VORONOI TILINGS, TORIC ARRANGEMENTS AND DEGENERATIONS OF LINE BUNDLES III

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Abstract. We describe limits of line bundles on nodal curves in terms of toric arrangements associated to Voronoi tilings of Euclidean spaces. These tilings encode information on the relationship between the possibly infinitely many limits, and ultimately give rise to a new definition of *limit linear series*. This article and the first two that preceded it are the first in a series aimed to explore this new approach.

In Part I, we set up the combinatorial framework and showed how graphs weighted with integer lengths associated to the edges provide tilings of Euclidean spaces by certain polytopes associated to the graph itself and to its subgraphs.

In Part II, we described the arrangements of toric varieties associated to the tilings of Part I in several ways: using normal fans, as unions of orbits, by equations and as degenerations of tori.

In the present Part III, we show how these combinatorial and toric frameworks allow us to describe all *stable limits* of a family of line bundles along a degenerating family of curves. Our main result asserts that the collection of all these limits is parametrized by a connected 0-dimensional closed substack of the Artin stack of all torsion-free rank-one sheaves on the limit curve. Moreover, we thoroughly describe this closed substack and all the closed substacks that arise in this way as certain torus quotients of the arrangements of toric varieties of Part II determined by the Voronoi tilings of Euclidean spaces studied in Part I.

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1. Introduction

This is a sequel to our previous works [AE20a, AE20b] whose aim is to achieve the description of stable limits of line bundles on nodal curves by means of graph theory and toric geometry. This is motivated by the desire to understand all the possible limits of linear series $\mathfrak{g}^r_d$ over any sequence $X_1, X_2, \dots$ of smooth projective curves of genus g whose corresponding points $x_1, x_2, \dots$ in $\mathscr{M}_g$ converge to a given point x on the Deligne-Mumford compactification $\overline{\mathscr{M}}_g$.

In this introduction, after providing an overview of the previous works, we focus on the contribution of this paper and its preceding companions and briefly discuss the content of our forthcoming work.

1.1. **Overview.** Nodal curves are curves (one-dimensional, reduced, connected but not necessarily irreducible projective schemes over an algebraically closed field) that fail to be smooth in the weakest possible form: the singularities are normal crossings, that is, ordinary nodes. Among them, the most important are (Deligne–Mumford) stable curves, characterized as those having ample canonical bundle. This is in fact the property that allows for the construction of their moduli space, $\overline{\mathscr{M}}_g$, where g stands for the (arithmetic) genus. One of the key properties of stable curves is the Stable Reduction Theorem, which says that a family of stable curves parameterized by the germ of a punctured smooth curve can be completed, after a finite base change, in a unique way to a family over the whole germ. It implies that the moduli of stable curves is complete, in fact projective, and is thus a compactification for the moduli of smooth curves of genus at least two. However, no such thing holds, in general, for line bundles and linear series over curves.

A linear series over a curve X is simply the data of a line bundle and a linear subspace of the space of sections of that bundle. If d is the (total) degree of the line bundle and r is the projective dimension of the subspace, we say we have a $\mathfrak{g}^r_d$. If $r \geq 0$, it corresponds to a rational map $X \dashrightarrow \mathbf{P}^r$, whence the importance of their study for understanding the projective geometry of curves. Line bundles over a curve are parametrized by the Picard scheme, and the linear series by fibrations by Grassmannians over the Picard scheme. It is all well if X is smooth, as the Picard scheme has projective connected components indexed by the degree d; the one with degree 0 is even an algebraic group, the Jacobian.

However, the Picard scheme of a stable curve has projective components only if the curve is of compact type. Furthermore, whereas for smooth curves, a $\mathfrak{g}^r_d$ gives rise to a rational map that can even be uniquely extended to the whole curve, the map induced by a $\mathfrak{g}^r_d$ on a stable curve may fail to be defined on whole components of the curve. Finally, there is nothing similar to stable reduction; quite to the contrary, the trivial line bundle over the total space of a family of smooth curves parameterized by the germ of a punctured smooth curve can be completed in infinitely many ways over the whole germ if the family of curves is completed by adding a reducible stable curve.

A compactification of the Picard scheme of an irreducible curve was suggested by Mumford in the sixties, carried out by D'Souza [D'S79] in the seventies, and thoroughly studied by Altman and Kleiman [AK80, AK79] and others in the years that followed. As it was later observed by Eisenbud and Harris [EH83], it is possible to complete the proof of the celebrated Brill–Noether Theorem by considering this compactification over rational irreducible cuspidal curves. The compactification parameterizes torsion-free, rank-one sheaves, and the notion of linear series extends simply by just considering spaces of sections of these sheaves. In fact, in this setup it is even possible to extend the notion of divisors: these are the pseudo-divisors studied by Hartshorne [Har06]. Finally, there are fine proper moduli spaces parameterizing torsion-free, rank-one sheaves of a given degree for families of irreducible curves.

