathbf{p}$ in $\mathcal{S}^{hyb}$, then necessarily*

$$\lim_{n\to\infty} t_n = \mathbf{t} \qquad \text{in } B^{hyb}. \tag{4.25}$$

Moreover, the convergence can be characterized as follows.

(i) Suppose that $\mathbf{p}$ belongs to the Riemann surface part of its fiber $\mathcal{S}_{\mathbf{t}}^{hyb}$. That is, $\mathbf{p} \in \mathbf{P}_\pi^0$ and formally $\mathbf{p} = (p, x)$ for some non-node point p in the fiber $\mathcal{S}_t$ (see (4.19)). Then $\lim_{n\to\infty} \mathbf{p}_n = \mathbf{p}$ in $\mathcal{S}^{hyb}$ exactly when (4.25) holds and

$$\lim_{n\to\infty} \mathbf{p}_n = p \qquad \text{in } \mathcal{S}.$$

(ii) Suppose that $\mathbf{p}$ belongs to the interior of an interval in its fiber $\mathcal{S}_{\mathbf{t}}^{hyb}$, that is, $\mathbf{p} \in \mathbf{P}_\pi^2$. Formally, $\mathbf{p}$ is a point $0 < \lambda_u^e < x_e$ in the interval $[0, x_e] \subseteq \mathcal{S}_{\mathbf{t}}^{hyb}$ representing an edge $e = uv$ of the dual graph for $\mathcal{S}_t$. Then $\lim_{n\to\infty} \mathbf{p}_n = \mathbf{p}$ in $\mathcal{S}^{hyb}$ exactly when (4.25) holds, $\lim_{n\to\infty} p_n = p^e(t)$ in the original family $\mathcal{S}$ and

$$\lim_{n\to\infty} \mathrm{Log}_e(p_n) = \lim_{n\to\infty} \frac{\log|z_u^e(p_n)|}{\log|z_e(p_n)|} = \frac{\lambda_u^e}{x_e}$$

for the Log-map on some (equivalently, on every) standard coordinate neighborhood (U, z) of $p^e(t)$ in $\mathcal{S}$.

(iii) Suppose that $\mathbf{p}$ is the attachment point of an interval in its fiber $\mathcal{S}_{\mathbf{t}}^{hyb}$. That is, $\mathbf{p} \in \mathbf{P}_\pi^1$ and $\mathbf{p} = \mathbf{p}_u^e(\mathbf{t})$ for some edge e and adjacent vertex u in the dual graph of $\mathcal{S}_t$ (see also(4.20)). Then $\lim_{n\to\infty} \mathbf{p}_n = \mathbf{p}$ in $\mathcal{S}^{hyb}$ exactly when (4.25) holds, $\lim_{n\to\infty} p_n = p^e(t)$ in the original family $\mathcal{S}$ and

$$\lim_{n\to\infty} \mathrm{Log}_e(p_n) = \lim_{n\to\infty} \frac{\log|z_u^e(p_n)|}{\log|z_e(p_n)|} = 0$$

for the Log-map on some (equivalently, on every) standard coordinate neighborhood (U, z) of $p^e(t)$ in $\mathcal{S}$.

Proof. The claims follow easily from the definition of the topology on $\mathcal{S}^{hyb}$. □

Now we proceed with the formal definition of the topology on $\mathcal{S}^{hyb}$. It suffices to specify for each point $\mathbf{p} \in \mathcal{S}^{hyb}$ its system of neighborhoods $\mathcal{V}(\mathbf{p})$. In fact, in the following we will define a neighborhood subbase $\mathcal{V}_0(\mathbf{p})$ for $\mathbf{p}$. The total neighborhood system $\mathcal{V}(\mathbf{p})$ is then obtained by taking first all intersection of finitely many sets in $\mathcal{V}_0(\mathbf{p})$ and then all supersets of such sets.

Fix some point $\mathbf{p} \in S^{hyb}$. Assume further that $\mathbf{p} \in S^{hyb}_{\pi}$ for the ordered partition π of the subset $F = E_\pi$ of E (here, $S^{hyb}_{\pi_\varnothing} := \mathcal{S}^*$ for the empty partition $\pi_\varnothing$ of $F = \varnothing$). Let $\mathbf{t} = (t, x) := \mathfrak{p}(\mathbf{p}) \in B^{hyb}$ be the hybrid base point of the fiber containing $\mathbf{p}$. The neighborhood subbase $\mathcal{V}_0(\mathbf{p})$ will contain two different types of sets W.

First of all, for each neighborhood U of the hybrid base point $\mathbf{t}$ in B^{hyb}, we add its preimage $W = \mathfrak{p}^{-1}(U)$ to $\mathcal{V}_0(\mathbf{p})$ (see (4.23)). Clearly, this ensures the continuity of the hybrid projection $\mathfrak{p}\colon \mathcal{S}^{hyb} \to D^{hyb}$.

The second type of sets in $\mathcal{V}_0(\mathbf{p})$ clarifies the topological relation between the subfamilies in (4.5), defined over different strata of the hybrid base. These sets have the structure

$$W = \bigsqcup_{\pi' \preceq \pi} W_{\pi'} \tag{4.26}$$

where each $W_{\pi'}$ is a subset of $\mathcal{S}^{hyb}_{\pi'}$. Their precise form depends on the type of the point $\mathbf{p}$ according to the decomposition (4.18). We will proceed by case distinction. Assume that...

(i) $\mathbf{p}$ is a smooth point in its fiber, that is, $\mathbf{p}$ belongs to $\mathbf{P}^0_\pi$ (see (4.19)). Then $\mathbf{p} = (p, x)$ for some normal smooth point $p \in P^0$. In this case, the neighborhood $W_{\pi'}$ should simply reflect the topology on the original family $\mathcal{S}$. Hence we will transfer neighborhoods from $\mathcal{S}$ in a suitable way.

Suppose W_p is a neighborhood of p in $\mathcal{S}$ containing only normal smooth points, that is, $W_p \subseteq P^0$. Then for any ordered partition π' with $\pi' \preceq \pi$, we set

$$W_{\pi'} := \big(\bigcup_{s \in D^\circ_{F'}} W_p \cap \mathcal{S}_s \big) \times \sigma^\circ_{\pi'} = (W_p \cap \mathcal{S}_{F'}) \times \sigma^\circ_{\pi'} \subseteq \mathcal{S}^{hyb}_{\pi'} \tag{4.27}$$

where $F' = E_{\pi'}$. In particular, notice that $W_{\pi'}$ is naturally a subset of $\mathcal{S}^{hyb}_{\pi'}$ (see (4.12) and (4.16)). Altogether, we define the neighborhood W of $\mathbf{p}$ in $\mathcal{S}^{hyb}$ by (4.26). We remark that W contains only smooth points of the corresponding hybrid curves, that is, $W_{\pi'} \subseteq \mathbf{P}^0_{\pi'}$ for all $\pi' \preceq \pi$.

(ii) $\mathbf{p}$ is the interior point of an interval $\mathcal{I}_e$ in its fiber, that is, $\mathbf{p}$ belongs to $\mathbf{P}^2_\pi$ (see (4.21)). The definition of $W_{\pi'}$ then depends on whether the hybrid curves in $\mathcal{S}_{\pi'}$ contain the edge e or not. If they do, then we can easily relate the corresponding intervals by rescaling the lengths. In the other case, we use the Log-maps as described above.

Let us now make this idea rigorous. Formally (see (4.21)), we can write the point $\mathbf{p}$ as $\mathbf{p} = (t, x, \lambda^e_u, \lambda^e_v)$ for some edge $e \in F$ and $0 < \lambda^e_u, \lambda^e_v < x_e$ with $\lambda^e_u + \lambda^e_v = x_e$.

Fix an ordered partition π' of some subset F' with $\pi' \preceq \pi$. Suppose that...

- π' does not contain the edge e (that is, $e \notin F'$). Then we fix a standard coordinate neighborhood (U, z) of the point $p^e(t)$ in the original family $\mathcal{S}$. Moreover, we take a small neighborhood W_p of $p^e(t)$ in $\mathcal{S}$ with $W_p \subseteq U$ and some $\varepsilon > 0$. In terms of the Log-map on U, we define

$$W_{\pi'} := \Big\{ q \in \mathcal{S}_{F'} \cap U \;\Big|\; |\mathrm{Log}_e(q) - \lambda_u^e / x_e| < \varepsilon \text{ and } q \text{ belongs to } W_p \Big\} \times \sigma_{\pi'}^{\circ}$$

That is, we take the points in fibers over $D_{\pi'}^{hyb}$ which correspond to points on Riemann surfaces, are close to the node $p^e(t)$ in the original family $\mathcal{S}$, and whose logarithmic image is close to the point in question on the interval. This clearly formalizes the use of the Log-map as outlined at the beginning of the section. Moreover, by definition we have $W_{\pi'} \subseteq \mathbf{P}_{\pi'}^0$ (see (4.19)).

- π' contains the edge e (that is, $e \in F$). Then we define

$$W_{\pi'} := \Big\{ \big(s, y, \tilde{\lambda}_u^e, \tilde{\lambda}_v^e\big) \in D_{F'}^{\circ} \times \mathcal{I}_{\pi'}^e \;\Big|\; |\tilde{\lambda}_u^e / y_e - \lambda_u^e / x_e| < \varepsilon \Big\}.$$

Notice that $W_{\pi'}$ is naturally a subset of $\mathcal{S}_{\pi'}^{hyb}$. It only contains parts of the interval representing the edge e in the corresponding fibers. In particular, $W_{\pi'} \subseteq \mathbf{P}_{\pi'}^2$.

Finally, as in the previous case we obtain W by combining the pieces $W_{\pi'}$ according to (4.26).

(iii) $\mathbf{p}$ is the attachment point of an interval $\mathcal{I}_e$ in its fiber, that is, $\mathbf{p}$ belongs to $\mathbf{P}_\pi^1$ (see (4.20)). Again, the definition of $W_{\pi'}$ will depend on whether the partition π' contains the edge e.

If π' does not contain the edge e, then we use the Log-maps in the same way as above. If π' contains the edge e, then the singularity along the section p^e was resolved also in the family $\mathcal{S}_{\pi'}^{hyb}$. Thus, roughly speaking, $\mathcal{S}_{\pi'}^{hyb}$ looks locally like the normalized family $\breve{\mathcal{S}}_e$ (see (4.11)) with some interval attached. In this case, we want the neighborhoods of $\mathbf{p}$ to contain both parts of the stable Riemann surfaces and some pieces of intervals.

In the following, we formalize this idea. Let us assume that $p = \mathbf{p}_u^e(\mathbf{t})$ for an edge e with adjacent vertex u in G_F.

Fix an ordered partition π' of some subset F' with $\pi' \preceq \pi$. Suppose that...

- π' does not contain the edge e (that is, $e \notin F'$). Then we fix a standard coordinate neighborhood (U, z) of the point $p^e(t)$ in the original family $\mathcal{S}$. Moreover, we take a small neighborhood W_p of $p^e(t)$ in $\mathcal{S}$ with $W_p \subseteq U$ and some $\varepsilon > 0$. Using the Log-map on U, we then set

$$W_{\pi'} := \Big\{ q \in \mathcal{S}_{F'} \cap U \,\big|\, \mathrm{Log}_e(q) < \varepsilon \text{ and } q \text{ belongs to } W_p \} \times \sigma_{\pi'}^{\circ}.$$

Informally speaking, we simply take the points in fibers over $D_{\pi'}^{hyb}$ which correspond to points on Riemann surfaces, are close to the node $p^e(t)$ in the original family $\mathcal{S}$, and whose logarithmic image is close to the left endpoint of the interval. Again, $W_{\pi'}$ is naturally a subset of $\mathcal{S}_{\pi'}^{hyb}$ (see (4.19)). It contains only smooth points of the corresponding fibers, that is, $W_{\pi'} \subseteq \mathbf{P}_{\pi'}^0$.

- π' contains the edge e (that is, $e \in F'$). Then $W_{\pi'}$ will consist of two parts. To define the first one, we use the normalized family $\check{\mathcal{S}}_e$ defined in (4.11). Notice that we can see
$$\check{\mathcal{S}}_e|_{D^\circ_{F'}} \times \sigma^\circ_{\pi'}$$
naturally as a subset of $\mathcal{S}^{hyb}_{\pi'}$ (similar to (4.27)). More precisely, the above set is contained in $\mathbf{P}^0_{\pi'} \sqcup \mathbf{P}^1_{\pi'}$. For a neighborhood $\check{W}$ of $p^e_u(t)$ in $\check{\mathcal{S}}_e$ we can again see
$$\Big(\bigcup_{s \in D^\circ_{F'}} \check{W} \cap (\check{\mathcal{S}}_e|_s)\Big) \times \sigma^\circ_{\pi'} = (\check{W} \cap (\check{\mathcal{S}}_e|_{D^\circ_{F'}})) \times \sigma^\circ_{\pi'}$$
naturally as a subset of $\mathcal{S}^{hyb}_{\pi'}$. To define the second part of $W_{\pi'}$, we use the subset
$$\Big\{(s, y, \lambda^e_u, \lambda^e_v) \in D^\circ_{F'} \times \mathcal{I}^e_{\pi'} \,|\, \lambda^e_u/x_e < \varepsilon\Big\} \subseteq \mathbf{P}^1_{\pi'} \sqcup \mathbf{P}^2_{\pi'},$$
which corresponds to taking interval parts $[0, \varepsilon]$ in the edge e of the metrized complexes. Altogether, the subset $W_{\pi'} \subseteq \mathcal{S}^{hyb}_{\pi'}$ is defined as the union of the above two sets. By construction, $W_{\pi'}$ contains points of all three types described in (4.18).

Finally, the whole neighborhood W of $\mathbf{p}$ is defined by combining the pieces $W_{\pi'}$, $\pi' \preceq \pi$, according to (4.26).

Now we can finish the definition of the neighborhood subbase $\mathcal{V}_0(\mathbf{p})$. That is, we add to it all sets W which can be constructed in the way described above (ranging over all $\varepsilon > 0$ and neighborhoods U, W_p, $\check{W}$ and so on).

Finally, the topology on $\mathcal{S}^{hyb}$ is the unique one such that $\mathcal{V}_0(\mathbf{p})$ is a neighborhood subbase for each $\mathbf{p} \in \mathcal{S}^{hyb}$. In fact, after a routine calculation we see that the related neighborhood system $(\mathcal{V}(\mathbf{p}))_{\mathbf{p}}$ satisfies the axioms of a neighborhood system on $\mathcal{S}^{hyb}$. Hence such a topology indeed exists. Moreover, we stress once again that, by construction, the hybrid projection map $\mathfrak{p}\colon \mathcal{S}^{hyb} \to B^{hyb}$ is continuous.

Remark 4.4. The topological space $\mathcal{S}^{hyb}$ has the following properties.

(i) $\mathcal{S}^{hyb}$ is a second countable, locally compact Hausdorff space and in particular, $\mathcal{S}^{hyb}$ is metrizable.
(ii) $\mathcal{S}^{hyb}$ is a continuous family of hybrid curves over B^{hyb}.
(iii) On each subfamily $\mathcal{S}^{hyb}_\pi \subseteq \mathcal{S}^{hyb}$ (see (4.5)), the induced topology from $\mathcal{S}^{hyb}$ coincides with the one introduced on $\mathcal{S}^{hyb}_\pi$ in the previous section.

$\diamond$

4.4. **Hybrid moduli space of stable curves.** Let g and n be two non-negative integers verifying $3g - 3 + g + n \geqslant 0$. If $g = 1$, we impose moreover that $n \geqslant 1$.

4.4.1. *Stratification of $\overline{\mathscr{M}}_{g,n}$ by stable graphs with markings.* A nice reference for the materials which follow in this short subsection is [ACP15] which already makes connection to the moduli space of tropical curves. Once again, we emphasize that our notion of tropical curve differs from the one considered in [ACP15]. The tropical moduli space underlying our tropical curves should be a refinement of the one studied in [ACP15].

Let $\mathscr{M}_{g,n}$ be the (coarse) moduli space of stable curves of genus g with n markings. Denote by $\overline{\mathscr{M}}_{g,n}$ the Deligne-Mumford compactification of $\mathscr{M}_{g,n}$, obtained by adding stable curves of genus g with n marked points to $\mathscr{M}_{g,n}$.

There is a stratification of $\overline{\mathscr{M}}_{g,n}$ by combinatorial types of genus g stable dual graphs with n marked points G. Denoting by $\mathscr{M}_G$ the stratum corresponding to the stable graph with n marked points G of genus g, we have

$$\overline{\mathscr{M}}_{g,n} = \bigsqcup_{G\colon \text{stable marked graph of genus } g} \mathscr{M}_G.$$

The stratum $\mathscr{M}_G$ can be further described as follows. Consider a stable graph with n marked points $G = (V, E, \mathfrak{g}, \mathfrak{m})$. Let $\mathfrak{n} : V \to \mathbb{N} \cup \{0\}$ be the counting function which at each vertex v counts the number of labels placed at v.

For each vertex v, consider the moduli space $\mathscr{M}_{\mathfrak{g}(v),\deg(v)+\mathfrak{n}(v)}$, and the product

$$\widetilde{\mathscr{M}}_G := \prod_{v\in V} \mathscr{M}_{\mathfrak{g}(v),\deg(v)+\mathfrak{n}(v)}.$$

Denote by $\mathrm{Aut}(G)$ the automorphism group of the the marked stable graph G. The group $\mathrm{Aut}(G)$ naturally acts on $\widetilde{\mathscr{M}}_G$, through the decomposition as a product of factors, and we have

$$\mathscr{M}_G \simeq \widetilde{\mathscr{M}}_G/\mathrm{Aut}(G).$$

We now describe the closure of $\mathscr{M}_G$ in $\overline{\mathscr{M}}_{g,n}$. Recall that in Section 2 we defined a partial order on stable graphs (with marking) of given genus g by saying that $G \leqslant H$ if G can be obtained by a sequence of edge-contractions from H. We refer to that section for more details.

The closure of $\mathscr{M}_G$ in $\overline{\mathscr{M}}_{g,n}$ is given by

$$\overline{\mathscr{M}_G} = \bigsqcup_{G\leqslant H} \mathscr{M}_H.$$

We denote by $\mathscr{C}_{g,n}$ the universal family of curves with n markings over $\mathscr{M}_{g,n}$. By an abuse of the notation, the universal family of stable curves with n markings over $\overline{\mathscr{M}}_{g,n}$ is also denoted by $\mathscr{C}_{g,n}$. (Notice again that we use the terminology of universal curve slightly imprecisely, as we consider the coarse moduli space and the fibres of the family are quotients of stable Riemann surfaces by the action of their automorhpism groups.)

4.4.2. *Moduli space of tropical curves of given combinatorial type.* Let $G = (V, E, \mathfrak{g}, \mathfrak{m})$ be a stable graph of genus g with n marked points. We define the *moduli space of tropical curves of combinatorial type* G as follows. First, we define

$$\widetilde{\mathscr{M}}_G^{\mathrm{trop}} := \bigsqcup_{\pi\in\Pi(E)} \sigma_\pi^\circ$$

endowed with a natural topology, compatible with the hybrid topology on B^{hyb} from Section 3.3, that we describe in Section 6. (So here the vertices and the edges of the graph G are all labelled.) The group $\mathrm{Aut}(G)$ naturally acts on $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ by permutation of the edges, and the moduli space of tropical curves of combinatorial type G is defined by taking the quotient

$$\mathscr{M}_G^{\mathrm{trop}} := \widetilde{\mathscr{M}}_G^{\mathrm{trop}}/\mathrm{Aut}(G).$$

4.4.3. *Hybrid moduli space.* Let $G = (V, E, \mathfrak{g}, \mathfrak{m})$ be a stable graph of genus g with n marked points. The group $\mathrm{Aut}(G)$ acts naturally on $\widetilde{\mathscr{M}}_G$ and on $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$. From these two actions, we get a diagonal action of $\mathrm{Aut}(G)$ on the product $\widetilde{\mathscr{M}}_G \times \widetilde{\mathscr{M}}_G^{\mathrm{trop}}$, and define the *hybrid stratum* $\mathscr{M}_G^{hyb}$ associated to G as

$$\mathscr{M}_G^{hyb} := (\widetilde{\mathscr{M}}_G \times \widetilde{\mathscr{M}}_G^{\mathrm{trop}})/\mathrm{Aut}(G).$$

By construction, the elements of the hybrid stratum $\mathscr{M}_G^{hyb}$ correspond to isomorphism classes of hybrid curves with combinatorial type G (here, we consider isomorphisms of the underlying marked stable Riemann surfaces respecting the ordered partition and edge lengths).

We now give an alternative characterization of these hybrid strata and obtain a refined decomposition into hybrid orbifolds.