Though even in codimension one (in their moduli), stable curves fail to be irreducible, fortunately, general reducible stable curves are of compact type. Curves of compact type are those curves for which the connected components of the Picard scheme are compact, indeed projective. Eisenbud and Harris studied degenerations of line bundles along families of smooth curves degenerating to curves of compact type, and coined the term *limit linear series* [EH86]. Remarkably, they observed that it was actually useful that the family of line bundles could degenerate to many limit line bundles, as just one limit would rarely carry enough information about the degeneration. They found out that for many applications it was important, and enough, to consider as many limit line bundles as components of the limit stable curve, each focusing on one of the components. Their study allowed for a deep understanding of the moduli of stable curves up to codimension two, and found many applications in describing the geometry of a general curve and the theory of divisors of $\overline{\mathscr{M}}_g$ [EH86, EH87a, EH87b]. It is thus no surprise they wished for an extension of their theory to more general curves; as they wrote in [EH89]: "The known special cases suffice for many applications, but there is probably a gold mine awaiting a general insight."

Already in the late seventies, the study of degenerations to more complicated nodal curves began, in the hands of Oda and Seshadri [OS79]. To do away with the problem of many limits line bundles, they introduced the notion of stability, one that had naturally arisen in the study of vector bundles over smooth curves and the construction of their moduli space by Mumford's Geometric Invariant Theory. Out of the many limits a family of line bundles could have, at most one would be stable, and a finite nonzero number of them would be semistable. Moreover, the semistable limits would be equivalent in the sense they would all have the same quotients in their Jordan–Hölder filtrations. The convenient notion of stability has been centerpiece in all the studies that followed, including in the landmark construction of a compactification of the relative Picard scheme over the whole moduli of curves by Caporaso in the early 90's [Cap94].

However, as far as limits of linear series are considered, stability appears to be very restrictive. The line bundles Eisenbud and Harris considered, one for each component of the stable curve, would rarely be semistable; even if one could force one of them to be semistable, by deforming the notion of stability, the others would seldom be. Moreover, generally semistable line bundles in the same equivalence class have almost completely unrelated spaces of sections.

On the other hand, an approach similar to that by Eisenbud and Harris, choosing for each component of the limit curve a "best" limit line bundle, was tentatively carried out by many: Ziv Ran had an early draft on this already in the 80's, whereas the second author, in collaboration with Medeiros and Salehyan [EM02, ES07], used this approach to study limits of Weierstrass points for a wide class of stable curves in the nineties. However, one could not carry the approach further.

In the middle of the first decade of the present century, Osserman [Oss06] introduced a new idea, that not only extended the theory to include curves in positive characteristic, but also modified substantially the approach by Eisenbud and Harris. Osserman's idea was remarkably simple, even though it seemed at first to complicate further the study: One should consider not only the "best" limit line bundle for each component of the limit curve, but also all of those in between, more precisely, all of those with nonnegative degree on every component; the intermediate limits carried only partial information over each component, but perhaps crucial information nonetheless. And, in fact, if Eisenbud's and Harris's limit linear series was not refined, Osserman found out that the extra limits did carry more information. For instance, Osserman and the second author discovered later that they described limits of divisors of the family of linear series that cannot be accounted for only by the "best" line bundles [EO13].

The approach by Osserman introduced a new challenge: How to account for, how to keep track of all the data that show up when considering all effective limit line bundles? Osserman introduced his approach first only for two-component curves of compact type, the simplest case, and recently extended the approach to more general curves [Oss19b, Oss16], specifically, curves of pseudo-compact type where a parallel theory to that by Eisenbud and Harris can be established.

This was not the only challenge: another was to consider one-parameter families of curves whose total space was not smooth. For such families, when the limit curve is not of compact type, it might as well happen that a family of line bundles has no line bundle in the limit. One can blow up the total space and replace the stable curve by one of its semistable models, splitting apart each node to introduce a chain of a variable number of rational smooth curves. This way one obtains a limit line bundle, but on a different curve! Worse, one has to deal with a whole set of new variables, one for each node: the number of rational curves in each chain. This procedure, semistable reduction, has already appeared when dealing with curves of compact type; it did not lead to many difficulties in applications because in them, a class of curves was considered, and the class did not change by semistable reduction. For more general curves, as it became apparent in the work by Esteves and Medeiros [EM02] on curves with two components, the specific type of reduction that appeared had enormous influence on, for instance, the limits of Weierstrass points.

The whole study has been combinatorially intensive from the start, already in Oda–Seshadri [OS79]. Farther than obstructing the study of the problem, combinatorics has been one of the main tools used by all those that ventured in the field. It has even spanned a new line of approach, using tropical and non-Archimedean geometry, that has been very fruitful. In fact, the first author in a joint work with Baker [AB15] has introduced a notion of limit linear series which also extends, in a somewhat more combinatorial way originating from non-Archimedean analysis, the definition by Eisenbud and Harris. He has used this approach as an alternative tool in the study of reduction of Weierstrass points and their distributions [Ami14]. Moreover, this circle of ideas has been used by Jensen and Payne to prove specific cases of the Maximal Rank Conjecture [JP14, JP16, JP17], as well as in the recent work by Farkas, Jensen and Payne on the Kodaira dimension of the moduli spaces of curves of genus 22 and 23 [FJP20]. A comparison of Osserman's approach [Oss19b] to the work of Amini and Baker [AB15] can be found in [Oss19a].

The overall approach we take in these series of works features a new interplay between the combinatorics and the geometry. We do not do away with any data (as it is done, sometimes harmlessly for the applications in mind), but we use the combinatorics to organize all the information. In a nutshell we prove in the present paper that the collection of all limit torsion-free, rank-one sheaves of a family of line bundles along a family of smooth curves degenerating to a nodal curve X is parameterized by a connected 0-dimensional closed substack $\mathfrak{I}$ of the Artin stack $\mathbf{J}$ of all torsion-free, rank-one sheaves on X.

More significatively, we characterize combinatorially all the closed substacks $\mathfrak{I}$ that arise from degenerations. The characterization is given by Theorems 4.7 and 5.1, which are the main results in the article. In short, the $\mathfrak{I}$ are certain torus quotient of a combinatorial arrangement of toric varieties. Thus our theorems provide a far-reaching generalization to all nodal curves of the combinatorial-toric approach carried out by Esteves and Medeiros in [EM02] for the case of stable curves with two irreducible components.

In a future work, if a family of linear series is given, in addition to the substack $\mathfrak{I}$, we will show the existence of a closed substack $\mathfrak{I}'$ of $\mathbf{G}$, the Grassmann fibration over $\mathbf{J}$ parameterizing vector spaces of sections of torsion-free, rank-one sheaves on X, satisfying two conditions: First, the stack $\mathfrak{I}'$ will lie over $\mathfrak{I}$. Second, the induced relative torsion-free rank-one sheaf $\mathcal{I}$ on $X \times \mathfrak{I}'$ over $\mathfrak{I}'$ and the induced locally free subsheaf $\mathcal{V}$ of the pushforward $p_{2*}\mathcal{I}$ will be such that their restrictions over the points of $\mathfrak{I}'$ parameterize all the limits of the family of linear series. We will describe $\mathfrak{I}'$ combinatorially as well; it is actually very similar in esprit to the description we give of $\mathfrak{I}$.

The connection with the work by Osserman will be elaborated in our future study. In the case of a two-component curve of compact type, this has been carried out in detail in the doctoral thesis by Rizzo [Riz13], who explains how the collection of linear series considered by Osserman as a limit linear series for a two-component curve of compact type can actually be viewed as members of a $\mathbf{G_m}$-equivariant family of linear series on X parameterized by a 2-punctured chain of rational curves. The quotient of this chain by $\mathbf{G_m}$ is the stack $\mathfrak{I}'$ truncated in nonnegative degrees.

1.2. **The present work.** We will now proceed to describe the work we do in this paper, in continuation to what we did in [AE20a] and [AE20b]. Later we will point out what lies ahead in the path we are taking.

Fix a connected nodal curve X over an algebraically closed field κ. Consider its associated dual graph $G = (V, E)$, which is a pair consisting of a vertex set V in one-to-one correspondence with the set of irreducible components of X, and an edge set E in one-to-one correspondence with the set of nodes of X. An edge connects two vertices if the corresponding node lies on the two corresponding components. For our purposes, we discard those edges that form a loop.

Let now $\mathbb{E}$ be the set of all the oriented edges (also called arcs) obtained out of E: for each edge, there are two possible arcs, pointing to the two different vertices connected by the edge. For $e \in \mathbb{E}$, we write $e = uv$ to mean that e is an arc connecting u to v, even if it might not be the only one. Also, we let t_e denote the tail and h_e the head of e. In addition, $\overline{e}$ denotes the same edge with the reverse orientation.

Recall that given a commutative ring A, one associates to the graph G the complex of A-modules

$$d_A\colon C^0(G, A) \to C^1(G, A).$$

Here, $C^0(G, A)$ is the A-module of functions $V \to A$, and $C^1(G, A)$ is the A-module of all functions $f\colon \mathbb{E} \to A$ satisfying $f(\overline{e}) = -f(e)$ for each $e \in \mathbb{E}$. And $d_A(f)(e) = f(v) - f(u)$ for each $e = uv \in \mathbb{E}$.

The characteristic functions χ_v, for v in V, form a basis of the A-module $C^0(G, A)$, whereas the functions $\chi_e - \chi_{\overline{e}}$, for a collection of $e \in \mathbb{E}$ giving an orientation to the whole G, form a basis of $C^1(G, A)$. There are bilinear forms $\langle\, ,\rangle$ on $C^0(G, A)$ and $C^1(G, A)$ satisfying

$$\begin{aligned} &\langle \chi_v, \chi_w \rangle = \delta_{v,w} \quad \text{for } v, w \in V; \\ &\langle \chi_e - \chi_{\overline{e}}, \chi_f - \chi_{\overline{f}} \rangle = \delta_{e,f} - \delta_{\overline{e},f} \quad \text{for } e, f \in \mathbb{E}. \end{aligned}$$

Define the homomorphism $d_A^*\colon C^1(G, A) \to C^0(G, A)$ by putting $d_A^*(\chi_e - \chi_{\overline{e}}) := \chi_v - \chi_u$ for each $e = uv \in \mathbb{E}$. Then d_A^* is the adjoint to d_A, that is, $\langle f, d_A^*(h)\rangle = \langle d_A(f), h\rangle$ for each $f \in C^0(G, A)$ and $h \in C^1(G, A)$. In addition, the degree map $\deg : C^0(G, A) \to A$, sending f to $\sum_{v\in V} f(v)$, is a cokernel for d_A^*. The kernel is $H^1(G, A)$.