Let $G = (V, E, \mathfrak{g}, \mathfrak{m})$ be a stable dual graph with n markings. Consider an ordered partition $\pi = (\pi_1, \dots, \pi_r)$ of E and denote by (G, π) the corresponding layered stable graph with markings. Denote by $\mathrm{Aut}(G, \pi)$ the subgroup of $\mathrm{Aut}(G)$ consisting of those elements which preserve the ordered partition given by layers, i.e., which respect the corresponding filtration $\mathscr{F}^\pi$. The hybrid orbifold associated with π is the quotient

$$\mathscr{M}_{G,\pi}^{hyb} := (\widetilde{\mathscr{M}}_G \times \sigma_\pi^\circ)/\mathrm{Aut}(G, \pi).$$

Notice that the hybrid orbifold $\mathscr{M}_{G,\pi}^{hyb}$ only depends on the orbit $[\pi]$ of the ordered partition $\pi \in \Pi(E)$ under the action of $\mathrm{Aut}(G)$, that is, $\mathscr{M}_{G,\pi}^{hyb}$ and $\mathscr{M}_{G,\pi'}^{hyb}$ are isomorphic whenever $[\pi] = [\pi']$. The hybrid stratum $\mathscr{M}_G^{hyb}$ can then be identified as

$$\mathscr{M}_G^{hyb} = \bigsqcup_{[\pi] \in \Pi(E)/\mathrm{Aut}(G)} \mathscr{M}_{G,\pi}^{hyb}$$

where the union is over all orbits of ordered partitions under the action of $\mathrm{Aut}(G)$ (here, for each orbit in $\Pi(E)/\mathrm{Aut}(G)$ we have fixed an arbitrary representative π).

Definition 4.5 (Hybrid moduli spaces as a set)**.** The *moduli space* $\mathscr{M}_{g,n}^{hyb}$ *of hybrid curves of genus* g *with* n *markings* is defined by

$$\mathscr{M}_{g,n}^{hyb} := \bigsqcup_{G} \bigsqcup_{[\pi] \in \Pi(E(G))/\mathrm{Aut}(G)} \mathscr{M}_{G,\pi}^{hyb}$$

where the first union is over all stable graphs G of genus g with n markings (again, π is an arbitrary ordered partition representing the corresponding orbit). $\diamond$

By construction, the points $\mathbf{t}$ of $\mathscr{M}_{g,n}^{hyb}$ are in bijection with the isomorphism classes of marked hybrid curves. We denote by $\mathscr{C}_{g,n}^{hyb}$ the *universal hybrid curve* over $\mathscr{M}_{g,n}^{hyb}$, that is, its fiber over a point $\mathbf{t}$ in $\mathscr{M}_{g,n}^{hyb}$ is the marked hybrid curve represented by $\mathbf{t}$. In fact, since we work with coarse moduli spaces, this is slightly incorrect and the terminology of universal hybrid curves is used by an abuse of the notation. To be precise, the (set-theoretic) fibre $\mathscr{C}_{\mathbf{t}}^{hyb}$ over some base point $\mathbf{t} \in \mathscr{M}_{g,n}^{hyb}$ is the quotient of the hybrid curve represented by $\mathbf{t}$ by the action of its automorphism group.

4.5. **Hybrid topology.** In this section, we describe the hybrid topology on the hybrid moduli spaces.

Consider a point s_0 in $\overline{\mathscr{M}}_{g,n}$, and denote by B the analytified versal deformation space of the corresponding stable curve (with n marked points) $S_0 := \mathscr{C}_{b_0}$. This way, we get an étale open chart (for the fine moduli space) $B \to \overline{\mathscr{M}}_{g,n}$ around s_0. Let B^{hyb} be the hybrid deformation space, defined in Section 4.2, obtained by the general constructions worked out in Section 3. Denote by G the dual stable graph of S_0 with n markings.

By construction, the automorphism group $\mathrm{Aut}(S_0)$ of the marked stable Riemann surface S_0 acts on B. Since every automorphism of S_0 defines an automorphism of the dual stable graph G, this extends to a natural action of $\mathrm{Aut}(S_0)$ on B^{hyb}. We equip the quotient $B^{hyb}/\mathrm{Aut}(S_0)$ with the quotient topology induced from the hybrid topology of B^{hyb}. Combining the above, we thus get a map from $B^{hyb}/\mathrm{Aut}(S_0)$ to $\mathscr{M}_{g,n}^{hyb}$.

Proposition 4.6. *Notations as above, the collection of sets consisting of images of $B^{hyb}/\mathrm{Aut}(S_0)$ in $\mathscr{M}_{g,n}^{hyb}$ form a covering of $\mathscr{M}_{g,n}^{hyb}$.*

Proof. By definition of a covering, we must show that the union of these images is equal to $\mathscr{M}_{g,n}^{hyb}$. The points of $\mathscr{M}_{g,n}^{hyb}$ bijectively correspond to isomorphism classes of hybrid curves. Hence the claim is obvious, since (the isomorphism class of) each hybrid curve trivially belongs to the image of $B^{hyb}/\mathrm{Aut}(S_0)$ for its stable Riemann surface S_0. □

Definition 4.7 (Hybrid topology)**.** We say that all the images of sets of the form $B^{hyb}/\mathrm{Aut}(G)$ are open in $\mathscr{M}_{g,n}^{hyb}$, and define the hybrid topology on $\mathscr{M}_{g,n}^{hyb}$ as the one generated by this open covering. ◇

4.6. **Universal hybrid curve.** It follows that locally, for the local charts $B^{hyb}/\mathrm{Aut}(S_0)$ considered in the previous section, the universal hybrid curve $\mathscr{C}_{g,n}^{hyb}$ restricts to the versal hybrid curve $\mathcal{S}^{hyb}/B^{hyb}$ (see Section 4.3) descended over $B^{hyb}/\mathrm{Aut}(S_0)$. The latter means that we first extend the action of $\mathrm{Aut}(S_0)$ from the base B^{hyb} to the versal hybrid curve $\mathcal{S}^{hyb}$ and then take the quotient. We endow the universal hybrid curve $\mathscr{C}_{g,n}^{hyb}$ with the topology induced by these local charts.

4.7. **Towers of hybrid moduli spaces.** Using the constructions outlined in Section 3.4, we can actually go further and define a tower of hybrid spaces

$$\mathscr{M}_{g,n}^{hyb(0)} \longleftarrow \mathscr{M}_{g,n}^{hyb(1)} \longleftarrow \mathscr{M}_{g,n}^{hyb(2)} \longleftarrow \dots \longleftarrow \mathscr{M}_{g,n}^{hyb(N-1)} \longleftarrow \mathscr{M}_{g,n}^{hyb} \tag{4.28}$$

interpolating between $\overline{\mathscr{M}}_{g,n}$ and $\mathscr{M}_{g,n}^{hyb}$. Here $N = 3g-3+n$ is the dimension of $\mathscr{M}_{g,n}$.

Moreover, over each member of the tower, we get the corresponding universal hybrid curve $\mathscr{C}_{g,n}^{hyb(r)}$, forming a tower

$$\mathscr{C}_{g,n}^{hyb(0)} \longleftarrow \mathscr{C}_{g,n}^{hyb(1)} \longleftarrow \mathscr{C}_{g,n}^{hyb(2)} \longleftarrow \dots \longleftarrow \mathscr{C}_{g,n}^{hyb(N-1)} \longleftarrow \mathscr{C}_{g,n}^{hyb} \tag{4.29}$$

compatible with (4.28).

5. Canonical measures

The aim of this section is to introduce canonical measures on different types of objects (metric graphs, Riemann surfaces and hybrid curves), which are the main objects of study in this paper.

5.1. **Canonical measure on metric graphs.**

In this section, we collect basic facts about the canonical measure on compact metric graphs.

Let $G = (V, E)$ be a finite connected graph of genus h and $\ell\colon E \to (0, \infty)$ an edge length function. The corresponding metric graph is denoted by $\mathscr{G}$ (see Section 2.9). Recall that as a topological space, $\mathscr{G}$ is obtained by identifying each edge with a closed interval $\mathcal{I}_e$ of length ℓ_e and identifying endpoints of intervals corresponding to the same vertex. The topology is metrizable by the associated path metric. There are several equivalent ways to introduce the canonical measure on $\mathscr{G}$. The original definition of Zhang is based on the use of the metric graph Laplacian and its associated Green functions [Zha93], and generalizes the previous constructions of Chinburg and Rumley [CR93] (note however that what is called canonical measure in [CR93] is different from the one considered here, which is that of [Zha93]). In the following, we give three different expressions which are convenient for our purposes.

First of all, recall the following expression in terms of *spanning trees*: For a spanning tree $T = (V(T), E(T))$ of G, define its weight $\omega(T)$ as

$$\omega(T) = \prod_{e \in E \setminus E(T)} \ell_e. \tag{5.1}$$

The set of spanning trees of G is denoted by $\mathcal{T}(G)$. The *Foster coefficients* $\mu(e)$, $e \in E$ of the metric graph $\mathscr{G}$ are defined as (see, e.g., [BF11, equation (TC1)])

$$\mu(e) = \frac{1}{\sum_{T \in \mathcal{T}(G)} \omega(T)} \sum_{T \in \mathcal{T}(G)\colon e \notin E(T)} \omega(T). \tag{5.2}$$

The canonical measure on $\mathscr{G}$ is the measure

$$\mu_{\mathrm{Zh}} = \sum_{e \in E} \frac{\mu(e)}{\ell_e}\, d\theta_e, \tag{5.3}$$

with $d\theta_e$ the uniform Lebesgue measure on the edge $e \in E$.

Moreover, the canonical measure can be expressed using the *cycle space* of G. Let $H = H_1(G, \mathbb{Z})$ be the first homology of G. By definition we have an exact sequence

$$0 \to H_1(G, \mathbb{Z}) \longrightarrow \mathbb{Z}^E \longrightarrow \mathbb{Z}^V \to 0.$$

For each edge e of the graph, denote by $\langle \cdot\,, \cdot \rangle_e$ the bilinear form on $\mathbb{R}^E$ defined by

$$\langle x\,, y \rangle_e := x_e y_e$$

for any pair of elements $x = (x_f)_{f \in E}, y = (y_f)_{f \in E} \in \mathbb{R}^E$. We denote by q_e the corresponding quadratic form. Notice that any edge length function $\ell\colon E \to (0, \infty)$ defines an inner product on $\mathbb{R}^E$ given by

$$\langle x, y \rangle_\ell := \sum_{e \in E} \ell_e \langle x\,, y \rangle_e = \sum_{e \in E} \ell_e x_e y_e, \qquad x, y \in \mathbb{R}^E.$$

The corresponding quadratic form is denoted by q_ℓ. Let $\pi : \mathbb{R}^E \to H_1(G,\mathbb{R})$ be the orthogonal projection (w.r.t. $\langle \cdot , \cdot \rangle_\ell$) of $\mathbb{R}^E$ onto its subspace $H_1(G,\mathbb{R})$. The canonical measure μ_{Zh} is the measure on $\mathscr{G}$ given by (see, e.g., [SW19, Proposition 5.2])

$$\mu_{\mathrm{Zh}} = \sum_{e \in E} \frac{1}{\ell_e^2} q_\ell\Big(\pi(e)\Big)\, d\theta_e. \tag{5.4}$$

Finally, we derive another expression for μ_{Zh} which is closely related to (5.8). After fixing a basis $\gamma_1, \dots, \gamma_h$ of H, we can identify the quadratic form q_e restricted to $H_1(G,\mathbb{R})$ with an $h \times h$ symmetric matrix M_e of rank at most one so that, thinking of elements of H as column vectors, we have

$$q_e(x) = {}^{\mathrm{t}}x M_e x. \tag{5.5}$$

For an edge length function $\ell \colon E \to (0,\infty)$, the restriction of q_ℓ to $H_1(G,\mathbb{R})$ can be identified with the $h \times h$ symmetric matrix

$$M_\ell = \sum_{e\in E} \ell_e M_e. \tag{5.6}$$

We denote its inverse matrix by M_ℓ^{-1}. Recall that the (i,j)-coordinate of a matrix A is denoted by $A(i,j)$.

Theorem 5.1. *The canonical measure on $\mathscr{G}$ is the measure*

$$\mu_{\mathit{Zh}} = \sum_{e\in E} \frac{\mu(e)}{\ell_e}\, d\theta_e, \tag{5.7}$$

where $d\theta_e$ is the uniform Lebesgue measure on the edge e and the Foster coefficients $\mu(e)$, $e \in E$, are given by

$$\mu(e) = \ell_e \sum_{i,j=1}^h M_\ell^{-1}(i,j)\gamma_i(e)\gamma_j(e),$$

with $\gamma_1, \dots, \gamma_h$ a basis of $H_1(G,\mathbb{R})$.

Proof. Taking into account (5.4), it will be enough to show that for all edges $e \in E$,

$$q_\ell\Big(\pi(e)\Big) = \ell_e^2 \sum_{i,j=1}^h \gamma_i(e)\gamma_j(e) M_\ell^{-1}(i,j).$$

Writing $\pi(e)$ in the basis $\gamma_1, \dots, \gamma_h$ gives

$$\pi(e) = \sum_{i=1}^h a_{e,i}\gamma_i,$$

where the real coefficents $a_{e,i}$ are determined by the system of linear equations

$$\sum_j a_{e,j}\langle \gamma_j, \gamma_i\rangle = \langle \pi(e), \gamma_i\rangle = \langle e, \gamma_i \rangle = \ell_e \gamma_i(e)$$

for $e \in E$. This gives

$$M_\ell\Big(a_{e,j}\Big)_{j=1}^h = \Big(\ell_e\gamma_i(e)\Big)_{i=1}^h,$$

from which it follows that

$$\Big(a_{e,j}\Big)_{j=1}^{h} = M_\ell^{-1}\Big(\ell_e\gamma_i(e)\Big)_{i=1}^{h}.$$

We infer that

$$\begin{aligned} q_\ell\Big(\pi(e)\Big) = \langle \pi(e), \pi(e)\rangle &= \Big(a_{e,j}\Big)^T M_\ell\Big(a_{e,j}\Big) \\ &= \Big(\ell_e\gamma_i(e)\Big)^T M_\ell^{-1} M_\ell M_\ell^{-1}\Big(\ell_e\gamma_i(e)\Big) \\ &= \ell_e^2 \sum_{i,j=1}^{h} \gamma_i(e)\gamma_j(e) M_\ell^{-1}(i,j), \end{aligned}$$

and the result follows. □

5.2. **Canonical measure on a Riemann surface.** Let S be a compact Riemann surface of positive genus g. Denote by $\Omega^1(S)$ the vector space of holomorphic one-forms ω on S. It has complex dimension g. We have a hermitian inner product on $\Omega^1(S)$ defined by

$$\langle \omega_1, \omega_2\rangle := \frac{i}{2}\int_S \omega_1 \wedge \bar\omega_2.$$

Choosing an orthonormal basis $\eta_1, \dots, \eta_g$ of $\Omega^1(S)$, we define the canonical measure of S by

$$\mu_{\mathrm{Ar}} = \frac{i}{2}\sum_{j=1}^{g} \eta_j \wedge \bar\eta_j,$$

which is a positive density measure of total mass g on S.

An alternate description of μ_{Ar} can be given as follows. Let $a_1, \dots, a_g, b_1, \dots, b_g$ be a symplectic basis of $H_1(S, \mathbb{Z})$. This means that for the intersection pairing $\langle\cdot,\cdot\rangle$ between 1-cycles in S, we have

$$\langle a_i, a_j\rangle = \langle b_i, b_j\rangle = 0$$

and

$$\langle a_i, b_j\rangle = \delta_{i,j}.$$

Let $\omega_1, \dots, \omega_g$ be an adapted basis of $\Omega^1(S)$ in the sense that

$$\int_{a_j} \omega_i = \delta_{i,j}$$

and define the period matrix of S by

$$\Omega = \Big(\Omega(i,j)\Big) = \Big(\int_{b_j}\omega_i\Big)_{i,j=1}^{g}.$$

Denote by $\mathrm{Im}(\Omega)$ the imaginary part of Ω. This is a symmetric positive definite real matrix. Denote by $\mathrm{Im}(\Omega)^{-1}$ the inverse of $\mathrm{Im}(\Omega)$.

Then we have

$$\mu_{\mathrm{Ar}} = \frac{i}{2}\sum_{j,k=1}^{g} \mathrm{Im}(\Omega)^{-1}(i,j)\,\omega_i \wedge \bar\omega_j \tag{5.8}$$

where $\mathrm{Im}(\Omega)^{-1}(i,j)$ is the (i,j)-coordinate of the matrix $\mathrm{Im}(\Omega)^{-1}$.

5.3. **Metrized curve complexes.** Let S be a semistable curve of arithmetic genus g and denote by $G = (V, E, \mathfrak{g})$ its augmented dual graph (which has genus g). For each vertex v of G, let C_v be the (normalization of the) corresponding irreducible component of S, which is thus a smooth curve of genus $\mathfrak{g}(v)$. For each edge $e = uv$, denote by p_u^e the point of C_u which corresponds to the edge e.

Given a length function $\ell : E \to \mathbb{R}_+$ which to any edge e associates the positive real length ℓ_e, recall that the metrized curve complex $\mathcal{MC}_\ell = \mathcal{MC}_\ell(S)$ is the topological space defined by taking the complex analytification of the complex C_v, for $v \in V$, and inserting a segment $\mathcal{I}_e$ of length ℓ_e between the two marked points p_u^e and p_v^e of C_u and C_v, respectively.

Let $\mathscr{G} = \mathscr{G}_\ell$ be the metric graph associated to (G, ℓ), and let $\mu_{\mathrm{Zh}} = \sum_{e \in E} \frac{\mu(e)}{\ell_e} d\theta_e$ be the canonical measure on $\mathscr{G}$.

The canonical measure $\mu_{\mathrm{Zh\text{-}Ar}}$ on the metrized complex $\mathcal{MC}_\ell$ is the measure of total mass g whose restriction to the interval $\mathcal{I}_e$ coincides with $\frac{\mu(e)}{\ell_e} d\theta_e$, for each edge $e \in E$, and whose restriction to the component C_v is equal to the canonical measure μ_{Ar} on C_v for any vertex $v \in V$ with $\mathfrak{g}(v) \neq 0$. By definition, $\mu_{\mathrm{Zh\text{-}Ar}}$ does not have any mass on components of the metrized complex which are of genus zero.

5.4. **Tropical curves.** Let $\mathscr{C}^{\mathrm{trop}} = (\mathscr{G}, \pi)$ be a tropical curve. Recall that this means $\mathscr{G}$ is an augmented metric graph which has a finite graph model $G = (V, E)$ with an edge length function $\ell : E \to \mathbb{R}_+$, and we have an ordered partition $\pi = (\pi_1, \dots, \pi_r)$ of the edge set E, for some integer r. Moreover, we have the normalization property $\sum_{e \in \pi_j} \ell_e = 1$. Denote by $\mathfrak{g} : V \to \mathbb{N} \cup \{0\}$ the corresponding genus function.

Denote by $\mathrm{gr}_\pi^j(\mathscr{G})$ the metric graph associated to the graded minor $\mathrm{gr}_\pi^j(G)$ and the edge length function $\ell^j : \pi_j \to \mathbb{R}_+$, the restriction of ℓ to π_j. Denote by μ_{Zh}^j the canonical measure of the metric graph $\mathrm{gr}_\pi^j(\mathscr{G})$. By an abuse of the notation, we denote by μ_{Zh}^j the corresponding measure on $\mathscr{G}$ with support in the intervals $\mathcal{I}_e$ for $e \in \pi_j$. We can write

$$\mu_{\mathrm{Zh}}^j = \sum_{e \in \pi_j} \frac{\mu^j(e)}{\ell_e} d\theta_e$$

for the uniform Lebesgue measure $d\theta_e$ on the interval $\mathcal{I}_e$ and the Foster coefficients $\mu^j(e)$, $e \in \pi_j$ of the metric graph $\mathrm{gr}_\pi^j(\mathscr{G})$.

The canonical measure μ^{can} of the tropical curve $\mathscr{C}^{\mathrm{trop}}$ is the measure on $\mathscr{G}$ given by

$$\mu^{can} := \sum_{v \in V} \mathfrak{g}(v) \delta_v + \sum_{j=1}^r \sum_{e \in \pi_j} \frac{\mu^j(e)}{\ell_e} d\theta_e.$$

5.5. **Hybrid curves.** Consider now a hybrid curve $\mathscr{C}^{hyb}$, which, we recall, by definition is a metrized complex $\mathcal{MC}$ with the underlying graph $G = (V, E)$, the edge length function ℓ and an ordered partition π of the edge set E, with the normalization property that the sum of edge lengths in each layer is equal to one. Denote by $\mathscr{C}^{\mathrm{trop}}$ the corresponding tropical curve. For each vertex v, let C_v be the corresponding smooth compact Riemann surface of genus $\mathfrak{g}(v)$. The canonical measure μ^{can} on $\mathscr{C}^{hyb}$ is by definition the measure which restricts to the canonical measure of the tropical curve $\mathscr{C}^{\mathrm{trop}}$ on the intervals $\mathcal{I}_e$, for each edge $e \in E$, and which coincides with the canonical measure on each Riemann surface C_v of positive genus $\mathfrak{g}(v) > 0$.

The following is straightforward.

Proposition 5.2. *Let $\mathfrak{q} : \mathscr{C}^{hyb} \to \mathscr{C}^{\mathrm{trop}}$ be the natural forgetful projection map which contracts each Riemann surface C_v to the vertex v of $\mathscr{C}^{\mathrm{trop}}$. The push-out of the canonical measure on $\mathscr{C}^{hyb}$ by $\mathfrak{q}$ coincides with the canonical measure of $\mathscr{C}^{\mathrm{trop}}$.*

Note finally that canonical measures on hybrid curves provide a common generalization of all the previously introduced notions of canonical measures on metric graphs, Riemann surfaces and metrized curve complexes.