Let $H_{0,A} := \{f \in C^0(G, A) \,|\, \deg(f) = 0\}$ and $F_A := \mathrm{Im}(d_A)$. Let $\Delta_A := d_A^* d_A$, the Laplacian of G. The homomorphism d_A^* induces an injection $F_A \to H_{0,A}$. For $A = \mathbb{R}$, it is a bijection, and the bilinear form $\langle\, ,\rangle$ on $C^1(G, \mathbb{R})$ induces by restriction a norm on $F_{\mathbb{R}}$ corresponding via $d_{\mathbb{R}}^*$ to the quadratic form q on $H_{0,\mathbb{R}}$ satisfying $q(f) = \langle f, \Delta_{\mathbb{R}}(f)\rangle$ for each $f \in C^0(G, \mathbb{R})$.

In [AE20a], we described a certain family of tilings of $H_{0,\mathbb{R}}$ by polytopes. Each tiling consists of a family of polytopes covering $H_{0,\mathbb{R}}$ such that

- each face of a polytope which is in the tiling belongs itself to the tiling; and
- the intersection of a finite number of polytopes in the tiling is a face of each of the polytopes.

By removing from a polytope in the tiling all the faces of positive codimension it contains, we get the corresponding open face. The open faces form then a stratification of the whole space $H_{0,\mathbb{R}}$. We call tiles the polytopes of maximum dimension.

For instance, let $\Lambda_A := \mathrm{Im}(d_A^*)$. Then $\Lambda_{\mathbb{R}} = H_{0,\mathbb{R}}$ and $\Lambda_{\mathbb{Z}}$ is a sublattice of $H_{0,\mathbb{Z}}$ of finite index equal to the number of spanning trees of G, by the Kirchhoff Matrix Tree Theorem. The *standard Voronoi tiling* of G is the Voronoi decomposition Vor_G of $H_{0,\mathbb{R}}$ with respect to $\Lambda_{\mathbb{Z}}$ and q. The tiles are

$$\mathrm{Vor}_G(\beta) := \big\{\eta \in H_{0,\mathbb{R}} \,|\, q(\eta-\beta) \leq q(\eta-\alpha) \text{ for every } \alpha \in \Lambda_{\mathbb{Z}} - \{\beta\}\big\}$$

for $\beta \in \Lambda_{\mathbb{Z}}$.

This is one of the infinitely many tilings we consider. There are variants of it, that we call *twisted mixed Voronoi tilings* and denote $\mathrm{Vor}^{\mathfrak{m}}_{G,\ell}$. Though the standard Voronoi tiling is homogeneous, meaning all tiles are translates of the tile centered at the origin, $\mathrm{Vor}_G(O)$, a twisted mixed Voronoi tiling is obtained by putting together translations of the tiles $\mathrm{Vor}_H(O)$ associated to connected spanning subgraphs H of G. More precisely, the twisted Voronoi tiling $\mathrm{Vor}^{\mathfrak{m}}_{G,\ell}$ depends on $\mathfrak{m} \in C^1(G,\mathbb{Z})$ (the "twisting") and an edge length function $\ell\colon E \to \mathbb{N}$; its tiles are the polytopes $d^*_{\mathbb{R}}(\mathfrak{d}^{\mathfrak{m}}_f) + \mathrm{Vor}_{G^{\mathfrak{m}}_f}(O)$ for $f \in C^0(G,\mathbb{Z})$ with $G^{\mathfrak{m}}_f$ connected, where $\mathfrak{d}^{\mathfrak{m}}_f \in C^1(G,\mathbb{R})$ is a modification of $d_{\mathbb{Z}}(f)$, namely

$$\mathfrak{d}^{\mathfrak{m}}_f(e) := \frac{\delta^{\mathfrak{m},\ell}_e(f) - \delta^{\mathfrak{m},\ell}_{\bar{e}}(f)}{2}, \text{ where } \delta^{\mathfrak{m},\ell}_e(f) := \Big\lfloor \frac{f(v)-f(u)+\mathfrak{m}(e)}{\ell(e)} \Big\rfloor \text{ for each } e = uv \in \mathbb{E},$$

and $G^{\mathfrak{m}}_f$ is the spanning subgraph of G obtaining by removing the edges $e \in \mathbb{E}$ for which $\mathfrak{d}^{\mathfrak{m}}_f(e) \notin \mathbb{Z}$. We refer to [AE20a] for a thorough presentation of these tilings.

In the present article we establish a correspondence between the stratifications of $H_{0,\mathbb{R}}$ associated to the $\mathrm{Vor}^{\mathfrak{m}}_{G,\ell}$ and the stable limits of line bundles in one-parameter smoothings of X. Properties of each stratum, and the way they fit together in the stratification of $H_{0,\mathbb{R}}$ are reflected in properties of the limits and the relationship between them.

More explicitly, let $\pi\colon \mathcal{X} \to B$ be a (one-parameter) smoothing of X. Here, B is the spectrum of $\kappa[[t]]$ and π is a projective flat morphism whose generic fiber is smooth and special fiber is isomorphic to X. We fix such an isomorphism. Let η and o be the generic and special points of B. The total space $\mathcal{X}$ is regular except possibly at the nodes of X. For $e \in E$, the completion of the local ring of $\mathcal{X}$ at the corresponding node N_e is $\kappa[[t]]$-isomorphic to $\kappa[[u,v,t]]/(uv-t^{\ell_e})$ for a certain $\ell_e > 0$, called the *singularity degree* of π at N_e. If $\ell_e = 1$, then $\mathcal{X}$ is regular at N_e. If all $\ell_e = 1$, then π is said to be *Cartier*; it is *regular* if $\mathcal{X}$ is regular at *all* the nodes of X. A finite base change is obtained by sending

t to t^n for some n. The resulting family is similar to the original one: the special fiber is the same, the generic fiber is a base field extension of the original one, but the singularity degrees ℓ_e change to $n\ell_e$.

Let L_η be an invertible sheaf on the generic fiber. If π is Cartier, it extends to an *almost invertible* sheaf $\mathcal{L}$ on $\mathcal{X}$, a sheaf that is invertible at all N_e. It is not unique, as $\mathcal{L}\otimes\mathcal{O}_{\mathcal{X}}(\sum f(v)X_v)$ is another extension for each $f \in C^0(G,\mathbb{Z})$. (Here X_v is the component of X corresponding to $v \in V$, which can and will be viewed as a Cartier divisor of $\mathcal{X}$ because π is Cartier.) For general π, the sheaf L_η extends to a relatively torsion-free, rank-one sheaf $\mathcal{I}$ on $\mathcal{X}/B$, that is, a B-flat coherent sheaf on $\mathcal{X}$ whose fibers over B are torsion-free, rank-one. Again, it is not the unique extension: in [Est01] a procedure similar to the one explicited above shows how to change $\mathcal{I}$ into other extensions. Furthermore, one could do a finite base change to π, extend L_η to the new generic fiber and consider its extensions. Of course, they will be extensions on a different total space. But the special fibers are the same, and thus the restrictions of all these extensions to X are torsion-free, rank-one sheaves that we call the *stable limits* of L_η.

Let $\mathbf{J}$ denote the Artin stack parameterizing torsion-free, rank-one sheaves on X. It is the disjoint union of the closed and open substacks $\mathbf{J}^d$, each parameterizing sheaves I with degree d, that is, with $\chi(I) - \chi(\mathcal{O}_X) = d$. Letting $d := \deg(L_\eta)$, we may consider the subset $\mathfrak{I}$ of $\mathbf{J}^d$ parameterizing all the stable limits of L_η. We proceed to describe $\mathfrak{I}$ thoroughly.

First, we give a meaningful structure to $\mathbf{J}$ as a quotient stack, as follows. Fixing $\mathbf{b} \in C^0(G,\mathbb{Z})$, we let

$$\mathbf{J}^{\mathbf{b}} := \prod_{v\in V} \mathbf{J}_v^{\mathbf{b}(v)},$$

where $\mathbf{J}_v^{\mathbf{b}(v)}$ parameterizes torsion-free, rank-one sheaves of degree $\mathbf{b}(v)$ on the component X_v for each $v \in V$. Over $\mathbf{J}^{\mathbf{b}}$ we construct a scheme $\mathbf{R}_{\mathbf{J}^{\mathbf{b}}}$ parameterizing gluings along the nodes N_e associated to $e \in E$ of the sheaves given by points of $\mathbf{J}^{\mathbf{b}}$, their modifications and their degenerations.

More precisely, given torsion-free, rank-one sheaves K_v on X_v for each $v \in V$, we view their gluings along the nodes as subsheaves of the direct sum $\oplus_v K_v$ whose quotients are supported with length 1 on each and every node N_e. This allows for "degenerate" gluings, and thus for torsion-free sheaves that may fail to be invertible at part or all of the N_e. The parameter space for these subsheaves is a product of $\mathbf{P}^1$, one for each $e \in E$, and describes an irreducible component of the fiber of $\mathbf{R}_{\mathbf{J}^{\mathbf{b}}}$ over the point on $\mathbf{J}^{\mathbf{b}}$ parameterizing the K_v.

The other components are obtained by modifying the K_v as follows. Fix $\mathbf{c} \in C^1(G,\mathbb{Z})$. For each $c \in C^1(G,\mathbb{Z})$, put

$$K_v^{\mathbf{c}-c} := K_v \otimes \mathcal{O}_{X_v}\Big(\sum_{\substack{e\in\mathbb{E}\\ \mathrm{h}_e=v}} (\mathbf{c}(e) - c(e))N_e\Big)$$

and do the same gluing as above; we obtain another irreducible component (if $c \neq \mathbf{c}$) of the fiber of $\mathbf{R}_{\mathbf{J^b}}$ over the same point on $\mathbf{J^b}$ parameterizing the K_v, and all components are obtained this way.

The fibers of $\mathbf{R}_{\mathbf{J^b}}$ over $\mathbf{J^b}$ are thus arrangements of an infinite number of simple toric varieties. Each fiber is the same: what we call the arrangement of toric varieties or simply toric tiling $\mathbf{R}$ associated to the Voronoi decomposition of $C^1(G,\mathbb{R})$ in hypercubes with respect to $C^1(G,\mathbb{Z})$. More precisely, each Voronoi tile is a rational polytope, to each rational polytope we may associate its normal fan, and to the normal fan the corresponding toric variety. The polytopes in the Voronoi decomposition form a complex, in the sense that each two of them intersect in a common face, when they intersect. We may thus glue the toric varieties associated to each two polytopes by identifying the orbit closures of the common face in each variety with the toric variety associated to the face itself, viewed as a polytope of smaller dimension. This gives us $\mathbf{R}$.

There is nothing special about the above Voronoi decomposition with regard to the above construction. It can be carried out for any tiling of an Euclidean space by rational polytopes intersecting each other in faces. As we have already mentioned, we consider other tilings in the present article.