6. Continuity of the universal canonically measured tropical curve

The aim of this section is to prove that the universal family $\mathscr{C}_G^{\mathrm{trop}}$ of canonically measured tropical curves with a given combinatorial type $G = (V, E, \mathfrak{g})$ form a continuous family over the moduli space $\mathscr{M}_G^{\mathrm{trop}}$, introduced in Section 4.4.2. The result will be needed in the last section to conclude the proof of the main theorem.

The theorem we prove in this section reads as follows.

Theorem 6.1 (Continuity: tropical curves)**.** *The universal family of canonically measured tropical curves $\mathscr{C}_G^{\mathrm{trop}}$ of given combinatorial type G over $\mathscr{M}_G^{\mathrm{trop}}$ is a continuous family. That is, for every continuous function $f : \mathscr{C}_G^{\mathrm{trop}} \to \mathbb{R}$, the function $F : \mathscr{M}_G^{\mathrm{trop}} \to \mathbb{R}$ defined by integration along fibers*

$$F(x) := \int_{\mathscr{C}_x^{\mathrm{trop}}} f_{|_{\mathscr{C}_x^{\mathrm{trop}}}} \, d\mu_x^{can}, \qquad x \in \mathscr{M}_G^{\mathrm{trop}}$$

satisfies the continuity condition

$$\lim_{y \to x} F(y) = F(x)$$

for all points $x \in \mathscr{M}_G^{\mathrm{trop}}$.

In the above theorem, $\mathscr{C}_x^{\mathrm{trop}}$ denotes the fibre over a point $x \in \mathscr{M}_G^{\mathrm{trop}}$ in the universal family $\mathscr{C}_G^{\mathrm{trop}}$ (which, intuitively, we think of as the tropical curve defined by x). Since we consider coarse moduli spaces, actually $\mathscr{C}_x^{\mathrm{trop}}$ coincides with the quotient of the tropical curve associated to x, which we denote by $\widetilde{\mathscr{C}}_x^{\mathrm{trop}}$, by the action of its automorphism group. Each fibre $\mathscr{C}_x^{\mathrm{trop}}$ is naturally equipped with its corresponding canonical measure μ_x^{can} (i.e., the push-out of the canonical measure on the tropical curve $\widetilde{\mathscr{C}}_x^{\mathrm{trop}}$ through the quotient map).

We start by recalling the definition of $\mathscr{M}_G^{\mathrm{trop}}$, making precise its topology, and defining the universal tropical curve $\mathscr{C}_G^{\mathrm{trop}}$.

6.1. **The moduli space $\mathscr{M}_G^{\mathrm{trop}}$.** Let $G = (V, E, \mathfrak{g})$ be a fixed augmented graph. Denote by $\mathscr{M}_G^{\mathrm{trop}}$ the moduli space of tropical curves of combinatorial type G (see Section 4.4.2). Recall that, as a set, $\mathscr{M}_G^{\mathrm{trop}}$ is defined by taking the quotient

$$\mathscr{M}_G^{\mathrm{trop}} = \widetilde{\mathscr{M}}_G^{\mathrm{trop}}/\mathrm{Aut}(G), \tag{6.1}$$

where

$$\widetilde{\mathscr{M}}_G^{\mathrm{trop}} = \bigsqcup_{\pi\in\Pi(E)} \sigma_\pi^\circ.$$

In the following, we equip the spaces $\mathscr{M}_G^{\mathrm{trop}}$ and $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ with suitable topologies. We will define a topology on $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ by specifying for each point $x \in \widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ its system of neighborhoods $\mathcal{U}(x)$ and then put the quotient topology on $\mathscr{M}_G^{\mathrm{trop}}$.

It suffices to define a neighborhood base $\big(U^\varepsilon(x)\big)_{\varepsilon>0}$ around each $x \in \widetilde{\mathscr{M}}_G^{\mathrm{trop}}$. Hence, assume that $x \in \widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ and $\varepsilon > 0$ are fixed. Suppose that $x \in \sigma_\pi^\circ \subset \widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ for the ordered partition π of E. We will define $U^\varepsilon(x)$ as a union

$$U^\varepsilon(x) := \bigcup_{\pi'\preceq\pi} U^\varepsilon_{\pi'},$$

where the subset $U^\varepsilon_{\pi'} \subseteq \sigma^\circ_{\pi'}$ is defined as follows:
Assume that $\pi' = (\pi'_i)_{i=1}^r$ is an ordered partition of E and that $\pi' \preceq \pi$. Then, in this case, π has the following form

$$\pi = \Big((\varrho^i)_{i=1}^r\Big), \tag{6.2}$$

where $\varrho^i = (\varrho^i_k)_k$ is an ordered partition of π'_i for $i = 1, \dots, r$.
A point $y \in \sigma^\circ_{\pi'}$ belongs to $U^\varepsilon_{\pi'}$ if the following condition is satisfied:

- For each $i \in \{1, ..., r\}$, consider the ordered partition $\varrho^i = (\varrho^i_k)_{k=1}^{r_i}$ of π'_i. Then

$$\max_k \left\{ \frac{\sum_{e\in\varrho^i_{k+1}} y_e}{\sum_{e\in\varrho^i_k} y_e} \right\} < \varepsilon \qquad \text{and} \qquad \max_k \max_{e\in\varrho^i_k} \left| \frac{y_e}{\sum_{\hat e\in\varrho^i_k} y_{\hat e}} - x_e \right| < \varepsilon. \tag{6.3}$$

This defines a topology on $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ as stated in the following theorem.

Theorem 6.2. *There is a unique topology on $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ such that $\mathcal{U}(x)$ coincides with the system of neighborhoods for any $x \in \widetilde{\mathscr{M}}_G^{\mathrm{trop}}$.*

It is also clear that $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ is a second countable and compact Hausdorff space. We now characterize the convergence of sequences, similar to Proposition 3.3.

Proposition 6.3. *Let $x \in \widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ and assume $x \in \sigma_\pi^\circ$ for an ordered partition $\pi \in \Pi(E)$. Suppose that $(x_n)_n$ is a sequence in $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$. Then:*

(a) If x_n converges to x in $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$, then almost all x_n belong to a stratum $\sigma^\circ_{\pi'}$ of ordered partitions π' of E with $\pi' \preceq \pi$.

(b) Assume that $(x_n)_n \subseteq \sigma^\circ_{\pi'}$ for some fixed ordered partition $\pi' \preceq \pi$. In particular, π is of the form (6.2). Then x_n converges to x in $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ if and only if the following condition holds:

- *Let $\varrho^i = (\varrho^i_k)_k$ be one of the ordered partitions in (6.2). For all k and $e \in \varrho^i_k$,*

$$\lim_{n\to\infty} \frac{x_{n,e}}{\sum_{\hat{e}\in\varrho^i_k} x_{n,\hat{e}}} = x_e. \tag{6.4}$$

Moreover, if $e \in \varrho^i_k$ and $e' \in \varrho^i_{k'}$ with $k < k'$, then

$$\lim_{n\to\infty} \frac{x_{n,e'}}{x_{n,e}} = 0 \tag{6.5}$$

Proof. The claims are simple consequences of the definition of the topology on $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$. □

6.2. **Universal tropical curve of given combinatorial type.** Each element $x \in \mathscr{M}_G^{\mathrm{trop}}$ corresponds to a tropical curve $\widetilde{\mathscr{C}}_x^{\mathrm{trop}}$ with underlying graph of combinatorial type $G = (V, E, \mathfrak{g})$. That is, $\widetilde{\mathscr{C}}_x^{\mathrm{trop}}$ consists of the augmented metric graph $\mathscr{G}_x$ (i.e., the metric realization of the pair (G, x)) and the layers of $\mathscr{G}_x$ are given by the ordered partition $\pi = (\pi_i)_{i=1}^r \in \Pi(E)$ associated to x (i.e., such that $\sigma_\pi^\circ \ni x$).

Informally, we define the universal tropical curve $\mathscr{C}_G^{\mathrm{trop}}$ over $\mathscr{M}_G^{\mathrm{trop}}$ as the one having fiber $\widetilde{\mathscr{C}}_x^{\mathrm{trop}}$ over any point $x \in \mathscr{M}_G^{\mathrm{trop}}$. However, in our coarse moduli space setting this is not completely accurate. To be more precise, the fibre $\mathscr{C}_x^{\mathrm{trop}}$ in the family $\mathscr{C}_G^{\mathrm{trop}}$ over some point $x \in \mathscr{M}_G^{\mathrm{trop}}$ should coincide with the quotient of the tropical curve $\widetilde{\mathscr{C}}_x^{\mathrm{trop}}$ by the action of its automorphism group (i.e., automorphisms of G which fix the ordered partition and edge lengths of x).

In order to formalize this, notice that the original space $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ (see (6.1)) actually supports a natural family of tropical curves $\widetilde{\mathscr{C}}_G^{\mathrm{trop}}$ (whose fibres indeed coincide with the corresponding tropical curves). The action of the automorphism group $\mathrm{Aut}(G)$ on $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ then extends naturally to an action on $\widetilde{\mathscr{C}}_G^{\mathrm{trop}}$. This allows to introduce the universal tropical curve as the topological quotient

$$\mathscr{C}_G^{\mathrm{trop}} = \widetilde{\mathscr{C}}_G^{\mathrm{trop}} / \mathrm{Aut}(G).$$

By construction, there is a natural projection map from $\mathscr{C}_G^{\mathrm{trop}}$ to $\mathscr{M}_G^{\mathrm{trop}} = \widetilde{\mathscr{M}}_G^{\mathrm{trop}}/\mathrm{Aut}(G)$ whose fibres are the quotients described above.

The construction of the family $\widetilde{\mathscr{C}}_G^{\mathrm{trop}}$ follows along standard lines and we include it for the sake of completeness. First, for each edge $e \in E$, we define the interval fiber bundle $\mathcal{I}_e$ over $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ as the sum

$$\mathcal{I}_e = \bigsqcup_{\pi\in\Pi(E)} \mathcal{I}_\pi^e.$$

Here $\mathcal{I}_\pi^e$ is the interval fiber bundle over σ_π° introduced in (4.13). By its structure as a sum, $\mathcal{I}_e$ comes with a natural projection map to $\widetilde{\mathscr{M}}_G^{\mathrm{trop}} = \bigsqcup_{\pi\in\Pi(E)} \sigma_\pi^\circ$. To obtain the family of tropical curves $\widetilde{\mathscr{C}}_G^{\mathrm{trop}}$, we then glue the extremities of the intervals $\mathcal{I}_e$ above any point of the base $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ using the incidence relations between vertices and edges in G. In other words, $\widetilde{\mathscr{C}}_G^{\mathrm{trop}}$ is defined as the topological quotient

$$\widetilde{\mathscr{C}}_G^{\mathrm{trop}} = \bigsqcup_{e\in E} \mathcal{I}_e / \sim,$$

where over each base point $x \in \mathscr{M}_G^{\mathrm{trop}}$, we identify the interval endpoints $s_u^e(x)$ (see Section 4.3.4) for all edges e adjacent to some common vertex $u \in V(G)$.

6.3. **Tropical canonical measures.** Let $\pi = (\pi_1, \dots, \pi_r)$ be an ordered partition of E. Recall the definition of the graded minors in this setting (see Section 2.3): we get the filtration

$$\mathscr{E}_\pi^\bullet : \quad E_\pi^1 = E \supsetneq E_\pi^2 \supsetneq \cdots \supsetneq E_\pi^r \supsetneq E_\pi^{r+1} = \varnothing,$$

where $E_\pi^j := \bigcup_{i \geqslant j} \pi_i$. We then define $G_\pi^j = (V, E_\pi^j)$ and the j^{th} graded minor of G is

$$\mathrm{gr}_\pi^j(G) := G_\pi^j / E_\pi^{j+1}, \tag{6.6}$$

that is, $\mathrm{gr}_\pi^j(G)$ is the graph obtained from G_π^j by contracting all edges of E_π^{j+1}. It has vertex set denoted by V_π^j and edge set π_j.

For each $x \in \sigma_\pi^\circ$, denote by $\mathscr{G}_x$ the metric graph obtained as the metric realization of $(V, E, \mathfrak{g}, x)$. We introduced in Section 5.4 a canonical measure on the tropical curve corresponding to $\mathscr{G}_x$ with the layering induced by π. Recall that it is of the form

$$\mu^{can} = \sum_{v \in V} \mathfrak{g}(v) \delta_v + \sum_{j=1}^r \sum_{e \in \pi_j} \frac{\mu^j(e)}{\ell_e} d\theta_e,$$

where $d\theta_e$ denotes the uniform Lebesgue measure on the edge $e \in E$ and $\mu^j(e)$ is the Foster coefficient of the metric graph $\mathrm{gr}_\pi^j(\mathscr{G}_x)$ (i.e., the metric realization of the graded minor $\mathrm{gr}_\pi^j(G)$ with the edge length function x restricted on π_j). This means that the canonical measure on the metric graph $\mathrm{gr}_\pi^j(\mathscr{G}_x)$ is

$$\mu_{\mathrm{Zh}}^j = \sum_{e \in \pi_j} \frac{\mu^j(e)}{\ell_e} d\theta_e.$$

6.4. **Proof of Theorem 6.1.** The continuity property for tropical curves stated in Theorem 6.1 will follow from the following lemma:

Lemma 6.4. *Let ℓ be in $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ and $\pi = (\pi_j)_{j=1}^r$ its ordered partition (i.e., we have $\ell \in \sigma_\pi^\circ$). Suppose that $(\ell_n)_n \subseteq \widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ is a sequence with $\lim_{n\to\infty} \ell_n = \ell$ in $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ and $\ell_n \in \sigma_{(E)}^\circ$ for all n. Then for all $j = 1, \dots, r$ and $e \in \pi_j$,*

$$\lim_{n \to \infty} \mu_n(e) = \mu_\ell^j(e). \tag{6.7}$$

Here $\mu_n(e)$, $e \in E$ are the Foster coefficients of the metric graphs $\mathscr{G}_n$ (i.e., the metric realizations of (V, E, ℓ_n)) and $\mu_\ell^j(e)$, $e \in \pi_j$ are the Foster coefficients of the j^{th} graded minor $\mathrm{gr}_\pi^j(\mathscr{G}_\ell)$.

Proof. Let $\pi = (\pi_1, \dots, \pi_r)$ with $E = \pi_1 \sqcup \cdots \sqcup \pi_r$. To prove (6.7), we use the definition of the canonical measure on $\mathscr{G}_n$ given in (5.2). Let $T \in \mathcal{T}(G)$ be a fixed spanning tree of G and recall that its weight $\omega_n(T)$ in $\mathscr{G}_n$ is given by (see (5.1))

$$\omega_n(T) = \prod_{e \notin E(T)} \ell_n(e) = \prod_{i=1}^r \prod_{e \in \pi_i \setminus E(T)} \ell_n(e).$$

We claim that the following limit

$$\omega_\infty(T) := \lim_{n\to\infty} \omega_n(T) \prod_{i=1}^{r} \Big(\sum_{e\in\pi_i} \ell_n(e) \Big)^{-h_\pi^i} \tag{6.8}$$

exists for all $T \in \mathcal{T}(G)$. Here h_π^i is the genus of the graded minor $\mathrm{gr}_\pi^i(G)$.

Assume first that T is of the form $T = \bigcup_{i=1}^r T_i$, where each T_i is a spanning tree of $\mathrm{gr}_\pi^i(G)$. Recall from Section 2, Proposition 2.4, that every union of spanning trees of the graded minors $\mathrm{gr}_\pi^i(G)$ is indeed a spanning tree of T, and the set of such spanning trees is denoted by $\mathcal{T}_\pi(G)$. In this case, for all $i = 1, \dots, r$, we get

$$\big|\, \pi_i \backslash E(T) \,\big| = \big|\, E(\mathrm{gr}_\pi^i(G)) \backslash E(T_i) \,\big| = h_\pi^i.$$

Moreover, since $\lim_{n\to\infty} \ell_n = \ell$ in $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$, the limit in (6.8) exists and equals

$$\omega_\infty(T) = \prod_{i=1}^{r} \prod_{e\in\pi_i\backslash E(T_i)} \ell(e) = \prod_{i=1}^{r} \omega_{\mathrm{gr}_\pi^i}(T_i), \tag{6.9}$$

where $\omega_{\mathrm{gr}_\pi^i}(T_i)$ denotes the weight of T_i in the graded minor $\mathrm{gr}_\pi^i(G)$, equipped with the edge lengths of $\ell \in \widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ restricted to $\pi_i = E(\mathrm{gr}_\pi^i(G))$.

We now prove for all other spanning trees T of G, i.e., with $T \notin \mathcal{T}_\pi(G)$, the limit defining $\omega_\infty(T)$ is zero.

Let $T \in \mathcal{T}(G)$ be an arbitrary spanning tree of G. Since $G_\pi^i \cap T$ contains no cycles, we get the inequality (see Proposition 2.2 for the last equality)

$$\big|\, E_\pi^i \backslash E(T)\big| = \sum_{j=i}^{r} \big|\, \pi_j \backslash E(T) \,\big| \geqslant h(G_\pi^i) = \sum_{j=i}^{r} h_\pi^j, \qquad i = 1, \dots, r. \tag{6.10}$$

If equality holds true in (6.10) for all $i = 1, \dots, r$, then $(V, E_\pi^i \cap E(T))$ is a spanning tree of G_π^i for all i. However, it is easy to see that then $T^i := \mathrm{gr}_\pi^i \cap T$ is a spanning tree of gr_π^i for all $i = 1, \dots, r$. That is, $T \in \mathcal{T}_\pi(G)$ and this case was already treated above. Hence we can assume that the inequality in (6.10) is strict for some $i_0 \in \{1, \dots, r\}$. We now show that the limit in (6.8) is zero in this case.

To prove this, we will find a decomposition of the edges of $E(G)\backslash E(T)$ into a disjoint union

$$E\backslash E(T) = \bigsqcup_{i=1}^{r} \varrho_i$$

where each ϱ_i is a subset of $\bigcup_{j\geqslant i} \pi_j \backslash E(T)$ and contains exactly h_π^i edges. Such a decomposition can indeed be easily constructed inductively, starting from ϱ_r: by (6.10), there are at least h_π^r edges in $\pi_r\backslash E(T)$ and choosing h_π^r of them, we define ϱ_r. Now assume we already constructed edge sets $\varrho_{i+1}, \cdots, \varrho_r$ with the claimed properties. By (6.10), the total number of edges in $\bigcup_{j\geqslant i} \pi_j \backslash E(T)$ which are not already contained in one of the ϱ_j with $j \geqslant i+1$, is

$$\sum_{j=i}^{r} \big|\, \pi_j \backslash E(T) \,\big| - \sum_{j>i}^{r} \big|\varrho_j\big| = \sum_{j=i}^{r} \big|\pi_j \backslash E(T)\big| - \sum_{j>i}^{r} h_\pi^j \geqslant h_\pi^i.$$

Hence, ϱ_i can be defined by choosing h^i_π many of the remaining edges of $\bigcup_{j\geqslant i}\pi_j\backslash E(T)$. Proceeding inductively, we end up with a sequence $(\varrho_i)_{i=1}^r$ with the claimed properties.

Notice that by our assumption on T, one of the edge sets ϱ_i must contain an edge from some π_j with $j > i$. Since $\lim_{n\to\infty}\ell_n = \ell$ in $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$, it follows from the characterization of the convergence of sequences in $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$, given in Proposition 6.3, that

$$\omega_\infty(T) = \lim_{n\to\infty}\prod_{i=1}^{r}\prod_{e\in\varrho_i}\frac{\ell_n(e)}{\sum_{\hat e\in\pi_i}\ell_n(\hat e)} = 0,$$

and we have proved our claim.

Taking into account now the representation (5.2), this implies that for all $e\in E$,

$$\lim_{n\to\infty}\mu_n(e) = \sum_{\substack{T\in\mathcal{T}_\pi(G):\\ e\notin E(T)}}\omega_\infty(T)\Big/\sum_{T\in\mathcal{T}_\pi(G)}\omega_\infty(T).$$

Assume now that the edge $e\in E$ belongs to the set π_i of the partition π. Then it follows from (6.9) that

$$\begin{aligned}\lim_{n\to\infty}\mu_n(e) &= \Big(\sum_{\substack{T_i\in\mathcal{T}(\mathrm{gr}^i_\pi(G)):\\ e\notin E(T_i)}}\omega_{\mathrm{gr}^i_\pi}(T_i)\Big/\sum_{T_i\in\mathcal{T}(\mathrm{gr}^i_\pi(G))}\omega_{\mathrm{gr}^i_\pi}(T_i)\Big)\\ &\qquad\times\frac{\prod_{j\neq i}\sum_{T_j\in\mathcal{T}(\mathrm{gr}^j_\pi(G))}\omega_{\mathrm{gr}^j_\pi}(T_j)}{\prod_{j\neq i}\sum_{T_j\in\mathcal{T}(\mathrm{gr}^j_\pi(G))}\omega_{\mathrm{gr}^j_\pi}(T_j)}\\ &= \Big(\sum_{\substack{T_i\in\mathcal{T}(\mathrm{gr}^i_\pi(G)):\\ e\notin E(T_i)}}\omega_{\mathrm{gr}^i_\pi}(T_j)/\sum_{T_i\in\mathcal{T}(\mathrm{gr}^i_\pi(G))}\omega_{\mathrm{gr}^i_\pi}(T_j)\Big).\end{aligned}$$

Using (6.9) and the representation (5.2) once again, we have arrived at the Foster coefficients of the graded minor $\mathrm{gr}^i_\pi(G)$ and (6.7) is proved. ☐

Remark 6.5. One can obtain an alternative proof of Lemma 6.4 by using the representation of μ_{Zh} given in Theorem 5.1 and arguing similar as in Section 9.4. ◇

Proof of Theorem 6.1. We will prove the stronger claim that the family $\widetilde{\mathscr{C}}_G^{\mathrm{trop}}$, where each fibre $\widetilde{\mathscr{C}}_x^{\mathrm{trop}}$ is equipped with the corresponding canonical measure for tropical curves, is a continuous family of measured spaces over $\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$. Since $\mathscr{M}_G^{\mathrm{trop}}$ and $\mathscr{C}_G^{\mathrm{trop}}$ are defined as topological quotients, Theorem 6.1 then follows as a direct consequence.