The scheme $\mathbf{R}_{\mathbf{J^b}}$ parameterizes torsion-free, rank-one sheaves with a lot of redundancy. Two groups act on it independently. First, there is a natural action of $\mathbf{G}_{\mathbf{m}}^V/\mathbf{G}_{\mathbf{m}}$, where we view $\mathbf{G}_{\mathbf{m}}$, the multiplicative group of κ, embedded diagonally in $\mathbf{G}_{\mathbf{m}}^V$. The action is given by observing that $\mathbf{G}_{\mathbf{m}}^V$ is the automorphism group of $\oplus_v K_v^{\mathbf{c}-c}$ for each point on $\mathbf{J^b}$ parameterizing the K_v and each $c \in C^1(G,\mathbb{Z})$, and that a subsheaf is fixed under the diagonal action of $\mathbf{G}_{\mathbf{m}}$. The action preserves each fiber of $\mathbf{R}_{\mathbf{J^b}}$ over $\mathbf{J^b}$ and each component of that fiber.

The second group action moves fibers around. Recall $H^1(G,\mathbb{Z})$, the group of cycles of G, the kernel of $d^*_{\mathbb{Z}}$. For each $\gamma \in C^1(G,\mathbb{Z})$ and torsion-free rank-one sheaves K_v on X_v for each $v \in V$, put:

$$K_v^{-\gamma} := K_v \otimes \mathcal{O}_{X_v}\Big(-\sum_{\substack{e\in\mathbb{E}\\ \mathrm{h}_e=v}} \gamma(e) N_e\Big).$$

If $\gamma \in H^1(G,\mathbb{Z})$ and the K_v are parameterized by a point $s \in \mathbf{J^b}$, so are the $K_v^{-\gamma}$, parameterized by a point we denote by $\tau^\gamma(s)$. Then τ^γ is an automorphism of $\mathbf{J^b}$. It lifts in a rather trivial way to an automorphism of $\mathbf{R}_{\mathbf{J^b}}$: Since $\mathbf{c}-(c+\gamma) = -\gamma + (\mathbf{c}-c)$, we may associate to a point on the component corresponding to $c+\gamma$ of the fiber of $\mathbf{R}_{\mathbf{J^b}}$ over s the point on the component corresponding to c of the fiber of $\mathbf{R}_{\mathbf{J^b}}$ over $\tau^\gamma(s)$ parameterizing the same subsheaf of the same direct sum, for each $c \in C^1(G,\mathbb{Z})$. The action of $H^1(G,\mathbb{Z})$ may move fibers and is free, as even if a fiber is fixed, its components are not.

The two actions are independent and $\mathbf{J}^d$ can be obtained as the quotient stack:

$$\mathbf{J}^d = \left[\frac{\mathbf{R}_{\mathbf{J^b}}}{H^1(G,\mathbb{Z}) \times \mathbf{G}_{\mathbf{m}}^V/\mathbf{G}_{\mathbf{m}}}\right]. \tag{1.1}$$

We may thus describe the collection $\mathfrak{I}$ of stable limits of a sheaf L_η by describing its inverse image in $\mathbf{R}_{\mathbf{J^b}}$. Our Theorem 4.7 claims that this inverse image is the disjoint union of certain connected subschemes of certain fibers of $\mathbf{R}_{\mathbf{J^b}}$ over $\mathbf{J^b}$, each subscheme isomorphic to its image in the quotient

$$\mathbf{S}^d := \left[\frac{\mathbf{R}_{\mathbf{J^b}}}{H^1(G,\mathbb{Z})}\right],$$

all the images being the same. Under chosen identifications we may view each subscheme as a subscheme of the arrangement of toric varieties $\mathbf{R}$. We prove this subscheme is $Y^{a,b}_{\ell,\mathfrak{m}}$ for certain choices of ℓ, $\mathfrak{m}$, a and b. Each $Y^{a,b}_{\ell,\mathfrak{m}}$ is itself an arrangement of toric varieties of dimension $|V|-1$ which was thoroughly described in [AE20b].

Here we explain the parameters defining $Y^{a,b}_{\ell,\mathfrak{m}} \subseteq \mathbf{R}$: characters $a\colon C^1(G,\mathbb{Z}) \to \mathbf{G}_{\mathbf{m}}(\kappa)$ and $b\colon H^1(G,\mathbb{Z}) \to \mathbf{G}_{\mathbf{m}}(\kappa)$, an edge length function $\ell\colon E \to \mathbb{N}$ and an element $\mathfrak{m} \in C^1(G,\mathbb{Z})$ we call a twisting. These data arise from the smoothing $\pi\colon \mathcal{X} \to B$ and from L_η, as follows: The length function is simply the collection of singularity degrees ℓ_e; and the character a keeps track of the infinitesimal data of the arc defined by π on the moduli $\overline{\mathscr{M}}_g$ or on a versal deformation space of X. The character b describes the gluing data of an almost invertible stable limit, if L_η admits one, and then we may set $\mathfrak{m}=0$, no twisting is necessary. If not, then L_η admits an almost invertible limit on X^ℓ, the semistable model of X obtained by splitting the branches of each N_e apart, and connecting them by a chain of $\ell_e - 1$ smooth rational curves for each $e \in E$. It even admits an admissible almost invertible limit, meaning an almost invertible limit whose restriction to each component of each chain has degree zero, but possibly one, where the degree is one. Then b is related to the gluing data of an admissible almost invertible limit and the twisting $\mathfrak{m}$ keeps track of where that limit has degree one on each added chain.

Different choices of a, b and $\mathfrak{m}$ may yield the same subscheme $Y^{a,b}_{\ell,\mathfrak{m}} \subseteq \mathbf{R}$. We have left this analysis for a later work. But the structure of $Y^{a,b}_{\ell,\mathfrak{m}}$ depends on ℓ and $\mathfrak{m}$ only, its equations being a deformation of the equations defining $Y^{1,1}_{\ell,\mathfrak{m}}$, which we denote by $Y^{\mathrm{bt}}_{\ell,\mathfrak{m}}$ and call the basic toric tiling, as pointed out in [AE20b], Prop. 4.6.

In [AE20b], Section 4, we explained how $Y^{\mathrm{bt}}_{\ell,\mathfrak{m}}$ is determined from the tiling $\mathrm{Vor}^{\mathfrak{m}}_{G,\ell}$ of $H_{0,\mathbb{R}}$. There we remark we can naturally view $Y^{\mathrm{bt}}_{\ell,\mathfrak{m}}$ and its deformations $Y^{a,b}_{\ell,\mathfrak{m}}$ as closed subschemes of $\mathbf{R}$.

Finally, we may consider the subscheme $Y^{a,b}_{\ell,\mathfrak{m}} \subseteq \mathbf{R}$ for arbitrary choices of a, b, ℓ and $\mathfrak{m}$, and under chosen identifications, as a subscheme of a fiber of $\mathbf{R}_{\mathbf{J^b}}$ over $\mathbf{J^b}$. If we denote by $\mathfrak{I}$ the image of this subscheme in the quotient $\mathbf{J}^d$, our Regeneration Theorem 5.1 claims that $\mathfrak{I}$ parameterizes the collection of stable limits of an invertible sheaf under a smoothing of X with singularity degrees ℓ_e.

Besides the above description, $Y^{a,b}_{\ell,\mathfrak{m}}$ is also described in [AE20b], Subsection 4.2, by giving the equations of each of its irreducible components in the corresponding component of $\mathbf{R}$, and in [AE20b], Thm. 5.3, by means of its orbits under the action of $\mathbf{G}^V_{\mathbf{m}}/\mathbf{G}_{\mathbf{m}}$. These characterizations are the ones we use in the present article.

But we have also described $Y^{a,b}_{\ell,\mathfrak{m}}$ in [AE20b], Thm. 6.3, globally by its (infinitely many) equations in $\mathbf{R}$, and in [AE20b], Thm. 7.2, as an equivariant degeneration of the torus $\mathbf{G}^V_{\mathbf{m}}/\mathbf{G}_{\mathbf{m}}$. The first description will be important in understanding the moduli of the $Y^{a,b}_{\ell,\mathfrak{m}}$, to be worked out later. And the second yields that the connected 0-dimensional substack $\mathfrak{I}$ of $\mathbf{J}^d$ parameterizing a collection of stable limits is the degeneration of a point!

1.3. **Future work.** To attain our goal of giving a new definition of limit linear series and constructing their moduli space, our series of articles will continue. We summarize now what comes ahead.

First of all, we have constructed $\mathbf{J}^d$ as the quotient expressed in (1.1), we have shown it parameterizes torsion-free, rank-one, degree-d sheaves on X, but we have not proved it is actually the moduli stack of these sheaves. That $\mathbf{J}^d$ is bound to be the moduli stack is an observation by Margarida Melo and Filippo Viviani, and the proof that it actually is will appear later.

Second, given a smoothing $\pi\colon \mathcal{X} \to B$ of X and an invertible sheaf L_η on its generic fiber, we may consider the punctured arc in the relative Artin stack $\mathbf{J}_{\mathcal{X}/B}$ parameterizing torsion-free rank-one sheaves on the fibers of π. It is natural to think that by adding the Artin substack $\mathfrak{I}$ of stable limits of L_η that the punctured arc will be completed to a B-flat closed substack $A \subseteq \mathbf{J}_{\mathcal{X}/B}$. To show this, we need to construct A, the degeneration of a point to $\mathfrak{I}$, whose existence we proved in [AE20b], Thm. 7.2, and moreover a relative torsion-free, rank-one sheaf on $\mathcal{X} \times_B A$ whose restriction to the generic fiber is L_η. This will also appear later.

Third, we will describe the moduli of all collections of stable limits, which is tantamount to describing the various ways the $Y^{a,b}_{\ell,\mathfrak{m}}$ appear as subschemes of fibers of $\mathbf{R}_{\mathbf{J}^\mathfrak{b}}$ over $\mathbf{J}^\mathbf{b}$. Already mentioned above, the global equations of $Y^{a,b}_{\ell,\mathfrak{m}}$ in $\mathbf{R}$, laid out by [AE20b], Thm. 6.3, and a thorough description of how $Y^{a,b}_{\ell,\mathfrak{m}}$ actually depends on a, b, ℓ and $\mathfrak{m}$ are the fundamental pieces in this construction. However, since the $Y^{a,b}_{\ell,\mathfrak{m}}$ have infinitely many components, the moduli will be formal. To get an actual moduli, we need to truncate $\mathbf{R}_{\mathbf{J}^\mathfrak{b}}$. The most obvious truncation is to restrict to the open subscheme parameterizing sheaves obtained as gluings of sheaves of nonnegative degrees on the components X_v, as for understanding limits of effective divisors these sheaves are enough.