Taking into account the definition of the canonical measure in terms of the Foster coefficients (see Section 5.4), Lemma 6.4 proves the continuity at each point $\ell\in\widetilde{\mathscr{M}}_G^{\mathrm{trop}}$ through the stratum $\sigma^\circ_{(E)}$. The proof in the general case follows by fixing an ordered partition $\pi'\preceq\pi$ and applying Lemma 6.4 to each graded minor $\mathrm{gr}^j_{\pi'}(G)$. ☐

7. Monodromy

This section recalls basic results about degenerations of Riemann surfaces. In particular, we will recall an interpretation of the rank one symmetric matrices M_e introduced in (5.5) in terms of the monodromy of a degenerating family of Riemann surfaces. In the next section, we relate these results to the asymptotic of the period mapping. Our main references are again [ABBF16, ACG11, Hof84]. We will use the notations of Section 4.1.

Let S_0 be a stable curve of genus g with dual graph $G = (V, E, \mathfrak{g})$. From the discussion of Section 4.1, we get an analytic family $\mathcal{S} \to B$ of stable Riemann surfaces over a polydisc B of dimension $N = 3g-3$ with fiber $\mathcal{S}_0 = S_0$ over $0 \in B$. This means that $B = \underbrace{\Delta \times \Delta \times \cdots \times \Delta}_{N \text{ times}}$ for Δ a small disk around 0 in $\mathbb{C}$. We have a collection of analytic divisors $D_e \subset B$ which are defined by the equations $\{z_e = 0\}$, where D_e is the locus of all points $t \in B$ such that in the family $\mathfrak{p} : \mathcal{S} \to B$, the fiber $\mathfrak{p}^{-1}(t)$ has a singular point corresponding to e.

Let $B^* := B \backslash \bigcup_{e\in E} D_e$ and set $\mathcal{S}^* := \mathfrak{p}^{-1}(B^*)$ to be the locus of points whose fibers in the family π are smooth.

Fix a base-point $\flat \in B^*$. The fundamental group $\pi_1\big(B^*, \flat\big)$ is isomorphic to $\mathbb{Z}^E$, with a generator λ_e per edge $e \in E$ corresponding to a simple loop $\lambda_e \subset B^*$ which is based at $\flat$ and turns once around the divisor D_e, and which is moreover contractible in the space $B \backslash \bigcup_{e' \neq e} D_{e'}$.

From the family $\mathfrak{p} : \mathcal{S}^* \to B^*$, we get the local system $\mathcal{H} := H_1(\mathcal{S}_\flat, \mathbb{Z})$ on B^* which is classified by the *monodromy action* of $\pi_1(B^*, \flat)$ on $H_1(\mathcal{S}_\flat, \mathbb{Z})$. Since λ_e for $e \in E$ form a system of generators for the fundamental group, it will be enough to describe the monodromy action of each λ_e on $H_1(\mathcal{S}_\flat, \mathbb{Z})$.

Picard-Lefschetz theory describes the action of λ_e on $H_1(\mathcal{S}_\flat, \mathbb{Z})$ as follows. There is a *vanishing cycle* a_e *in* $H_1(\mathcal{S}_\flat, \mathbb{Z})$ associated to the singular point p^e of S_0 such that the monodromy action of λ_e is given by sending

$$\lambda_e : H_1(\mathcal{S}_\flat, \mathbb{Z}) \longrightarrow H_1(\mathcal{S}_\flat, \mathbb{Z}) \tag{7.1}$$

$$\beta \mapsto \beta - \langle \beta, a_e \rangle a_e, \tag{7.2}$$

for any β in $H_1(S_0, \mathbb{Z})$. Here $\langle \cdot\, , \cdot \rangle$ denotes the intersection pairing between one-cycles in $H_1(\mathcal{S}_\flat)$.

7.1. **Specialization.** After possibly shrinking the polydisc B, the inclusion $S_0 \hookrightarrow \mathcal{S}$ admits a deformation retraction $\mathcal{S} \to S_0$. It follows that S_0 and $\mathcal{S}$ have the same homology groups. From the composition of the inclusion $\mathcal{S}_\flat \hookrightarrow \mathcal{S} \simeq_{\text{homotopy}} S_0$ with the homotopy equivalence, we get the *specialization map*

$$\mathrm{sp}\colon H_1(\mathcal{S}_\flat, \mathbb{Z}) \longrightarrow H_1(\mathcal{S}, \mathbb{Z}) \simeq H_1(S_0, \mathbb{Z}).$$

We admit the proof of the following proposition, cf. [ABBF16].

Proposition 7.1. *The specialization map* sp *is surjective.*

Let $A \subset H_1(\mathcal{S}_\flat, \mathbb{Z})$ denote the kernel of the specialization map. It corresponds precisely to the subspace spanned by the vanishing cycles a_e. We have an exact sequence

$$0 \to A \to H_1(\mathcal{S}_\flat, \mathbb{Z}) \xrightarrow{\mathrm{sp}} H_1(S_0, \mathbb{Z}) \to 0.$$

From the inclusion of subspaces $C_v \hookrightarrow S_0$, we get a short exact sequence

$$0 \to \bigoplus_{v\in V} H_1(C_v, \mathbb{Z}) \hookrightarrow H_1(S_0, \mathbb{Z}) \to H_1(G, \mathbb{Z}) \to 0.$$

Define
$$A' = A + \mathrm{sp}^{-1}\Big(\bigoplus_{v \in V} H_1(C_v, \mathbb{Z})\Big) \subseteq H_1(\mathcal{S}_\flat, \mathbb{Z}).$$
It follows that
$$H_1(\mathcal{S}_\flat, \mathbb{Z})/A' \simeq H_1(S_0, \mathbb{Z})/\bigoplus_{v \in V} H_1(C_v, \mathbb{Z}) \simeq H_1(G, \mathbb{Z}). \tag{7.3}$$

Recall that a subspace $H \subseteq H_1(S_\flat, \mathbb{Z})$ is called *isotropic* if for any pair of elements $a, b \in H$, the intersection pairing $\langle a, b\rangle$ vanishes.

Proposition 7.2. *The subspace $A \subset H_1(\mathcal{S}_\flat, \mathbb{Z})$ of vanishing cycles has rank equal to h and is isotropic. Moreover, we have $\langle A, A'\rangle = 0$.*

Proof. By surjectivity of the specialization map, we have
$$\mathrm{rank}\, A = \mathrm{rank}\, H_1(\mathcal{S}_\flat, \mathbb{Z}) - \mathrm{rank}\, H_1(S_0, \mathbb{Z}) = 2g - h - 2\sum_{v \in V} \mathfrak{g}(v) = h.$$

The vanishing cycles a_e are disjoint for $\flat$ chosen close to $0 \in B$, since they approach different singular points $p^e \in S_0$. Thus, $\langle a_e, a_{e'}\rangle = 0$, which shows that the subspace A is isotropic.

The space A' contains A and cycles which can be deformed to a cycle on a component C_v of S_0. Since the elements of A vanish in S_0, the pairing between A and cycles in any C_v is trivial, and the result follows. □

It follows that the intersection pairing involving vanishing cycles reduces to a pairing $A \times \Big(H_1(\mathcal{S}_\flat, \mathbb{Z})/A'\Big) \to \mathbb{Z}$.

7.2. Description of the monodromy action.

For every edge $e \in E$, define
$$N_e := \lambda_e - \mathrm{Id}\,.$$
By (7.1), for any $\beta \in H_1(\mathcal{S}_\flat, \mathbb{Z})$, we have $N_e(\beta) = \langle a_e, \beta\rangle a_e$. This shows that the image of N_e is contained in A, which combined by the previous proposition implies $N_e \circ N_e = 0$. We infer that $N_e = \log(\lambda_e)$. Note that (7.1) implies that actually N_e vanishes on A'.

It follows that $N_e : H_1(\mathcal{S}_\flat, \mathbb{Z}) \to A$ passes to the quotient by A' and induces
$$H_1(G, \mathbb{Z}) \simeq H_1(\mathcal{S}_\flat, \mathbb{Z})/A' \xrightarrow{N_e} A \simeq (H_1(\mathcal{S}_\flat, \mathbb{Z})/A')^\vee \simeq H_1(G, \mathbb{Z})^\vee. \tag{7.4}$$

Proposition 7.3. *The bilinear form on $H_1(G, \mathbb{Z})$ given by the composition of maps in (7.4) coincides with the bilinear form $\langle\cdot\,,\cdot\rangle_e$ restricted to $H_1(G, \mathbb{Z})$.*

Proof. For convenience, we reproduce the proof from [ABBF16]. For any $\beta \in H_1(\mathcal{S}_\flat, \mathbb{Z})$, the image $\mathrm{sp}(\beta)$ of β by the specialization map in the quotient $H_1(\mathcal{S}_\flat, \mathbb{Z})/A' \simeq H_1(G, \mathbb{Z})$, see (7.3), can be identified with a cycle γ in the graph. Choose an orientation of the edges and write $\gamma = \sum_e n_e e$. To find the coefficients n_e, we consider the pairing $\langle \gamma, a_e\rangle = \langle \beta, a_e\rangle$. For all edges $e' \neq e$, there is no intersection between e' and a_e, and for the edge e, e and a_e are transversal. This leads to the equation $n_e = \langle \beta, a_e\rangle$.

To conclude, note that the quadratic form on $H_1(\mathcal{S}_\flat, \mathbb{Z})$ associated to N_e sends β to $\langle\langle a_e, \beta\rangle a_e, \beta\rangle = n_e^2 = q_e(\gamma)$, where q_e is the quadratic form on $H_1(G, \mathbb{Z})$ associated to $\langle\cdot\,,\cdot\rangle_e$. From this the proposition follows. □

7.3. **Admissible symplectic basis.** Enumerate the vertices of the graph as $v_1, \dots, v_n$. Define $g_0, \dots, g_n$ by

$$g_i = h + \sum_{j=1}^{i} \mathfrak{g}(v_i).$$

Note that $g_0 = h$ and $g_n = g$

By an *admissible symplectic basis* $a_1, \dots, a_g, b_1, \dots, b_g$ for $H_1(\mathcal{S}_\flat, \mathbb{Z})$ we mean a symplectic basis such that

(1) $a_1, \dots, a_h$ form a basis of the space of vanishing cycles A, and

(2) for any $r = 1, \dots, n$, the collection of elements a_j and b_j for $j = g_{r-1} + 1, \dots, g_r$ gives a basis of $H_1(C_{v_r}, \mathbb{Z})$ in

$$H_1(C_{v_r}, \mathbb{Z}) \hookrightarrow H_1(S_0, \mathbb{Z}) \simeq H_1(\mathcal{S}_\flat, \mathbb{Z})/A.$$

Note that in (2) the elements a_j and b_j seen in $H_1(C_{v_r}, \mathbb{Z})$ necessarily form a symplectic basis of $H_1(C_{v_r}, \mathbb{Z})$. Indeed, the pairing between A and A' is trivial in $H_1(\mathcal{S}_\flat, \mathbb{Z})$, and so the intersection pairing in $H_1(C_{v_r}, \mathbb{Z})$ coincides with the one induced from $H_1(\mathcal{S}_\flat, \mathbb{Z})$ via the specialization map.

For any edge $e \in E$, the vanishing cycle a_e can be decomposed as a linear combination of the basis elements a_i of A:

$$a_e = \sum_{i=1}^{h} c_{e,i} a_i. \tag{7.5}$$

Let $B \subset H_1(\mathcal{S}_\flat, \mathbb{Z})$ be the subspace generated by the basis elements $b_1, \dots, b_h$. As we saw above, projection onto $H_1(\mathcal{S}_\flat, \mathbb{Z})/A'$ provides an isomorphism $B \simeq H_1(G, \mathbb{Z})$. The pairing $\langle \cdot, \cdot \rangle$ gives an isomorphism $B \simeq A^\vee$, and the monodromy operators λ_e give maps

$$N_e \colon B \longrightarrow A.$$

The following proposition is straightforward to prove.

Proposition 7.4. *In terms of the basis $b_1, \dots, b_h$ for $B \simeq H_1(G, \mathbb{Z})$, we can write $M_e = \Big(c_{e,i} c_{e,j}\Big)_{1 \leqslant i,j \leqslant h}$.*

In the following, we will write $\mathscr{A}$ (respectively $\mathscr{B}$) for the subspace of $H^1(\mathcal{S}_\flat, \mathbb{Z})$ generated by $a_1, \dots, a_g$ (respectively $b_1, \dots, b_g$). Then $\mathscr{A}$ is a maximal isotropic subspace with $A \subset \mathscr{A} \subset A'$.

Later on we will work with hybrid curves which thus come with a layering on their underlying dual graphs. In this situation, we impose the following additional condition in the definition of admissibility (see the next section for details).

(3) The elements $b_1, \dots, b_h$ form an admissible basis in the sense of Section 7.4 for the homology group $H_1(G, \mathbb{Z})$ of the layered dual graph G.

7.4. **Admissible basis for layered graphs and hybrid curves.** In this section, we provide a finer version of the notion of admissibility for the choice of a symplectic basis in context with hybrid curves. Later on, this will provide a refined control on the first $h \times h$ block matrices in the period matrices (corresponding to the spaces $\mathscr{A}$ and $\mathscr{B}$) and their limits.

Let $G = (V, E, \pi)$ be a layered graph of genus h with vertex set V, edge set E, and the ordered partition $\pi = (\pi_1, \dots, \pi_r)$ of E with depth r corresponding to the layers. Denote by $\mathscr{F}^\pi_\bullet$ and $\mathscr{E}^\bullet_\pi$ the corresponding increasing and decreasing filtrations of E (see Section 2.3),

$$\mathscr{F}^\pi_\bullet : \quad F_0 = \varnothing \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r = E$$
$$\mathscr{E}^\bullet_\pi : \quad E^1_\pi = E \supsetneq E^2_\pi \supsetneq \cdots \supsetneq E^r_\pi \supsetneq E^{r+1}_\pi = \varnothing.$$

Consider the associated decreasing sequence of spanning subgraphs of G,

$$G =: G^1_\pi \supset G^2_\pi \supset \cdots \supset G^r_\pi \supset G^{r+1}_\pi = (V, \varnothing)$$

with $G^j_\pi = (V, E^j_\pi)$, and let $\mathrm{gr}^j_\pi(G) : G^j_\pi / E^{j+1}_\pi$ be the corresponding graded minor of G.

Recall that $\mathscr{K}_j : G^j_\pi \to \mathrm{gr}^j_\pi(G)$ is the contraction map. By an abuse of the notation, we denote by $\mathscr{K}_j : H_1\big(G^j_\pi, \mathbb{Z}\big) \to H_1\big(\mathrm{gr}^j_\pi(G), \mathbb{Z}\big)$ the corresponding map on the level of homology groups. For future use, we record the following basic proposition.

Proposition 7.5. *The map $\mathscr{K}_j : H_1\big(G^j_\pi, \mathbb{Z}\big) \to H_1\big(\mathrm{gr}^j_\pi(G), \mathbb{Z}\big)$ is surjective.*

7.4.1. *Admissible basis for layered graphs.* Notations as in the previous section, let $G = (V, E)$ be a layered graph of genus h with the underlying ordered partition $\pi = (\pi_1, \dots, \pi_r)$.

By the genus formula, cf. Proposition 2.2, we have $h = h^1_\pi + \cdots + h^r_\pi$. Consider now the partition

$$[h] = J^1_\pi \sqcup J^2_\pi \sqcup \cdots \sqcup J^r_\pi \tag{7.6}$$

into intervals J^j_π of size h^j_π given by

$$J^j_\pi := \Big\{1 + \sum_{i=1}^{j-1} h^i_\pi,\ 2 + \sum_{i=1}^{j-1} h^i_\pi,\ \dots,\ \sum_{i=1}^{j} h^i_\pi \Big\}.$$

An *admissible basis* for a layered graph with the ordered partition $\pi = (\pi_1, \dots, \pi_r)$ is a basis $\gamma_1, \dots, \gamma_h$ of $H_1(G, \mathbb{Z})$ verifying the following additional condition.

- For each $j = 1, \dots, r$, we have the following properties.
  - (i) For each $k \in J^j_\pi$, the cycle γ_k lies in the spanning subgraph G^j_π of G. In other words, all the edges of γ_k belong to $E^j_\pi \subseteq E$.
  - (ii) The collection of cycles $\mathscr{K}_j(\gamma_k)$ for $k \in J^j_\pi$ forms a basis of $H_1(\mathrm{gr}^j_\pi(G), \mathbb{Z})$.

Proposition 7.6. *Any layered graph $G = (V, E)$ with ordered partition π admits an admissible basis.*

Proof. This is a consequence of the genus formula, cf. Proposition 2.2, and Proposition 7.5. We can indeed construct an admissible basis by first taking, for each $j = 1, \dots, r$, a basis γ^j_k of $\mathrm{gr}^j_\pi(G)$, for $k \in J^j_\pi$, and then lifting these cycles to cycles γ_k in G via the (surjective) projection maps $\mathscr{K}_j : G^j_\pi \to \mathrm{gr}^j_\pi(G)$. It is easy to see that the cycles

γ_k form a basis of $H_1(G, \mathbb{Z})$, and they obviously verify the two properties (i) and (ii) listed above. $\square$

7.4.2. *Admissible basis for hybrid curves.* Let $\mathscr{C}$ be a hybrid curve with underlying stable curve S_0 and layered dual graph $G = (V, E, \pi)$. Let $\pi = (\pi_1, \dots, \pi_r)$ be the ordered partition of E obtained by the layers, which is of depth r.

Enumerate the vertices of the graph as $v_1, \dots, v_n$. Define $g_0, \dots, g_n$ by

$$g_i = h + \sum_{j=1}^{i} \mathfrak{g}(v_i). \tag{7.7}$$

Note that $g_0 = h$ and $g_n = g$.

Let $\flat$ be a fixed base point in B^*. An *admissible symplectic basis* for $\mathscr{C}$ is a symplectic basis $a_1, \dots, a_g, b_1, \dots, b_g$ of $H_1(\mathcal{S}_\flat, \mathbb{Z})$ which verifies the following properties.

(1) The collection of elements $a_1, \dots, a_h$ form a basis of the space of vanishing cycles $A \subseteq H_1(\mathcal{S}_\flat, \mathbb{Z})$;

(2) for any $r = 1, \dots, n$, the collection of elements a_j and b_j for $j = g_{r-1}+1, \dots, g_r$ gives a basis of $H_1(C_{v_r}, \mathbb{Z})$ in
$$H_1(C_{v_r}, \mathbb{Z}) \hookrightarrow H_1(S_0, \mathbb{Z}) \simeq H_1(\mathcal{S}_\flat, \mathbb{Z})/A, \quad \text{ and}$$

(3) the elements $b_1, \dots, b_h \in B \simeq H_1(G, \mathbb{Z})$ form an admissible basis for the layered graph G, in the sense of the previous section.

In particular we infer from Proposition 7.6 that an admissible symplectic basis for $\mathscr{C}$ always exists.

8. Period map

The aim of this section is to describe the period map for the variation of Hodge structures $H_1(\mathcal{S}_t)$ for $t \in B^*$, for the family $\mathfrak{p} : \mathcal{S}^* \to B^*$ described in the previous section. We refer to [ABBF16, Hof84] for more details.

Recall first that $B = \Delta^{3g-3}$ and that $B^* = B \setminus \cup_{e \in E} D_e$. Shrinking the polydisk if necessary and making a choice of local parameters around 0 for D_e, we can write $B^* \simeq (\Delta^*)^E \times \Delta^{3g-3-|E|}$. Let $\widetilde{B^*}$ be the universal cover of B^*. We get an isomorphism $\widetilde{B^*} \simeq \mathbb{H}^E \times \Delta^{3g-3-|E|}$ where $\mathbb{H}$ is the Poincaré half-plane, and the map
$$\widetilde{B^*} \to B^*$$
is given by sending $\zeta_e \in \mathbb{H}$ to $\exp(2\pi i \zeta_e) \in \Delta^*$. (To be more precise, the map sends $\zeta_e \in \mathbb{H}$ to $c \exp(2\pi i \zeta_e) \in \Delta^*$ for some constant $c > 0$. To simplify, we assume $c = 1$.)

Denote by $\widetilde{\mathcal{S}}^*$ the family of Riemann surfaces over $\widetilde{B^*}$ induced from the family $\mathcal{S}^*/B^*$. In what follows, we use the notation of the preceding section. Recall that we have fixed a point $\flat \in B^*$ together with a lift $\tilde{\flat} \in \widetilde{B^*}$. We also choose an admissible symplectic basis
$$a_1, \dots, a_g, b_1, \dots, b_g \in H_1(\mathcal{S}_\flat, \mathbb{Z}) = \mathscr{A} \oplus \mathscr{B}.$$
Recall from Section 7.3 that the space of vanishing cycles A is generated by $a_1, \dots, a_h \in A$, and $b_1, \dots, b_h$ generate $H_1(\mathcal{S}_\flat, \mathbb{Z})/A' \simeq H_1(G, \mathbb{Z})$ as in (7.3), and the corresponding blocks of size $\mathfrak{g}(v_k)$ give a symplectic basis of $H_1(C_{v_k}, \mathbb{Z})$ for $k = 1, \dots, n$.

Denote by $\widetilde{\mathcal{H}}$ the local system $H^1(\widetilde{\mathcal{S}}_{\tilde{t}},\mathbb{Z}) = H_1(\mathcal{S}_t,\mathbb{Z})$ for any $t \in B^*$ and $\tilde{t} \in \widetilde{B^*}$ over it. The local system $\widetilde{\mathcal{H}}$ being trivial, we can spread out this symplectic basis to a symplectic basis

$$a_{1,\tilde{t}}, \dots, a_{g,\tilde{t}}, b_{1,\tilde{t}}, \dots, b_{g,\tilde{t}}$$

of $H_1(\widetilde{\mathcal{S}}_{\tilde{t}},\mathbb{Z})$ for any $t \in B^*$ and $\tilde{t} \in \widetilde{B^*}$ over it.

Since $\mathscr{A}$ is isotropic (see Proposition 7.2) and contains the subspace of vanishing cycles, the Picard-Lefschetz formula (7.1) implies that the elements $a_{i,\tilde{t}}$ only depend on t and not on $\tilde{t}$. Thus, we will denote them merely by $a_{i,t}$. The same remark actually applies for the basis elements $b_{j,\tilde{t}}$ with $j > h$ (see again Proposition 7.2). So the only non-trivial part of the symplectic basis regarding the monodromy action is $b_{1,\tilde{t}}, \dots, b_{h,\tilde{t}}$.

If there is no risk of confusion, we drop $\tilde{t}$, and simply use a_i and b_i for the elements, having in mind the dependency on t or $\tilde{t}$, with the above remarks.

The holomorphic vector bundle $\mathcal{H} \otimes_{\mathbb{Z}} \mathcal{O}_{B^*}$ admits a canonical extension to a holomorphic vector bundle over B. We denote this by $\mathcal{H}_{\mathbb{C}}$. Moreover, the Hodge filtration $F^0\mathcal{H} \otimes_{\mathbb{Z}} \mathcal{O}_{B^*}$ can be extended to a coherent subsheaf of $\mathcal{H}_{\mathbb{C}}$. This means we can find a collection of holomorphic one-forms $(\omega_i)_{i=1,\dots,g}$ on $\mathcal{S}$ such that, for each $t \in B$, the forms $\omega_{i,t} := \omega_i|_{\mathcal{S}_t}$, $i = 1, \dots, g$ form a basis of the holomorphic differentials on $\mathcal{S}_t$ and

$$\int_{a_{i,t}} \omega_{j,t} = \delta_{i,j} \tag{8.1}$$

holds for any $t \in B^*$. It follows that the period matrix for each fiber $\widetilde{\mathcal{S}}^*_{\tilde{t}}$ in the family of Riemann surfaces $\widetilde{\mathcal{S}}^*$ over $\widetilde{B^*}$ is given by

$$\Omega_{\tilde{t}} = \left(\int_{b_{i,\tilde{t}}} \omega_{j,t} \right).$$

Let $\mathbb{H}_g$ be the Siegel domain defined as

$$\mathbb{H}_g := \Big\{ g \times g \text{ complex symmetric matrix } \Omega \mid \mathrm{Im}(\Omega) > 0 \Big\}.$$

We get the following proposition.

Proposition 8.1. *The period map of the variation of Hodge structures $H_1(\mathcal{S}_t,\mathbb{Z})$ over B^* is given by*

$$\widetilde{\Phi} \colon \widetilde{B^*} \longrightarrow \mathbb{H}_g \tag{8.2}$$

$$\tilde{t} \longmapsto \Big(\int_{b_{i,\tilde{t}}} \omega_{j,t} \Big)_{i,j}. \tag{8.3}$$

The group $\mathrm{Sp}_{2g}(\mathbb{R})$ acts on $\mathbb{H}_g$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \Omega = (A\Omega + B)(C\Omega + D)^{-1}.$$

The Siegel moduli space is a smooth Deligne-Mumford stack defined as the quotient $\mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g$ and parametrizes principally polarized abelian varieties of dimension

g. As we will see below, the image of the monodromy map for the local system $\mathcal{H}$ lives in $\mathrm{Sp}_{2g}(\mathbb{Z})$.

We now explicitly describe the action of the logarithm of monodromy maps N_e, $e \in E$ on the entries in (8.3). As in (7.5), for each edge e, we can write

$$a_e = \sum_{i=1}^{g} c_{e,i} a_i. \tag{8.4}$$

Moreover, since $a_e \in A$, the coefficients $c_{e,i}$ are zero for $i > h$.

By the Picard-Lefschetz formula (7.1), we deduce that

$$N_e(b_i) = \langle a_e, b_i \rangle a_e = c_{e,i} a_e. \tag{8.5}$$

The forms ω_j are defined globally over $\mathcal{S}^*$, so they are invariant under monodromy. The integral of these forms with respect to the vanishing cycles is computed by

$$\int_{a_e} \omega_j = \sum_{i=1}^{h} c_{e,i} \int_{a_i} \omega_j = c_{e,j}. \tag{8.6}$$

Applying (8.5) and (8.6), we deduce that

$$N_e\Big(\int_{b_{i,\tilde{\mathfrak{b}}}} \omega_{j,\mathfrak{b}}\Big) = \langle a_e, b_i \rangle \int_{a_e} \omega_j = c_{e,i} c_{e,j}.$$

We introduce the $g \times g$ matrices $\widetilde{M}_e = \Big(\widetilde{M}_e(i,j)\Big)_{i,j=1}^{g}$ by

$$\widetilde{M}_e(i,j) := \begin{cases} c_{e,i} c_{e,j} & \text{if } i, j \leqslant h \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 7.4, the matrix $\widetilde{M}_e$ is the $h \times h$ matrix M_e from the previous sections which we have filled with zeros to a $g \times g$ matrix. In particular, the entries of this matrix are all integers.

Altogether, we have proven that the logarithm of the monodromy of the element λ_e is given by the following element of the Lie algebra $\mathrm{sp}_{2g}(\mathbb{Z})$

$$N_e = \begin{pmatrix} 0 & \widetilde{M}_e \\ 0 & 0 \end{pmatrix}. \tag{8.7}$$

It follows that the image of the monodromy lives in $\mathrm{Sp}_{2g}(\mathbb{Z})$ and the map $\widetilde{\Phi}$ descends to B^*, resulting in the following commutative diagram.

$$\begin{tikzcd} \widetilde{B^*} \arrow[r, "\widetilde{\Phi}"] \arrow[d] & \mathbb{H}_g \arrow[d] \\ B^* \arrow[r, "\Phi"] & \mathrm{Sp}_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g. \end{tikzcd} \tag{8.8}$$

8.1. **Asymptotic of the period map.** The asymptotic of the period map can be described by the nilpotent orbit theorem. We refer to the paper by Schmidt [Sch73] and Cattani-Kaplan-Schmidt [CKS86] for the basic case of variations of polarized pure Hodge structures needed here.

We separate the coordinate variables corresponding to the edges as z_E, and write the coordinates of any point t of B^* as $z_E \times z_{E^c}$. The coordinates in the universal cover $\widetilde{B^*}$ will be denoted by ζ_e. In these coordinates, the projection $\widetilde{B^*} \to B^*$ is given by

$$z_e = \begin{cases} \exp(2\pi i \zeta_e), & \text{for } e \in E, \\ \zeta_e, & \text{for } e \notin E. \end{cases} \tag{8.9}$$

The *twisted period map* $\widetilde{\Psi}$ on $\widetilde{B^*}$ given by

$$\widetilde{\Psi}(\tilde{t}) = \exp(-\sum_E \zeta_e N_e)\widetilde{\Phi}(\tilde{t}) \tag{8.10}$$

takes values in the compact dual $\check{D}$ of the period domain $D = \mathcal{H}_g$, and it is (essentially) a flag variety parametrizing filtrations $F^*\mathbb{C}^{2g}$ satisfying the conditions of being a Hodge filtration of weight -1 and genus g. The space $\check{D}$ contains $\mathbb{H}_g$ as an open subset and the action of the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$ can be extended to $\check{D}$.

The twisted period map $\widetilde{\Psi}$ is invariant under the transformation $\zeta_e \mapsto \zeta_e + 1$, and so descends to a map $\Psi : B^* \to \check{D}$.

The following result is the nilpotent orbit theorem that we need. Recall that Δ is a disk of small radius and $B = \Delta^{3g-3}$. As before, for any point $t \in B$, we denote by E_t the set of all $e \in E$ with $z_e(t) = 0$.

Theorem 8.2 (Nilpotent orbit theorem)**.** *After shrinking the radius of Δ if necessary, the map Ψ extends to a holomorphic map*

$$\Psi : B \longrightarrow \check{D}.$$

Moreover, there exists a constant T_0 such that for all $t \in B$, if for $e \in E_t$, we have $\mathrm{Im}(\zeta_e) \geqslant T_0$, then

$$\exp(\sum_{e \in E_t} \zeta_e N_e)\Psi(t) \in \mathbb{H}_g.$$

Furthermore, there are constants $C, \beta > 0$ so that we have the following estimate. For each $t \in B$, there exists a small open disc $U_t \subseteq B$ containing t such that

$$\mathrm{dist}\Big(\widetilde{\Phi}(\tilde{s}), \exp(\sum_{e \in E_t} \zeta_e(\tilde{s})N_e)\Psi(t)\Big) \leqslant C \sum_{e \in E_t} \mathrm{Im}(\zeta_e(\tilde{s}))^\beta \exp(-2\pi\,\mathrm{Im}(\zeta_e(\tilde{s})))$$

for any $s \in U_t \backslash D$ and $\tilde{s}$ above s in $\widetilde{B^*}$.

In what follows we denote the period map $\widetilde{\Phi}$ from Proposition 8.1 as

$$\begin{aligned} \widetilde{\Phi} \colon \widetilde{B^*} &\longrightarrow \mathbb{H}_g \\ \tilde{t} &\longmapsto \Omega_{\tilde{t}} = \Big(\int_{b_{i,\tilde{t}}} \omega_{j,t}\Big)_{i,j}. \end{aligned}$$

Proposition 8.3. *Notations as in the previous section, we have that*

$$\widetilde{\Psi}(\tilde{t}) = \Omega_{\tilde{t}} - \sum_{e \in E} \zeta_e(\tilde{t}) \widetilde{M}_e.$$

It follows from the nilpotent orbit theorem that we can write $\widetilde{\Psi}(\tilde{t}) = \Lambda_t \in \mathbb{H}_g$ for any $t \in B^*$ and $\tilde{t} \in \widetilde{B^*}$ above t. Moreover, the family of matrices $(\Lambda_t)_{t \in B^*}$ can be extended to a family over B by setting $\Lambda_t = \Psi(t)$ for the extension $\Psi : B \to \check{D}$.

In terms of this new notation, Theorem 8.2 implies that

$$\exp\Big(\sum_{e \in E_t} \zeta_e N_e\Big)\Psi(t) = \Lambda_t + \sum_{e \in E_t} \zeta_t \widetilde{M}_e \in \mathbb{H}_g$$

for any $t \in B$, provided $\mathrm{Im}(\zeta_e) \geqslant T_0$ for all $e \in E_t$.

Fixing constants C, β as in the statement of Theorem 8.2, for any point $t \in B$, we have the estimate

$$\mathrm{dist}\Big(\Omega_{\tilde{s}}, \Lambda_t + \sum_{e \in E_t} \zeta_e(\tilde{s}) \widetilde{M}_e\Big) \leqslant C \sum_{e \in E_t} \mathrm{Im}(\zeta_e(\tilde{s}))^\beta \exp(-2\pi \, \mathrm{Im}(\zeta_e(\tilde{s})))$$

for all points $\tilde{s} \in \widetilde{B^*}$ which live above a point s of $U_t \backslash D$ for a small neighborhood U_t of t in B. Here E_t denotes again the set of all edges $e \in E$ with $z_e(t) = 0$

In what follows, we will be interested in the imaginary parts of the matrices $\Omega_{\tilde{t}}$. By the description of the monodromy action, we see that the imaginary part $\mathrm{Im}(\Omega_{\tilde{t}})$ is actually invariant under the monodromy. This implies that $\mathrm{Im}(\Omega_{\tilde{t}})$ descends to a map from B^* to the space of positive definite symmetric $g \times g$ matrices. For a point $t \in B$, we denote this matrix by $\mathrm{Im}(\Omega_t)$.

From the results above, we get the following estimate for the imaginary parts.

Theorem 8.4. *For any point $t \in B$, there exists a neighborhood U_t of t in B such that*

$$\mathrm{dist}\Big(\mathrm{Im}(\Omega_s), \mathrm{Im}(\Lambda_t) + \sum_{e \in E_t} \ell_e(s) \widetilde{M}_e\Big) \leqslant C \sum_{e \in E_t} \ell_e(s)^\beta \exp(-\ell_e(s))$$

for all points $s \in U_t \backslash D$ with $\ell_e(s) := -\log|z_e(s)|$ for all $e \in E_t$.

Note that the limit period matrix Λ_0 at the origin $t = 0$ is defined only up to a sum of the form $\sum_{e \in E} \lambda_e \widetilde{M}_e$, because the parameters z_e are defined only up to multiplication by non-zero complex numbers.

8.2. **The limit Λ_0.** Notations as in the previous section, let $\mathfrak{p} : \mathcal{S} \to B$ be the family of Riemann surfaces over the polydisk B. The pullbacks of divisors $\mathfrak{p}^{-1}(D_e)$ for $e \in E$ form a simple normal crossing divisor in $\mathcal{S}$. Consider the sheaf $\omega_{\mathcal{S}/B}\big(\log(\mathfrak{p}^* D)\big)$ of holomorphic forms with logarithmic singularities along the divisor $\mathfrak{p}^* D$. Our forms $\omega_1, \dots, \omega_g$ form a basis of the space of global sections $H^0\Big(B, \mathfrak{p}_* \omega_{\mathcal{S}/B}\big(\log(\mathfrak{p}^* D)\big)\Big)$.

Let $a_1, \dots, a_g, b_1, \dots, b_g$ be an admissible symplectic basis of $H_1(\mathcal{S}_\flat, \mathbb{Z})$. For each $g_{r-1} + 1 \leqslant i \leqslant g_r$ (see (7.7) for the definition of g_r), we have

$$\int_{a_j} \omega_i = \delta_{i,j}$$

which implies that the restriction of $\omega_{g_{r-1}+1,0}, \dots, \omega_{g_r,0}$ to C_{v_r} form a basis of the space of holomorphic differentials on C_{v_r}. Moreover, integration $\int_{b_i} \omega_{j,t}$ gives a holomorphic function on B.

For each $r = 0, 1, \dots, n$, let I_r be the index set $I_r = \{g_{r-1}+1, \dots, g_r\}$. For a $g \times g$ matrix P, denote by $P[I_r, I_{r'}]$ the matrix with rows in I_r and columns in $I_{r'}$.

Theorem 8.5. *The limit period matrix Λ_0 at the origin $t = 0$ verifies the following properties.*

- *For any $1 \leqslant r \leqslant n$, the $\mathfrak{g}(v_r) \times \mathfrak{g}(v_r)$ matrix $\Lambda_0[I_r, I_r]$ coincides with the period matrix Ω_{v_r} of the component C_{v_r} with respect to the symplectic basis $a_{g_{r-1}+1}, \dots, a_{g_r}, b_{g_{r-1}+1}, \dots, b_{g_r}$ and the holomorphic forms $\omega_{g_{r-1}+1,0}, \dots, \omega_{g_r,0}$ restricted to the component C_{v_r}.*
- *All the matrices $\Lambda_0[I_r, I_{r'}]$ for distinct, non-zero values $1 \leqslant r, r' \leqslant n$, $r' \neq r$ are vanishing.*

In other words, the limit period matrix Λ_0 has the following form

$$
\Lambda_0 = \begin{pmatrix}
\Omega_G & * & * & * & \cdots & * \\
* & \Omega_{v_1} & 0 & 0 & \cdots & 0 \\
* & 0 & \Omega_{v_2} & 0 & \ddots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & 0 \\
* & 0 & \cdots & \cdots & \Omega_{v_{n-1}} & 0 \\
* & 0 & \cdots & \cdots & 0 & \Omega_{v_n}
\end{pmatrix} \tag{8.11}
$$

for some $h \times h$ matrix Ω_G.

9. Generic continuity

Let S_0 be a stable Riemann surface with dual graph $G = (V, E, \mathfrak{g})$, and denote by $\mathcal{S}/B$ the analytic versal deformation space and the versal family of Riemann surfaces over it. We fix a base point $\flat \in B^*$.

Consider the family of canonically measured hybrid curves $(\mathcal{S}_{\mathbf{t}}^{hyb}, \mu_{\mathbf{t}})$ over B^{hyb} with $\mu_{\mathbf{t}} = \mu_{\mathbf{t}}^{can}$. That is,

- over $t \in B^*$, the measure μ_t is the canonical measure μ_t^{can} on the smooth Riemann surface $\mathcal{S}_t$, and
- over $\mathbf{t} \in B^{hyb} \backslash B^*$, the measure $\mu_{\mathbf{t}}$ is the canonical measure $\mu_{\mathbf{t}}^{can}$ on the hybrid curve $\mathcal{S}_t^{hyb}$.

The aim of this section is to prove Theorem 1.5 *through the open subset $B^* \subset B^{hyb}$*. More precisely, we obtain the following result.

Theorem 9.1. *The family of canonically measured hybrid curves $(\mathcal{S}_{\mathbf{t}}^{hyb}, \mu_{\mathbf{t}}^{can})_{\mathbf{t} \in B^{hyb}}$ over the hybrid space B^{hyb} is continuous through the open subset $B^* \subset B^{hyb}$. That is, for every continuous function $f : \mathcal{S}^{hyb} \to \mathbb{R}$, the function $F : B^{hyb} \to \mathbb{R}$ defined by integration along fibers*

$$
F(\mathbf{t}) := \int_{\mathcal{S}_{\mathbf{t}}^{hyb}} f_{|_{\mathcal{S}_{\mathbf{t}}^{hyb}}} \, d\mu_{\mathbf{t}}, \qquad \mathbf{t} \in B^{hyb}
$$

satisfies the continuity condition

$$\lim_{\substack{t \to \mathbf{t} \\ t \in B^*}} F(t) = F(\mathbf{t})$$

for all points $\mathbf{t} \in B^{hyb}$.

Definition 9.2. Fix an admissible basis for $H_1(\mathcal{S}_\flat, \mathbb{Z})$, spread out through all $\widetilde{B^*}$, and let $(\omega_i)_{i=1}^g$ be the corresponding holomorphic one-forms on $\mathcal{S}$ (see Section 8). For each $t \in B^*$, the canonical measure on the smooth Riemann surface $\mathcal{S}_t$ can be written as

$$\mu_t := \frac{i}{2} \sum_{i,j=1}^{g} \mathrm{Im}(\Omega_t)^{-1}(i,j)\, \omega_i \wedge \bar{\omega}_j.$$

For each $t \in B^*$, we can decompose $\mu_t = \sum_{i,j} \mu_{i,j,t}$, where the complex-valued measures $\mu_{i,j,t}$, $1 \leqslant i,j \leqslant g$ on $\mathcal{S}_t$ are given by

$$\mu_{i,j,t} := \frac{i}{2} \mathrm{Im}(\Omega_t)^{-1}(i,j)\, \omega_i \wedge \bar{\omega}_j.$$

The proof of Theorem 9.1 is based on understanding the limits of the complex-valued measures $\mu_{i,j,t}$ when $t \in B^*$ approaches a limit point of the form $\mathbf{t} = (0,x)$ in B^{hyb}. There will be three regimes, each treated in a separate section below. For notational convenience, we introduce the following index sets (see (7.7) for the definition of g_k)

$$I_k = \{g_{k-1}+1, \dots, g_k\},$$

where C_{v_k}, $k \in [n]$, are the smooth components of $\mathcal{S}_t$.

(1) If $i,j > h$, we will show that $\mu_{i,j,t}$ converges
- either to a measure of the same form on the component C_{v_k} of S_0 (if i,j belong to the same block I_k for $k \in [n]$),
- or to zero (if i,j do not belong to the same block I_k).

(2) If $1 \leqslant i,j \leqslant h$, we will show that $\mu_{i,j,t}$ converges to a piece of the canonical measure on the underlying tropical curve of the hybrid curve $\mathcal{S}^{hyb}_{\mathbf{t}}$.

(3) If $i \leqslant h$ and $j > h$ (or vice versa), we will show that $\mu_{i,j,t}$ converges to zero.

In the following, we fix an ordered partition $\pi = (\pi_1, \dots, \pi_r)$ on the edge set E, and take a point $\mathbf{t} = (0,x) \in D^\circ_\pi \times \sigma^\circ_\pi$. We also fix an admissible basis for $H_1(S_\flat, \mathbb{Z})$ with respect to the layering π.

9.1. **Inverse lemma.** The proof of continuity in the above regimes requires to handle the inverse of block matrices with prescribed asymptotic on the blocks. The key property we use is the *inverse lemma* stated in this section. Its proof is given in Section 9.7

Lemma 9.3. *Let X be a topological space, $\mathbf{t} \in X$ a fixed point and $y_1, \dots, y_r \colon X \backslash \{\mathbf{t}\} \to \mathbb{C}$ functions such that*

$$\lim_{t \to \mathbf{t}} \frac{y_{k+1}(t)}{y_k(t)} = 0, \qquad i = 1, \dots, r-1. \tag{9.1}$$

Suppose $M\colon X\backslash\{\mathbf{t}\} \to \mathbb{C}^{n\times n}$ is a matrix-valued function. Assume that $M(t)$ has an (r,r) block decomposition (here, $n = \sum_{k=1}^{r} n_k$ for some $n_k \in \mathbb{N}$),

$$M(t) = \Big(\mathcal{A}_{kl}(t)\Big)_{1\leqslant k,l\leqslant r},$$

where, as t goes to $\mathbf{t}$ in X, the blocks $\mathcal{A}_{kl}\colon X\backslash\{\mathbf{t}\} \to \mathbb{C}^{n_k\times n_l}$ are asymptotically given by

$$\mathcal{A}_{kl}(t) = y_{\max\{k,l\}}(t)\big(\widehat{\mathcal{A}}_{kl} + o(1)\big)$$

for matrices $\widehat{\mathcal{A}}_{kl} \in \mathbb{R}^{n_k\times n_l}$ and all diagonal matrices $\widehat{\mathcal{A}}_{kk}$, $k = 1,...r$ are invertible.

Then, as t goes to $\mathbf{t}$ in X, the inverse $M(t)^{-1}$ has the (r,r) block decomposition

$$M(t)^{-1} = \Big(y_{\min\{k,l\}}(t)^{-1}\big(\mathcal{B}_{kl} + o(1)\big)\Big)_{1\leqslant k,l\leqslant r}$$

for some matrices $\mathcal{B}_{kl} \in \mathbb{C}^{n_k\times n_l}$, and the diagonal terms are given by

$$\mathcal{B}_{kk} = \widehat{\mathcal{A}}_{kk}^{-1}, \qquad k = 1,\dots r.$$

9.2. **The inverse of the period matrix.** In this section we apply the inverse lemma to $\mathrm{Im}(\Omega_t)$, the imaginary part of the period matrix, and describe the asymptotic behavior of $\mathrm{Im}(\Omega_t)^{-1}$ as $t \in B^*$ converges to $\mathbf{t}$.

In order to apply Lemma 9.3, we first need a detailed description of the asymptotics of $\mathrm{Im}(\Omega_t)$. We will derive those from the results of Section 8 based on the nilpotent orbit theorem. More precisely, recall from Theorem 8.4 that

$$\mathrm{dist}\Big(\mathrm{Im}(\Omega_t), \mathrm{Im}(\Lambda_0) + \sum_{e\in E} \ell_e(t)\widetilde{M}_e\Big) \leqslant C\sum_{e\in E} \ell_e(t)^{\beta}\exp(-\ell_e(t)) \tag{9.2}$$

for all $t \in B^*$ in a neighborhood of the origin $0 \in B$. Choosing an admissible basis for $H_1(\mathcal{S}_\flat, \mathbb{Z})$ (see Section 7.4.2 for details), $\mathrm{Im}(\Omega_t)$ has the following form (see (8.11))

$$\mathrm{Im}(\Omega_t) = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22}\end{pmatrix} = \begin{pmatrix} \mathrm{Im}(\Omega_G) + M_\ell & * & * & \cdots & * \\ * & \mathrm{Im}(\Omega_{v_1}) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ * & 0 & \cdots & \mathrm{Im}(\Omega_{v_{n-1}}) & 0 \\ * & 0 & \cdots & 0 & \mathrm{Im}(\Omega_{v_n})\end{pmatrix} + o(1),$$

as $t \in B^*$ converges to the origin $0 \in B$ in the standard topology on B.

Next, we rewrite $\mathrm{Im}(\Omega_t)$ as a square block matrix with $r+1$ blocks in each row/column (here, r is the depth of the ordered partition $\pi = (\pi_i)_{i=1}^{r}$ for our fixed hybrid point $\mathbf{t} = (0,x) \in B^{hyb}$). Recalling the index decomposition $[h] = J_\pi^1 \sqcup \cdots \sqcup J_\pi^r$ from (7.6), define

$$J_\pi^{r+1} := \{h+1,\dots,g\}$$

such that altogether, we have

$$[g] = J_\pi^1 \sqcup \cdots \sqcup J_\pi^{r+1}. \tag{9.3}$$

For a matrix $N \in \mathbb{C}^{g\times g}$, we write $N_{kl} := N|_{J_\pi^k\times J_\pi^l}$, $1 \leqslant k, \leqslant r+1$ for its (k,l)-th block with respect to the decomposition (9.3). In order to avoid confusion, we stress that the notation $N(i,j)$ is used for the (i,j)-th matrix entry.

Denoting the (k,l)-th block of $\mathrm{Im}(\Omega_t)$ by

$$\mathcal{A}_{kl} := \mathrm{Im}(\Omega_t)_{kl} = \mathrm{Im}(\Omega_t)|_{J_\pi^k \times J_\pi^l}, \qquad 1 \leqslant j,k \leqslant r+1,$$

we get the block decomposition

$$\mathrm{Im}(\Omega_t) = \Big(\mathcal{A}_{kl}\Big)_{1\leqslant k,l\leqslant r+1} = \begin{pmatrix} \Big(\mathcal{A}_{kl}\Big)_{1\leqslant k,l\leqslant r} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix}.$$

To describe the asymptotic behavior of the blocks $\mathcal{A}_{kl}$, we need the following functions $y_1, \dots y_{r+1}$ on B^*: for $k = 1, \dots, r$, define $y_k \colon B^* \to (0,\infty)$ by

$$y_k(t) := -\sum_{e\in\pi_k} \log|z_e(t)|, \qquad t \in B^*,$$

and moreover, set $y_{r+1}(t) \equiv 1$ on B^*. Taking into account the topology on $B^{^{hyb}}$, it is clear that

$$\lim_{\substack{t\to\mathbf{t}\\ t\in B^*}} \frac{y_{k+1}(t)}{y_k(t)} = 0$$

for all $k = 1, \dots, r$. In terms of these functions, we obtain the following asymptotic behavior when $t \in B^*$ goes to $\mathbf{t}$ in $B^{^{hyb}}$:

- Assume first that $1 \leqslant k \leqslant l \leqslant r$. Then

$$\begin{aligned}\mathcal{A}_{kl} = \mathcal{A}_{lk} &= \mathrm{Im}(\Lambda_0)_{kl} + \sum_{e\in E} \ell_e(t)(M_e)_{kl} + o(1) = \Big(\sum_{e\in E_\pi^l} \ell_e(t)(M_e)_{kl} + O(1)\Big)\\ &= y_l(t)\Big(\sum_{e\in\pi_l} \frac{\ell_e(t)}{y_l(t)}(M_e)_{kl} + o(1)\Big) = y_l(t)\Big(\sum_{e\in\pi_l} x_e(M_e)_{kl} + o(1)\Big),\end{aligned}$$

  where we have used that $(M_e)_{kl} = 0$ for all edges $e \in \pi_1 \sqcup \dots \pi_{l-1}$. The latter holds true since our fixed basis of $H_1(\mathcal{S}_\flat, \mathbb{Z})$ is admissible (see Section 7.4.2).
- On the other hand, if $k = r+1$ or $l = r+1$, then it follows from (9.2) that

$$\mathcal{A}_{kl} = \mathrm{Im}(\Lambda_0)_{kl} + o(1).$$

The results of the previous section now allow to describe $\mathrm{Im}(\Omega_t)^{-1}$. Denote by

$$M_{\pi,x}^k := \sum_{e\in\pi_j} x_e M_e \in \mathbb{R}^{h_\pi^k \times h_\pi^k} \tag{9.4}$$

the matrix (5.6) on the k-th graded minor $\mathrm{gr}_\pi^k(G)$ of G, equipped with the edge lengths of x and with respect to the basis of $H_1(\mathrm{gr}_\pi^k(G), \mathbb{Z})$ given by $\mathscr{K}_k(b_i)$, $i \in J_\pi^k$ (see Section 7.4.2 for details).

Theorem 9.4. *Let* $\mathrm{Im}(\Omega_t) \in \mathbb{R}^{g\times g}$*,* $t \in B^*$*, be the imaginary part of the period matrix. Consider the block matrix decomposition (with respect to (9.3)) of its inverse* $\mathrm{Im}(\Omega_t)^{-1}$*. If* $t \in B^*$ *converges to* $\mathbf{t} = (0,x)$ *in* $B^{^{hyb}}$*, then*

$$\mathrm{Im}(\Omega_t)^{-1} = \Big(y_{\min\{k,l\}}(t)^{-1}(\mathcal{B}_{kl} + o(1))\Big)_{1\leqslant k,l\leqslant r+1} \tag{9.5}$$

for matrices $\mathcal{B}_{kl} \in \mathbb{R}^{h_\pi^k \times h_\pi^l}$ *(where* $h_\pi^{r+1} := g - h$*). Moreover*

$$\mathcal{B}_{r+1,r+1} = \begin{pmatrix} \mathrm{Im}(\Omega_{v_1})^{-1} & 0 & 0 & \cdots & 0 \\ 0 & \mathrm{Im}(\Omega_{v_2})^{-1} & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \mathrm{Im}(\Omega_{v_{n-1}})^{-1} & 0 \\ 0 & \cdots & \cdots & 0 & \mathrm{Im}(\Omega_{v_n})^{-1} \end{pmatrix}.$$

and

$$\mathcal{B}_{kk} = (M_{\pi,x}^k)^{-1}, \qquad k = 1, \dots, r,$$

where $M_{\pi,x}^k \in \mathbb{R}^{h_\pi^k \times h_\pi^k}$ *is defined in* (9.4).

Proof. Taking into account the preceding discussion, the claim is an immediate consequence of Lemma 9.3. Indeed, the structure of $\mathcal{B}_{r+1,r+1}$ follows from (8.11) and it only remains to notice that $\sum_{e \in \pi_k} x_e (M_e)_{kk} = M_{\pi,x}^k$ for each $k = 1, \dots, r$. ☐

9.3. **Continuity I.** For each $k = 1, \dots, n$, let I_k be the set $\{g_{k-1} + 1, \dots, g_k\}$ (see also (7.7)). We prove the following result.

Theorem 9.5. *Assume that* t *converges to* $\mathbf{t} = (0, x)$ *through* B^**. Then the following holds true for each pair of indices* (i, j) *with* $h + 1 \leqslant i, j \leqslant g$*.*

- *If* i, j *belong to the same set* I_k *for some* $1 \leqslant k \leqslant n$*, then* $\mu_{i,j,t}$ *converges to the measure*

$$\mu_{i,j,\mathbf{t}} := \frac{i}{2} \mathrm{Im}(\Omega_{v_k})^{-1}(i,j) \omega_i \wedge \bar{\omega}_j$$

  on $\mathcal{S}_{\mathbf{t}}^{hyb}$*, supported on* $C_{v_k} \subset \mathcal{S}_{\mathbf{t}}^{hyb}$*. Here* Ω_{v_k} *denotes the period matrix of the component* $C_{v_k} \subseteq S_0$ *in the symplectic basis* $\{a_i, b_j\}_{i,j \in I_k}$*.*

- *If* i, j *belong to distinct sets* $I_k \neq I_l$*, then* $\mu_{i,j,t}$ *converges to the measure zero on* $\mathcal{S}_{\mathbf{t}}^{hyb}$*.*

Proof. The relative holomorphic forms ω_i, ω_j on $\mathcal{S}^*$ extend holomorphically to B. In particular, the complex valued-measure $\omega_i \wedge \bar{\omega}_j$ extends continuously over all points of B. It follows that when t approaches the point $\mathbf{t} = (0, x)$ through B^*, the family of measured spaces $(\mathcal{S}_t, \omega_{i,t} \wedge \bar{\omega}_{j,t})$ converges to $(\mathcal{S}_{\mathbf{t}}^{hyb}, \omega_i \wedge \bar{\omega}_j)$ (which is a measure with support in the component C_{v_k}).

Hence it will be enough to prove that

$$\lim_{\substack{t \to \mathbf{t} \\ t \in B^*}} \mathrm{Im}(\Omega_t)^{-1}(i,j) = \mathrm{Im}(\Omega_{v_k})^{-1}(i,j), \tag{9.6}$$

if i and j belong to the same interval I_k and otherwise, the limit in (9.6) is zero. However, this follows immediately from Theorem 9.4. ☐

9.4. **Continuity II.** Consider now a pair of indices $1 \leqslant i, j \leqslant h$. Remember that we have fixed an admissible basis for the hybrid curve $\mathcal{S}^{hyb}_{\mathbf{t}}$, whose underlying ordered partition is $\pi = (\pi_1, \dots, \pi_r)$. In particular, we have a decomposition of $[h]$ into intervals $J^1_\pi, \dots J^r_\pi$ of size $h^1_\pi, \dots, h^r_\pi$, respectively (see (7.6) and (9.3)). Here h^k_π is the genus of the graded minor $\mathrm{gr}^k_\pi(G)$. Recall that in (9.4), we introduced for $k = 1, \dots r$ the matrix

$$M^k_{\pi,x} = \sum_{e \in \pi_k} x_e M_e \in \mathbb{R}^{h^k_\pi \times h^k_\pi},$$

which coincides with (5.6) for the graded minor $\mathrm{gr}^k_\pi(G)$ (equipped with edge lengths $\ell^k = x|_{\pi_k}$), written in matricial form for the basis $\mathscr{K}_k(b_l)$, $l \in J^k_\pi$.

Theorem 9.6. *Assume that t converges to $\mathbf{t} = (0, x)$ through B^*. Then the following holds true for each pair of indices (i, j) with $1 \leqslant i, j \leqslant h$.*

- *If i, j belong to the same set $J^k_\pi \subset [h]$, then $\mu_{i,j,t}$ converges to a measure $\mu_{i,j,\mathbf{t}}$ on $\mathcal{S}^{hyb}_{\mathbf{t}}$ with support contained in the intervals of $\mathcal{S}^{hyb}_{\mathbf{t}}$. More precisely,*

$$\mu_{i,j,\mathbf{t}} := \sum_{e \in \pi_k} (M^k_{\pi,x})^{-1}(i,j) \gamma_i(e) \gamma_j(e)\, d\theta_e$$

  where $\gamma_i = \mathscr{K}_k(b_i)$ and $\gamma_j = \mathscr{K}_k(b_j)$ are the two cycles in $H_1(\mathrm{gr}^k_\pi(G), \mathbb{Z})$ corresponding to the two elements $b_i, b_j \in B \simeq H_1(G, \mathbb{Z})$ for the fixed admissible basis, $\mathscr{K}_k$ is the projection map $G^k_\pi \to \mathrm{gr}^k_\pi(G)$, and $d\theta_e$ is the uniform Lebesgue measure on the interval $\mathcal{I}_e \subseteq \mathcal{S}^{hyb}_{\mathbf{t}}$ representing the edge e.

- *If i, j belong to distinct sets $J^k_\pi \neq J^l_\pi$, then $\mu_{i,j,t}$ converges to the measure zero on $\mathcal{S}^{hyb}_{\mathbf{t}}$.*

The rest of this section is devoted to the proof of this theorem.

9.4.1. *The behavior near singular points.* Fix a small neighborhood U_0 of the origin $0 \in B$. Let $e = uv$ be an edge of the graph G, and consider the singular point $p^e(0)$ of the fiber $\mathcal{S}_0 = S_0$. We find a small neighborhood U_e of $p^e(0)$ in $\mathcal{S}$ lying above U_0, and put coordinates $\underline{z} = \Big((z_i)_{i \neq e}, z^e_u, z^e_v\Big)$ on U_e with the equation $z^e_u z^e_v = z_e$ (more precisely, we fix a standard coordinate neighborhood (U_e, z), see (4.8)).

Since ω_i and ω_j are global sections over $\mathcal{S}$ of the dualizing sheaf $\omega_{\mathcal{S}/B}$, locally in a small neighborhood U_e as above, we can write

$$\omega_i = \frac{1}{2\pi i} f_i(\underline{z}) \frac{dz^e_u}{z^e_u}, \qquad \omega_j = \frac{1}{2\pi i} f_j(\underline{z}) \frac{dz^e_u}{z^e_u} \tag{9.7}$$

for holomorphic functions f_i and f_j on U_e.

For the vanishing cycle a_e, we have $\int_{a_e} \omega_i = \gamma_i(e)$ and $\int_{a_e} \omega_j = \gamma_j(e)$ (see also (8.6)). It follows by the residue formula that $f_i(\underline{z}) = \gamma_i(e)$ provided that $z^e_u = z^e_v = 0$. The same holds true for f_j.

Now, we write $z_u^e = \exp(2\pi i \zeta_u^e)$, and pass to the polar coordinates $\ell_u^e = -\log|z_u^e|$ and $\tau_u^e = \mathrm{Re}(\zeta_u^e)$ on $U_e \backslash \mathfrak{p}^{-1}(D_e)$. In these coordinates, we can write

$$\frac{dz_u^e}{z_u^e} = 2\pi d\ell_u^e + (2\pi i) d\tau_u^e.$$

It follows that on $U_e \backslash \mathfrak{p}^{-1}(D_e)$,

$$\omega_i \wedge \bar{\omega}_j = f_i(\underline{z})\bar{f}_j(\underline{z})\frac{2(2\pi i)2\pi}{(2\pi i)^2} d\tau_u^e \wedge d\ell_u^e = -2i f_i(\underline{z})\bar{f}_j(\underline{z}) d\tau_u^e \wedge d\ell_u^e.$$

Let now $t \in B^*$ be a base point close to $\mathbf{t}$ in $B^{^{hyb}}$. Restricting the measure $\mu_{i,j,t}$ to $U_e \cap \mathcal{S}_t$, we get the expression

$$\mu_{i,j,t} = \mathrm{Im}(\Omega_t)^{-1}(i,j) f_i(\underline{z})\bar{f}_j(\underline{z}) d\tau_u^e \wedge d\ell_u^e \qquad \text{on } U_e \cap \mathcal{S}_t.$$

Assume now that i and j belong to the intervals J_π^k and J_π^l with $1 \leqslant k \leqslant l \leqslant r$ (see (9.3)). Then Theorem 9.4 implies that

$$\mu_{i,j,t} = \frac{f_i(\underline{z})\bar{f}_j(\underline{z})}{y_k(t)}\Big((\mathcal{B}_{kl})(i,j) + o(1)\Big) d\tau_u^e \wedge d\ell_u^e \qquad \text{on } U_e \cap \mathcal{S}_t,$$

and the $o(1)$-term goes to zero uniformly on $U_e \cap S_t$ as $t \in B^*$ goes to $\mathbf{t}$ in $B^{^{hyb}}$.

Finally, suppose that the edge e belongs to the m-th set π_m of the ordered partition π. Normalizing the coordinates by the respective lengths, we get

$$\mu_{i,j,t} = \frac{y_m(t)}{y_k(t)} f_i(\underline{z})\bar{f}_j(\underline{z})\Big((\mathcal{B}_{kl})(i,j) + o(1)\Big) d\tau_u^e \wedge d\Theta_e$$

in $U_e \cap \mathcal{S}_t$, where $\Theta_e := \ell_u^e / y_m(t)$. Moreover, f_i is holomorphic on U_e with $f_i(\underline{z}) = \gamma_i(e)$ for $z_u^e = z_v^e = 0$, and hence

$$f_i(\underline{z}) = \gamma_i(e) + O(|z_u^e|) + O(|z_v^e|) \qquad \text{in } U_e$$

and the same holds true for f_j. Altogether, we have shown that

$$\mu_{i,j,t} = c_t(p)\, d\tau_u^e \wedge d\Theta_e \qquad \text{in } \mathcal{S}_t \cap U_e, \tag{9.8}$$

where the function $c_t \colon \mathcal{S}_t \cap U_e \to \mathbb{C}$ has the form

$$c_t(p) = \frac{y_m(t)}{y_k(t)}\Big(\gamma_i(e)\gamma_j(e) + O(|z_u^e|) + O(|z_v^e|)\Big)\Big((B_{kl})(i,j) + o(1)\Big), \tag{9.9}$$

as $t \in B^*$ goes to $\mathbf{t}$ with uniform estimates for the error terms. More precisely, we mean that the $o(1)$ term goes to zero uniformly on $U_e \cap \mathcal{S}_t$ for $t \to \mathbf{t}$ and the constants in the $O(|z_u^e|)$ and $O(|z_u^e|)$ terms can be chosen independent of t.

9.4.2. *Proof of Theorem 9.6.* Let f be a continuous function on $\mathcal{S}^{^{hyb}}$. We are concerned with the limit behavior of

$$\int_{\mathcal{S}_t} f(p)\, d\mu_{i,j,t}(p) \tag{9.10}$$

as $t \in B^*$ converges to $\mathbf{t} = (0, x)$ in $B^{^{hyb}}$. Using the same notation as in the preceding section, we consider small neighborhoods U_e around the singular points of $\mathcal{S}_0 = S_0$ and choose coordinates as above.

Outside the open neighborhoods U_e, the measures $\mu_{i,j,t}$ extend continuously by zero. Indeed, the measures $\omega_i \wedge \bar{\omega}_j$ extend continuously outside these open sets, and $\mathrm{Im}(\Omega)^{-1}(i,j)$ converges to zero by Theorem 9.4. In particular,

$$\lim_{\substack{t\to\mathbf{t}\\ t\in B^*}} \int_{\mathcal{S}_t\setminus(\bigcup_e U_e\cap\mathcal{S}_t)} f(p)\,d\mu_{i,j,t}(p) = 0. \tag{9.11}$$

Hence it suffices to analyze the behavior on each of the sets U_e separately. Suppose as before that i and j belong to the intervals J_π^k and J_π^l with $1 \leqslant k \leqslant l \leqslant r$. Assume further that the edge e belongs to m-th set π_m of the ordered partition π.

If $m < k$, then $\gamma_e(i) = \gamma_e(j) = 0$ since our fixed basis of $H_1(\mathcal{S}_\flat, \mathbb{Z})$ is admissible (see Section 7.4). In this case, ω_i and ω_j do not have logarithmic poles at $p^e(0)$. Using the same argument as in (9.11), we see that

$$\lim_{\substack{t\to\mathbf{t}\\ t\in B^*}} \int_{\mathcal{S}_t\cap U_e} f(p)\,d\mu_{i,j,t}(p) = 0$$

for such edges $e \in E$. Hence, in the following we may suppose that $m \geqslant k$.

For the sake of concreteness, suppose that the coordinates z_u^e and z_v^e on U_e have the range $0 \leqslant |z_u^e|, |z_v^e| \leqslant \varepsilon$. We decompose $U_e \setminus \mathfrak{p}^{-1}(D_e) = U_e^1 \sqcup U_e^2$, where

$$U_e^1 = \Big\{p \in U_e\setminus \mathfrak{p}^{-1}(D_e);\ |z_u^e(p)| > \frac{1}{|\log|z_e(p)||} \text{ or } |z_v^e(p)| > \frac{1}{|\log|z_e(p)||}\Big\},$$
$$U_e^2 = \Big\{p \in U_e\setminus \mathfrak{p}^{-1}(D_e);\ |z_u^e(p)| \leqslant \frac{1}{|\log|z_e(p)||} \text{ and } |z_v^e(p)| \leqslant \frac{1}{|\log|z_e(p)||}\Big\}.$$

Taking into account that $z_u^e z_v^e = z_e$, we readily compute that for $p \in U_e^1 \cap \mathcal{S}_t$, the coordinate $\Theta_e(p)$ lies precisely in the interval

$$\Big[\frac{-\log(\varepsilon)}{y_m(t)}, g_1(t)\Big] \cup \Big[g_2(t), \frac{-\log(|t_e|/\varepsilon)}{y_m(t)}\Big].$$

where for a base point $t \in U_0$, we have introduced the values

$$g_1(t) := \frac{\log\big(|\log(|t_e|)|\big)}{y_m(t)}, \qquad\qquad g_2(t) := \frac{-\log\big(|t_e \log(|t_e|)|\big)}{y_m(t)}.$$

As $t \in B^*$ goes to $\mathbf{t} = (0,x)$ in $B^{^{hyb}}$, the above intervals shrink to the set $\{0\} \cup \{x_e\}$. Using the boundedness of f and the coefficient c_t (see (9.8) and (9.9), here it is crucial that $m \geqslant k$), this implies

$$\lim_{\substack{t\to\mathbf{t}\\ t\in B^*}} \int_{\mathcal{S}_t\cap U_e^1} f(p)\,d\mu_{i,j,t}(p) = 0.$$

Hence it remains to understand the behavior on the sets U_e^2. Taking into account the topology on $\mathcal{S}^{^{hyb}}$, we get

$$\lim_{\substack{t\to\mathbf{t}\\ t\in B^*}} \int_{U_e^2\cap\mathcal{S}_t} f(p)\,d\tau_u^e \wedge d\Theta_e(p) = \int_{\mathcal{I}_e} f(\lambda)\,d\theta_e(\lambda) \tag{9.12}$$

where $\mathcal{I}_e$ is the interval of length x_e representing the edge $e \in E$ in the hybrid curve $\mathcal{S}^{hyb}_{\mathbf{t}}$ and θ_e denotes the uniform Lebesgue measure on $\mathcal{I}_e$. Indeed, we can explicitly write

$$\int_{U_e^2 \cap \mathcal{S}_t} f \, d\tau_u^e \wedge d\Theta_e = \int_{g_1(t)}^{g_2(t)} \int_0^1 f\left((z_i)_{i \neq e}, \tau_u^e, \Theta_e\right) \, d\tau_u^e \, d\Theta_e \, .$$

Notice that if $t \in B^*$ converges to $\mathbf{t} = (0, x)$, then $[g_1(t), g_2(t)]$ converges to the interval $[0, x_e]$. In particular, (9.12) is a straightforward consequence of the definition of the topology on $\mathcal{S}^{hyb}$. (Remark that this argument fails for U_e^1 instead of U_e^2 and hence the above decomposition was necessary).

Finally, in view of (9.8), it remains to describe the behavior of the function $c_t(p)$. If $k = l = m$, then it follows from (9.9) and the inequality $|z_u^e|, |z_v^e| \leqslant 1/|\log(|z_e|)|$ on U_e^2 that (see also Theorem 9.4)

$$\lim_{\substack{t \to \mathbf{t} \\ t \in B^*}} \sup_{p \in U_e^2 \cap \mathcal{S}_t} \| c_t(p) - \gamma_i(e)\gamma_j(e)(M^k_{\pi,x})^{-1}(i,j) \| = 0.$$

This in turn implies that

$$\lim_{\substack{t \to \mathbf{t} \\ t \in B^*}} \int_{U_e^2 \cap \mathcal{S}_t} f(p) \, d\mu_{i,j,t}(p) = \gamma_i(e)\gamma_j(e)(M^k_{\pi,x})^{-1}(i,j) \int_{I_e} f(\lambda) \, d\theta_e(\lambda).$$

On the other hand, if $m > k$, then $y_m(t)/y_k(t)$ goes to zero as $t \in B^*$ goes to $\mathbf{t}$ in B^{hyb}. This means that

$$\lim_{\substack{t \to \mathbf{t} \\ t \in B^*}} \sup_{p \in U_e^2 \cap \mathcal{S}_t} \| c_t(p) \| = 0$$

and in particular

$$\lim_{\substack{t \to \mathbf{t} \\ t \in B^*}} \int_{U_e^2 \cap \mathcal{S}_t} f(p) \, d\mu_{i,j,t}(p) = 0.$$

Notice that this always occurs if $l > k$ (since we supposed that $k \leqslant l \leqslant m$). Finally, combining all the above considerations, we can compute the limit in (9.10) and arrive at Theorem 9.6.

9.5. **Continuity III.** Consider now a pair of indices (i, j) with $1 \leqslant i \leqslant h$ and $j > h$.

Theorem 9.7. *The measure $\mu_{i,j,t}$ converges to the measure zero on $\mathcal{S}^{hyb}_{\mathbf{t}}$, as t converges to $\mathbf{t} = (0, x)$ through B^*.*

The proof proceeds in the exact same way as in the previous section.

Taking open sets U_e around singular points p^e, one sees that outside the union of the U_e's the measure $\omega_i \wedge \bar{\omega}_j$ extends continuously while $\mathrm{Im}(\Omega_t)^{-1}(i,j)$ converges to zero (see Theorem 9.4).

Again, this remains true for the subsets U_e of edges $e \in E$ which are in a set π_m of the partition π with $m < k$ (here, we suppose that $i \in J^k_\pi$).

On the other hand, using the same arguments and notations as in the previous section, on each of the open sets U_e,

$$\mu_{i,j,t} = \frac{\bar{f}_j(\underline{z}) g(\underline{z})}{y_k(t)} \, d\tau_u^e \wedge d\ell_u^e \qquad \text{on } U_e \cap \mathcal{S}_t,$$

where $g\colon U_e \to \mathbb{C}$ is a bounded function. To conclude, we observe that ω_j does not have logarithmic poles at the p^e's. Proceeding as in the proof of Theorem 9.6, we arrive at Theorem 9.7.

9.6. **Proof of Theorem 9.1.** Recall that in Definition 9.2, the canonical measures $\mu_t = \mu_t^{can}$ for points $t \in B^*$ were decomposed as

$$\mu_t = \sum_{i,j=1}^{g} \mu_{i,j,t}.$$

The proof consists in understanding the limits of the complex-valued measures $\mu_{i,j,t}$ when t approaches a limit point of the form $\mathbf{t} = (0, x)$ in $B^{^{hyb}}$ through the open part B^*. This was precisely the content of the three preceding subsections, each treating one of the three regimes described in the paragraph right after Definition 9.2, and the theorem follows. In particular, the link between the measures $\mu_{i,j,\mathbf{t}}$ in Theorem 9.6 and the canonical measures on the graded minors is given by Theorem 5.1.

9.7. **Proof of the inverse lemma.** In this section, we prove the inverse Lemma 9.3.

Proof of Lemma 9.3. We prove the lemma by induction on the number of blocks in each row/column, which is denoted by r. If $r = 1$, then $M(t) = \mathcal{A}_{11}(t)$ and the claim is trivial. So suppose that $M(t)$ is an (r, r) block matrix and we have already proven the claim for $(r-1, r-1)$ block matrices. Then we can write

$$M(t) = \begin{pmatrix} \Pi_{11}(t) & \Pi_{12}(t) \\ \Pi_{21}(t) & \Pi_{22}(t) \end{pmatrix},$$

where $\Pi_{11}(t) = \mathcal{A}_{11}(t) \in \mathbb{R}^{n_1 \times n_1}$, $\Pi_{22}(t)$ is the $(r-1, r-1)$ block matrix

$$\Pi_{22}(t) = (\mathcal{A}_{kl}(t))_{k,l \geqslant 2} \in \mathbb{R}^{n' \times n'}$$

with $n' = \sum_{j=2}^{r} n_k$ and the off-diagonal matrices $\Pi_{21}(t) \in \mathbb{R}^{n' \times n_1}$, $\Pi_{12}(t) \in \mathbb{R}^{n_1 \times n'}$ are

$$\Pi_{21} = \begin{pmatrix} \mathcal{A}_{21} \\ \vdots \\ \mathcal{A}_{r1} \end{pmatrix} = \begin{pmatrix} y_2(\widehat{\mathcal{A}}_{21} + o(1)) \\ \vdots \\ y_r(\widehat{\mathcal{A}}_{r1} + o(1)) \end{pmatrix}, \qquad \Pi_{12} = \begin{pmatrix} \mathcal{A}_{12} \\ \cdots \\ \mathcal{A}_{1r} \end{pmatrix}^T = \begin{pmatrix} y_2(\widehat{\mathcal{A}}_{12} + o(1)) \\ \cdots \\ y_r(\widehat{\mathcal{A}}_{1r} + o(1)) \end{pmatrix}^T.$$

Moreover, introducing the Schur complement $S\colon X \backslash \{\mathbf{t}\} \to \mathbb{R}^{n_1 \times n_1}$ by

$$S(t) := \Pi_{11}(t) - \Pi_{12}(t)\Pi_{22}(t)^{-1}\Pi_{21}(t),$$

the inverse of $M(t)$ can be written as

$$M(t)^{-1} = \begin{pmatrix} \Psi_{11}(t) & \Psi_{12}(t) \\ \Psi_{21}(t) & \Psi_{22}(t) \end{pmatrix} = \begin{pmatrix} S^{-1} & -S^{-1}\Pi_{12}\Pi_{22}^{-1} \\ -\Pi_{22}^{-1}\Pi_{21}S^{-1} & \left(\Pi_{22} - \Pi_{21}\Pi_{11}^{-1}\Pi_{12}\right)^{-1} \end{pmatrix}.$$

We begin by determining the asymptotic behavior of $\Psi_{11}(t) = S(t)^{-1}$. The $(r-1, r-1)$ block matrix $\Pi_{22}(t)$ satisfies the assumptions of the induction hypothesis and hence

$$\Pi_{22}^{-1}(t) = \left(\frac{1}{y_{\min\{k,l\}}(t)} (N_{kl} + o(1)) \right)_{2 \leqslant k,l \leqslant r}$$

for some matrices $N_{kl} \in \mathbb{R}^{n_k \times n_l}$, $2 \leqslant k, l \leqslant r$. Since $\lim_{t \to \mathbf{t}} y_k(t)/y_{\min\{k,l\}}(t)$ exists, the following limit

$$N := \lim_{t \to \mathbf{t}} \Pi_{22}^{-1}\Pi_{21} = \lim_{t \to \mathbf{t}} \sum_{j \geqslant 2} \left(\frac{y_j(t)}{y_{\min\{i,j\}}(t)} (N_{ij} + o(1))(\widehat{\mathcal{A}}_{j1} + o(1)) \right)_{i \geqslant 2} \in \mathbb{R}^{n' \times n_1}$$

exists as well. Moreover, (9.1) implies that $\Pi_{12}(t) = y_2(t)O(1)$. Hence, altogether,

$$\begin{aligned} S(t)^{-1} &= \Big(\Pi_{11}(t) - y_2(t)O(1)\Big)^{-1} = \frac{1}{y_1(t)}\Big(\widehat{\mathcal{A}}_{11} - \frac{y_2(t)}{y_1(t)}O(1) + o(1)\Big)^{-1} \\ &= \frac{1}{y_1(t)}\Big(\widehat{\mathcal{A}}_{11} + o(1)\Big)^{-1} = \frac{1}{y_1(t)}(\widehat{\mathcal{A}}_{11}^{-1} + o(1)). \end{aligned}$$

This proves the claimed asymptotics for $\Psi_{11}(t)$. In addition, it is now clear that

$$\Psi_{21}(t) = \frac{1}{y_1(t)}(N + o(1))(\widehat{\mathcal{A}}_{11}^{-1} + o(1)) = \frac{1}{y_1(t)}(N\widehat{\mathcal{A}}_{11}^{-1} + o(1)).$$

These are precisely the claimed asymptotics for $\Psi_{21}(t)$ and the second off-diagonal matrix $\Psi_{12}(t)$ can be treated similarly.

Turning finally to $\Psi_{22}(t) = (\Pi_{22} - \Pi_{21}\Pi_{11}^{-1}\Pi_{12})^{-1}$, notice that it is again a $(r-1, r-1)$ block matrix. Moreover,

$$\Pi_{21}\Pi_{11}^{-1}\Pi_{12} = \Big(\mathcal{A}_{k1}\mathcal{A}_{11}^{-1}\mathcal{A}_{1l}\Big)_{2 \leqslant k,l \leqslant r}$$

and it follows from (9.1) that for each index pair (k, l) with $2 \leqslant k, l \leqslant r$,

$$\lim_{t \to \infty} \frac{1}{y_{\max\{k,l\}}(t)} \mathcal{A}_{k1}(t)\mathcal{A}_{11}(t)^{-1}\mathcal{A}_{1l}(t) = \lim_{t \to \infty} \frac{y_k(t)y_l(t)}{y_1(t)y_{\max\{k,l\}}(t)}(\widehat{\mathcal{A}}_{k1}\widehat{\mathcal{A}}_{11}^{-1}\widehat{\mathcal{A}}_{1l} + o(1)) = 0.$$

However, this means that the (k, l)-th block of $\tilde{S}(t) := \Pi_{22} - \Pi_{21}\Pi_{11}^{-1}\Pi_{12}$ behaves like

$$\tilde{S}_{kl}(t) = y_{\max\{k,l\}}(t)(\widehat{\mathcal{A}}_{kl} + o(1)).$$

Applying the induction hypothesis to the $(r-1, r-1)$ block matrix $\tilde{S}(t)$, we obtain the asymptotic behavior of $\Psi_{22}(t) = \tilde{S}(t)^{-1}$ and this finishes the proof. □

10. Proof of the main theorem

In this section, we present the proof of our main theorem in the local case, i.e., in the case of a versal analytic family of Riemann surfaces. As we mentioned previously, this will be enough to conclude with the proof of Theorem 1.2.

Let S_0 be a stable curve of genus g with the stable dual graph $G = (V, E, \mathfrak{g})$ and let $\mathcal{S} \to B$ be the versal analytic family of stable curves over a polydisc B of dimension $N = 3g - 3$ with $\mathcal{S}_0 = S_0$, as in the previous sections.

Consider the hybrid space B^{hyb} and the family of hybrid curves $\mathcal{S}^{hyb}$ over B^{hyb}. For any $\mathbf{t} \in B^{hyb}$, its fiber $\mathcal{S}_{\mathbf{t}}^{hyb}$ is a hybrid curve and we equip it with its canonical measure $\mu_{\mathbf{t}}$ defined in Section 5.5. We have to prove that the family of canonically measured hybrid curves $(\mathcal{S}_{\mathbf{t}}^{hyb}, \mu_{\mathbf{t}})_{\mathbf{t} \in B^{hyb}}$ over B^{hyb} is continuous. Recall that this means that for any continuous function $f\colon \mathcal{S}^{hyb} \to \mathbb{R}$, the associated function

$$F(\mathbf{t}) := \int_{\mathcal{S}_{\mathbf{t}}^{hyb}} f_{|_{\mathcal{S}_{\mathbf{t}}^{hyb}}} \, d\mu_{\mathbf{t}}, \qquad \mathbf{t} \in B^{hyb}$$

is continuous on B^{hyb}.

By the definition of the hybrid space B^{hyb}, a point $\mathbf{t}$ of B^{hyb} consists of a pair $\mathbf{t} = (t, x)$ for a point $t \in B$ and $x \in \sigma_\pi^\circ$, where π is an ordered partition of the subset $E_t \subseteq E$. Here, as before, E_t is the set of all edges e in G such that t belongs to the divisor $D_e \subseteq B$.

We first remark that without loss of generality, we can reduce the proof of the continuity to the case where $t = 0 \in B$. In fact, for a general point $t \in B$, there exists an open neighborhood $B_t \subset B$ of t such that $(B_t, \mathcal{S}|_{B_t})$ is isomorphic to the pair consisting of the base of the versal deformation space of $\mathcal{S}_t$ and the corresponding versal family of Riemann surfaces. Moreover, by the definition of hybrid spaces, it is easy to check that for any point $\mathbf{t} = (t, x) \in B^{hyb}$ with $x \in \sigma_\pi^\circ$ for π an ordered partition of E_t, the hybrid space B_t^{hyb} is naturally an open neighborhood of $\mathbf{t}$ in B^{hyb}. Therefore, in order to show the continuity, by replacing S_0 with $\mathcal{S}_t$ and B by B_t, if necessary, we can assume that $t = 0 \in B$ and that $x \in \sigma_\pi^\circ$ for an ordered partition π of E. So let $\mathbf{t}_0 = (0, x_0)$ be a fixed point in B^{hyb}. We prove the continuity of the measured family at $\mathbf{t}_0$.

In the previous section, we showed the continuity of the canonical measures at $\mathbf{t}_0$ through the open part B^* of B^{hyb}. In order to prove the continuity more generally, it will be enough to proceed stratum by stratum and prove the continuity at $\mathbf{t}_0$ through any stratum $D_{\pi'}^{hyb}$ of B^{hyb} corresponding to an ordered partition $\pi' \preceq \pi$ of a subset $F \subseteq E$ (see Proposition 3.3). This means we need to show that the family of canonically measured hybrid curves $\mathcal{S}^{hyb}|_{D_{\pi'}^{hyb} \cup \{\mathbf{t}_0\}}$ is continuous at $\mathbf{t}_0$, with respect to the induced topology on $D_{\pi'}^{hyb} \cup \{\mathbf{t}_0\} \subseteq B^{hyb}$. (The generic continuity corresponds thus to the case where $F = \varnothing$ and $\pi' = (\varnothing)$.)

So for the remaining of this section, we fix a subset of edges $F \subseteq E$ and an ordered partition π' of F with $\pi' \preceq \pi$. We show how to reduce this to the continuity result proved in Section 9.1 and the one treated in Section 6.2 in the tropical curve case.

Recall that for an ordered partition $\rho = (\rho_1, \dots, \rho_k)$ of a subset of edges $E_\rho = \rho_1 \sqcup \dots \sqcup \rho_k \subset E$, the hybrid stratum D_ρ^{hyb} of B^{hyb} is defined by

$$D_\rho^{hyb} = D_\rho^\circ \times \sigma_\rho^\circ \qquad \text{where} \qquad D_\rho^\circ = D_{E_\rho}^\circ \text{ and } \sigma_\rho^\circ = \prod_{i=1}^k \sigma_{\rho_i}^\circ.$$

Each point $\mathbf{t} \in D_\rho^{hyb}$ can be written in the form $\mathbf{t} = (t, x)$ with $t \in D_\rho^\circ$ and $x \in \sigma_\rho^\circ$. We denote the corresponding stable Riemann surface by $\mathcal{S}_\mathbf{t} := \mathcal{S}_t$, the fiber of the versal family $\mathcal{S}/B$ at point $t \in B$.

The layered stable dual graph of $\mathcal{S}_\mathbf{t}$ of a point $\mathbf{t} \in D_\rho^{hyb}$ is the layered stable graph G_ρ with vertex set V_ρ and edge set E_ρ obtained from the stable dual graph G by contracting all the edges of $E \backslash E_\rho$ and by the layering of edges given by ρ. For each vertex $u \in V_\rho$, we denote by $C_{\mathbf{t},u}$ the component of $\mathcal{S}_\mathbf{t}$ which corresponds to u.

Consider now the point $\mathbf{t}_0 = (0, x_0) \in B^{hyb}$ with $\mathbf{t}_0 \in D_\pi^{hyb}$ for an ordered partition $\pi = (\pi_1, \dots, \pi_r)$ of the edge set E. We fix an ordered partition $\pi' = (\pi'_1, \dots, \pi'_k)$ coarser than π with $E_{\pi'} = F$ the set of edges of π'. Let G' be the stable dual graph of any point in the stratum $D_{\pi'}^{hyb}$, obtained from G by contracting all the edges in $E \backslash F$.

Since $\pi' \preceq \pi$, there exists $r_0 \leqslant r$ such that the initial interval $\pi_0 := (\pi_1, \dots, \pi_{r_0})$ of π forms an ordered partition of F, and so $E_{\pi_0} = F$. Let G_{π_0} be the layered stable graph obtained by contracting all the edges of $E\backslash F$ and taking π_0 as the layering.

We have a continuous forgetful map $\mathfrak{q} : D_{\pi'}^{hyb} \sqcup D_{\pi}^{hyb} \to \mathscr{M}_{G'}^{\mathrm{trop}}$, which sends any point $\mathbf{t} = (t, x)$ to the point $\mathfrak{q}(\mathbf{t}) = x|_F$ either in $\sigma_{\pi'}^{\circ}$ or in $\sigma_{\pi_0}^{\circ}$ depending on whether $\mathbf{t} \in D_{\pi'}^{hyb}$ or $\mathbf{t} \in D_{\pi}^{hyb}$, respectively.

Moreover, for any such point $\mathbf{t} \in D_{\pi'}^{hyb} \sqcup D_{\pi}^{hyb}$ and for any edge $e \in F$, the restriction of the canonical measure $\mu_{\mathbf{t}}$ on the interval $\mathcal{I}_e|_{\mathbf{t}}$ of the hybrid curve $\mathcal{S}_{\mathbf{t}}^{hyb}$ coincides with the restriction of the canonical measure $\mu_{\mathfrak{q}(\mathbf{t})}^{can}$ of the tropical curve $\mathscr{C}_{\mathfrak{q}(\mathbf{t})}^{\mathrm{trop}}$ to the interval $\mathcal{I}_e|_{\mathfrak{q}(\mathbf{t})} = \mathcal{I}_e|_{\mathbf{t}}$ of $\mathscr{C}_{\mathfrak{q}(\mathbf{t})}^{\mathrm{trop}}$. Indeed, the graded minors of the hybrid and tropical curves coincide for those parts which concern the edges of F.

By Theorem 6.1, the universal family of tropical curves $\mathscr{C}_{G'}^{\mathrm{trop}}$ of combinatorial type G' form a continuous family over $\mathscr{M}_{G'}^{\mathrm{trop}}$. We thus infer that for any edge $e \in F$, the family of measured intervals $(\mathcal{I}_{\mathbf{t},e}, \mu_{\mathbf{t}}|_{\mathcal{I}_{\mathbf{t},e}}) = \mathfrak{q}^*\Big(\big(\mathcal{I}_{\mathfrak{q}(\mathbf{t}),e}, \mu_{\mathfrak{q}(\mathbf{t})}^{can}|_{\mathcal{I}_{\mathfrak{q}(\mathbf{t}),e}}\big)\Big)$ over $D_{\pi'}^{hyb} \sqcup D_{\pi}^{hyb}$ forms a continuous family of measured spaces (see also Lemma 6.4). This proves half of the continuity statement.

To prove the second half, let $\mathscr{K}_{\pi > \pi'} : G_\pi \to G_{\pi'}$ be the contraction map, which contracts all the edges not in F.

For each vertex $v \in V_{\pi'}$, let $V_{\pi,v}$ be the set of all vertices of G_π which are mapped to v by $\mathscr{K}_{\pi > \pi'}$, and let $E_{\pi,v}$ be the set of all edges in $E\backslash F$ with both end-points in $V_{\pi,v}$.

The graph $G_{\pi,v} = (V_{\pi,v}, E_{\pi,v})$ is a subgraph of G_π and the partition π induces a partition π_v of the set $E_{\pi,v}$.

For a vertex $v \in V_{\pi'}$, consider the subcurve $S_v \subseteq S_0 = \mathcal{S}_{\mathbf{t}_0}$ consisting of the union of all the components $C_{\mathbf{t}_0,u} = C_u$ for vertices $u \in V_{\pi,v}$. The curve S_v comes with a marking given by all the points in correspondence with the edges $e \in F$ of the stable dual graph G_π which are incident to a vertex of $V_{\pi,v}$, and the marked curve S_v is stable.

Denote by ${}_vB$ the analytic versal deformation space of the stable marked curve S_v. Let ${}_vB^{hyb}$ be the corresponding hybrid space and denote by ${}_v\mathcal{S}^{hyb}$ the versal hybrid curve over ${}_vB^{hyb}$, endowed in each fiber with its canonical measure.

We have a natural projection map from the closed stratum $D_F = \bigsqcup_{F \subseteq \hat{F} \subseteq E} D_{\hat{F}}^{\circ}$ to B_v, which in turn gives rise to a natural hybrid projection map ${}_v\mathfrak{p} : B_{[\pi',\pi]}^{hyb} \to {}_vB^{hyb}$. Recall that the subspace $B_{[\pi',\pi]}^{hyb} \subseteq B^{hyb}$ was introduced in (3.12) and corresponds to ordered partitions ϱ with $\pi' \preceq \varrho \preceq \pi$. Under this last map, we have the obvious inclusion

$$D_{\pi'}^{hyb} \subseteq {}_v\mathfrak{p}^{-1}\big({}_vB^*\big).$$

In addition, the pull-back ${}_v\mathfrak{p}^*\Big({}_v\mathcal{S}^{hyb}\Big)$ can be identified naturally with a measured subspace of the hybrid family $\mathcal{S}^{hyb}$.

From the generic continuity theorem (see Theorem 9.1), that extends verbatim to the marked setting, we get the generic continuity of the measured hybrid curves ${}_v\mathcal{S}^{hyb}$ equipped in each fiber ${}_v\mathcal{S}^{hyb}_{\mathbf{t}}$ with the canonical measure of the hybrid curve ${}_v\mathcal{S}^{hyb}_{\mathbf{t}}$.

This means that the family of canonically measured curves ${}_v\mathcal{S}^{hyb}|_{{}_vB^{*}\cup\{{}_v\mathbf{t}_0\}}$ is continuous at the point ${}_v\mathbf{t}_0 := {}_v\mathfrak{p}(\mathbf{t}_0)$ in ${}_vB^{hyb}$.

We infer that the family of measured spaces ${}_v\mathfrak{p}^*\left({}_v\mathcal{S}^{hyb}\right)$ is continuous over the subspace $D^{hyb}_{\pi'} \cup \{\mathbf{t}_0\} \subset B^{hyb}$. Since this holds for all vertices of $V_{\pi'}$, we get the second half of the continuity we were looking for, and the theorem follows.

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References

[AB15] Omid Amini and Matthew Baker, *Linear series on metrized complexes of algebraic curves*, Mathematische Annalen **362** (2015), no. 1-2, 55–106.

[ABBF16] Omid Amini, Spencer Bloch, José Ignacio Burgos Gil, and Javier Fresán, *Feynman amplitudes and limits of heights*, Izvestiya: Mathematics **80** (2016), no. 5, 813–848, special volume in honor of Jean–Pierre Serre.

[ABBR15] Omid Amini, Matthew Baker, Erwan Brugallé, and Joseph Rabinoff, *Lifting harmonic morphisms I: metrized complexes and Berkovich skeleta*, Research in the Mathematical Sciences **2** (2015), no. 7, 67 pages.

[AC13] Omid Amini and Lucia Caporaso, *Riemann–Roch theory for weighted graphs and tropical curves*, Advances in Mathematics **240** (2013), 1–23.

[ACG11] Enrico Arbarello, Maurizio Cornalba, and Phillip Griffiths, *Geometry of algebraic curves: volume II*, Vol. 268, Springer Science & Business Media, 2011.

[ACGH84] Enrico Arbarello, Maurizio Cornalba, Phillip A. Griffiths, and Joe Harris, *Geometry of algebraic curves*, Vol. 267, Springer, 1984.

[ACP15] Dan Abramovich, Lucia Caporaso, and Sam Payne, *The tropicalization of the moduli space of curves*, Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), no. 4, 765–809.

[AI21] Omid Amini and Hernán Iriarte, *Geometry of higher rank quasi-monomial valuations*, Preprint (2021).

[Ami14] Omid Amini, *Equidistribution of Weierstrass points on curves over non-Archimedean fields*, preprint (2014), revised version 2021.

[AN21a] Omid Amini and Noema Nicolussi, *Algebraic geometry of hybrid curves*, Preprint in preparation (2021).

[AN21b] Omid Amini and Noema Nicolussi, *Hybrid Green functions*, Preprint in preparation (2021).

[AP20a] Omid Amini and Matthieu Piquerez, *Hodge theory for tropical varieties*, arXiv preprint arXiv:2007.07826 (2020).

[AP20b] Omid Amini and Matthieu Piquerez, *Tropical Clemens-Schmid sequence and existence of tropical cycles with a given cohomology class*, arXiv preprint arXiv:2012.13142 (2020).

[Ber09] Vladimir G Berkovich, *A non-archimedean interpretation of the weight zero subspaces of limit mixed Hodge structures*, Algebra, arithmetic, and geometry, 2009, pp. 49–67.

[Ber71] George M Bergman, *The logarithmic limit-set of an algebraic variety*, Transactions of the American Mathematical Society **157** (1971), 459–469.

[BF11] Matthew Baker and Xander Faber, *Metric propertis of the tropical Abel–Jacobi map*, Journal of Algebraic Combinatorics **33** (2011), 349–381.

[BFJ15] Sébastien Boucksom, Charles Favre, and Mattias Jonsson, *Solution to a non-Archimedean Monge-Ampère equation*, Journal of the American Mathematical Society **28** (2015), no. 3, 617–667.

[BG84] Robert Bieri and J.R.J. Groves, *The geometry of the set of characters induced by valuations*, Journal für die reine und angewandte Mathematik **1984** (1984), no. 347, 168–195.

[BGJK20] José Ignacio Burgos Gil, Walter Gubler, Philipp Jell, and Klaus Künnemann, *A comparison of positivity in complex and tropical toric geometry*, arXiv preprint arXiv:2003.08644 (2020).

[BJ16] Matthew Baker and David Jensen, *Degeneration of linear series from the tropical point of view and applications*, Nonarchimedean and tropical geometry, 2016, pp. 365–433.

[BJ17] Sébastien Boucksom and Mattias Jonsson, *Tropical and non-Archimedean limits of degenerating families of volume forms*, Journal de l'École Polytechnique – Mathématiques **4** (2017), 87–139.

[BJ18] Sébastien Boucksom and Mattias Jonsson, *A non-Archimedean approach to K-stability*, preprint arXiv:1805.11160 (2018).

[BM08] John Adrian Bondy and U. S. Ramachandra Murty, *Graph theory*, Vol. 244, Springer, 2008.

[Bol98] Béla Bollobás, *Modern graph theory*, Vol. 184, Springer, 1998.

[CGP16] Melody Chan, Søren Galatius, and Sam Payne, *The tropicalization of the moduli space of curves II: Topology and applications*, preprint arXiv:1604.03176 (2016).

[CGP18] Melody Chan, Søren Galatius, and Sam Payne, *Tropical curves, graph complexes, and top weight cohomology of M_g*, preprint arXiv:1805.10186 (2018).

[Cin11] Zubeyir Cinkir, *Zhang's conjecture and the effective Bogomolov conjecture over function fields*, Inventiones mathematicae **183** (2011), no. 3, 517–562.

[CKS86] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid, *Degeneration of Hodge structures*, Annals of Mathematics **123** (1986), no. 3, 457–535.

[CLD12] Antoine Chambert-Loir and Antoine Ducros, *Formes différentielles réelles et courants sur les espaces de Berkovich*, arXiv preprint arXiv:1204.6277 (2012).

[CLT10] Antoine Chambert-Loir and Yuri Tschinkel, *Igusa integrals and volume asymptotics in analytic and adelic geometry*, Confluentes Mathematici **2** (2010), no. 03, 351–429.

[CR93] Ted Chinburg and Robert Rumely, *The capacity pairing*, Journal für die reine und angewandte Mathematik **1993** (1993), no. 434, 1–44.

[DF19] Romain Dujardin and Charles Favre, *Degenerations of* $\mathrm{SL}(2,\mathbb{C})$ *representations and Lyapunov exponents*, Annales Henri Lebesgue **2** (2019), 515–565.

[DHL19] Antoine Ducros, Ehud Hrushovski, and François Loeser, *Non-archimedean integrals as limits of complex integrals*, preprint arXiv:1912.09162 (2019).

[Die00] Reinhard Diestel, *Graph theory*, Vol. 173, Springer, 2000.

[dJ19] Robin de Jong, *Faltings delta-invariant and semistable degeneration*, Journal of Differential Geometry **111** (2019), no. 2, 241–301.

[DKY20] Laura DeMarco, Holly Krieger, and Hexi Ye, *Uniform Manin-Mumford for a family of genus 2 curves*, Annals of Mathematics **191** (2020), no. 3, 949–1001.

[DM69] Pierre Deligne and David Mumford, *The irreducibility of the space of curves of given genus*, Publications Mathématiques de l'IHES **36** (1969), 75–109.

[EKL06] Manfred Einsiedler, Mikhail Kapranov, and Douglas Lind, *Non-Archimedean amoebas and tropical varieties*, Journal für die reine und angewandte Mathematik **2006** (2006), no. 601, 139–157.

[Fal20] Gerd Faltings, *Arakelov geometry on degenerating curves*, Journal für die reine und angewandte Mathematik **1** (2020), no. ahead-of-print.

[Fav18] Charles Favre, *Degeneration of endomorphisms of the complex projective space in the hybrid space*, Journal of the Institute of Mathematics of Jussieu (2018), 1–43.

[FJP20] Gavril Farkas, David Jensen, and Sam Payne, *The Kodaira dimensions of $\overline{M}_{22}$ and $\overline{M}_{23}$*, arXiv preprint arXiv:2005.00622 (2020).

[GKZ94] Israel M. Gelfand, Mikhail M. Kapranov, and Andrei V. Zelvinsky, *Discriminants, resultants, and multidimensional determinants*, Birkhäuser Boston Inc., Boston, MA, 1994.

[GS06] Mark Gross and Bernd Siebert, *Mirror symmetry via logarithmic degeneration data I*, Journal of Differential Geometry **72** (2006), no. 2, 169–338.

[GS10] Mark Gross and Bernd Siebert, *Mirror symmetry via logarithmic degeneration data II*, Journal of Algebraic Geometry **19** (2010), no. 4, 679–780.

[GTZ13] Mark Gross, Valentino Tosatti, and Yuguang Zhang, *Collapsing of abelian fibered Calabi–Yau manifolds*, Duke Mathematical Journal **162** (2013), no. 3, 517–551.

[GTZ16] Mark Gross, Valentino Tosatti, and Yuguang Zhang, *Gromov–Hausdorff collapsing of Calabi–Yau manifolds*, Communications in Analysis and Geometry **24** (2016), no. 1, 93–113.

[GW00] Mark Gross and Pelham Wilson, *Large complex structure limits of K3 surfaces*, Journal of Differential Geometry **55** (2000), no. 3, 475–546.

[Hof84] Jerome William Hoffman, *The Hodge theory of stable curves*, Vol. 308, American Mathematical Soc., 1984.

[IKMZ19] Ilia Itenberg, Ludmil Katzarkov, Grigory Mikhalkin, and Ilia Zharkov, *Tropical homology*, Mathematische Annalen **374** (2019), no. 1-2, 963–1006.

[Jon16] Mattias Jonsson, *Degenerations of amoebae and Berkovich spaces*, Mathematische Annalen **364** (2016), no. 1-2, 293–311.

[JP16] David Jensen and Sam Payne, *Tropical independence II: The maximal rank conjecture for quadrics*, Algebra & Number Theory **10** (2016), no. 8, 1601–1640.

[KS06] Maxim Kontsevich and Yan Soibelman, *Affine structures and non-archimedean analytic spaces*, The unity of mathematics, 2006, pp. 321–385.

[Lag12] Aron Lagerberg, *Super currents and tropical geometry*, Mathematische Zeitschrift **270** (2012), no. 3-4, 1011–1050.

[Li20] Yang Li, *Metric SYZ conjecture and non-Archimedean geometry*, arXiv preprint arXiv:2007.01384 (2020).

[Lyo03] Russell Lyons, *Determinantal probability measures*, Publications Mathématiques de l'IHÉS **98** (2003), 167–212.

[Mik04] Grigory Mikhalkin, *Decomposition into pairs-of-pants for complex algebraic hypersurfaces*, Topology **43** (2004), no. 5, 1035–1065.

[MS84] John W Morgan and Peter B Shalen, *Valuations, trees, and degenerations of hyperbolic structures* I, Annals of Mathematics **120** (1984), no. 3, 401–476.

[Nee84] Amnon Neeman, *The distribution of weierstrass points on a compact riemann surface*, Annals of Mathematics (1984), 317–328.

[NY20] Ngai-Fung Ng and Sai-Kee Yeung, *Limit of Weierstrass measure on stable curves*, arXiv preprint arXiv:2012.09642 (2020), to appear in Journal of Differential Geometry.

[Oda19] Yuji Odaka, *Tropical Geometric Compactification of Moduli, I–M_g Case*, Moduli of K-stable varieties, 2019, pp. 75–101.

[OO18] Yuji Odaka and Yoshiki Oshima, *Collapsing K3 surfaces, Tropical geometry and Moduli compactifications of Satake, Morgan-Shalen type*, preprint arXiv:1810.07685 (2018).

[PR04] Mikael Passare and Hans Rullgård, *Amoebas, Monge-Ampère measures, and triangulations of the Newton polytope*, Duke Mathematical Journal **121** (2004), no. 3, 481–507.

[PS19] Léonard Pille-Schneider, *Hybrid convergence of Kähler-Einstein measures*, preprint arXiv:1911.03357 (2019).

[Ric18] David Harry Richman, *The distribution of Weierstrass points on a tropical curve*, preprint arXiv:1809.07920 (2018).

[Rud10] Helge Ruddat, *Log Hodge groups on a toric Calabi-Yau degeneration*, Mirror Symmetry and Tropical Geometry, Contemporary Mathematics **527** (2010), 113–164.

[Rud20] Helge Ruddat, *A homology theory for tropical cycles on integral affine manifolds and a perfect pairing*, arXiv preprint arXiv:2002.12290, to appear in Geometry and Topology (2020).

[Rul01] Hans Rullgård, *Polynomial amoebas and convexity*, Ph.D. Thesis, 2001.

[Sch73] Wilfried Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Inventiones mathematicae **22** (1973), no. 3-4, 211–319.

[Shi19] Sanal Shivaprasad, *Convergence of volume forms on a family of log-Calabi-Yau varieties to a non-Archimedean measure*, preprint arXiv:1911.07307 (2019).

[Shi20] Sanal Shivaprasad, *Convergence of Bergman measures towards the Zhang measure*, preprint arXiv:2005.05753 (2020).

[Sus18] Dmitry Sustretov, *Gromov-Hausdorff limits of flat Riemannian surfaces and non-Archimedean geometry*, preprint arXiv:1802.03818 (2018).

[SW19] Farbod Shokrieh and Chenxi Wu, *Canonical measures on metric graphs and a Kazhdan's theorem*, Inventiones mathematicae **215** (2019), 819–862.

[Wen91] Richard Wentworth, *The asymptotics of the Arakelov-Green's function and Faltings' delta invariant*, Communications in Mathematical Physics **137** (1991), no. 3, 427–459.

[Zha10] Shou-Wu Zhang, *Gross–Schoen cycles and dualising sheaves*, Inventiones mathematicae **179** (2010), no. 1, 1.

[Zha93] Shou-Wu Zhang, *Admissible pairing on a curve*, Inventiones mathematicae **112** (1993), no. 1, 171–193.

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