Fourth, the moduli of collections of stable limits can be thought of as a new compactification of the Picard scheme of the curve X. To show it is natural we will construct a relative version of it over the moduli stack of stable curves $\overline{\mathscr{M}}_g$. The construction may require a local analysis and blowups of $\overline{\mathscr{M}_g}$, in the way carried out by Mainò in her construction of the moduli space of enriched curves [Mai98]. But further blowups will be necessary, infinitely many to get the formal moduli, but finitely many for the truncated moduli.

Finally, as mentioned before, our notion of limit linear series, and the construction of their moduli space will be similar to what is done for smooth curves: If $\mathfrak{I}$ is a closed substack of $\mathbf{J}^d$ parameterizing a collection of stable limits, a "limit linear series" of "sections" of this collection is a locally free subsheaf $\mathcal{V}$ of the pushforward $p_{2*}\mathcal{I}$ satisfying certain conditions, where $\mathcal{I}$ is the sheaf induced on $X \times \mathfrak{I}$ (or certain base extensions) by the universal sheaf over $\mathbf{J}^d$.

1.4. **Organization.** The layout of the paper is as follows. In Section 2 we construct the quotient stack $\mathbf{J}^d$ describing thoroughly its atlas and the group action. In Section 3 we consider smoothings of a nodal curve and describe limits of line bundles. Limits of the trivial bundle are considered in detail, and are explained in terms of the versal deformation space of the curve, expanding on work done in [Mai98] and [EM02]. In Section 4 we prove one of our main theorems, Theorem 4.7, describing collections of stable limits as quotients by $\mathbf{G}_\mathbf{m}^V/\mathbf{G}_\mathbf{m}$ of the $Y^{a,b}_{\ell,\mathfrak{m}}$. Finally, in Section 5 we show that any such a quotient is a collection of stable limits, our Theorem 5.1.

1.5. **Basic notations.** In addition to the notations already introduced, throughout the present article, we will denote by N_e the node of X associated to each $e \in E$ and by X_v the irreducible component of X associated to each $v \in V$.

Given a collection $A \subseteq E$ of edges, an orientation of A is simply a section $\mathfrak{o}\colon A \to \mathbb{E}$ over A of the forgetful map $\mathbb{E} \to E$. We denote by $A^{\mathfrak{o}}$ the image of the orientation. To simplify the presentation, we fix an orientation $\mathfrak{o}\colon E \to \mathbb{E}$ for G. Given $e \in \mathbb{E}$, we will write e as well for the (non-oriented) edge in E. Given $e \in E$, we denote $e^{\mathfrak{o}} := \mathfrak{o}(e)$.

We will drop the subscripts from d_A and d_A^* when appropriate. Given α in $C^0(G,\mathbb{Z})$ (resp. $C^1(G,\mathbb{Z})$), we write $\alpha_x := \alpha(x)$ for each vertex (resp. oriented edge) x of G. Similarly, given a character a of $C^0(G,\mathbb{Z})$ (resp. $C^1(G,\mathbb{Z})$), we write $a_v := a(\chi_v)$ for each $v \in V$ (resp. $a_e := a(\chi_e - \chi_{\overline{e}})$ for each $e \in \mathbb{E}$). The multiplicative group is denoted $\mathbf{G}_\mathbf{m}$.

2. The space of embedded sheaves

2.1. **Gluing sheaves.** For each $v \in V$ and each $d \in \mathbb{Z}$, let $\mathbf{J}_v^d$ denote the degree-d compactified Jacobian of X_v; it is a projective variety of dimension g_v, where g_v is the (arithmetic) genus of X_v, parameterizing torsion-free, rank-one sheaves on X_v of degree d. We refer to [Est01] for basic definitions and results concerning torsion-free rank-one sheaves.

Fix an integer d and let $\mathbf{b} \in C^0(G,\mathbb{Z})$ be of degree d, meaning $\sum_{v\in V} \mathbf{b}_v = d$. Define

$$\mathbf{J}^{\mathbf{b}} := \prod_{v\in V} \mathbf{J}_v^{\mathbf{b}_v}.$$

Note that $\mathbf{J}^{\mathbf{b}}$ is a projective variety of dimension $\sum g_v$, and comes with natural projection maps to $\mathbf{J}_v^{\mathbf{b}_v}$ for each $v \in V$.

For each vertex $v \in V$, let $\iota_v\colon X_v \hookrightarrow X$ denote the inclusion map, and $\mathcal{L}_v$ the pullback to $X_v \times \mathbf{J}^{\mathbf{b}}$ of a "Poincaré sheaf," or universal sheaf on $X_v \times \mathbf{J}_v^{\mathbf{b}_v}$. Observe that $\mathcal{L}_v$ is actually not well-defined, in the sense that a Poincaré sheaf is not unique. However, if we pick a point P_v on the smooth locus of X_v, then $\mathcal{L}_v$ may be defined by imposing the condition that it be rigid at P_v, that is, $\mathcal{L}_v|_{P_v\times\mathbf{J}^{\mathbf{b}}} \cong \mathcal{O}_{P_v\times\mathbf{J}^{\mathbf{b}}}$. At any rate, this is not relevant to us, so we just choose one Poincaré sheaf $\mathcal{L}_v$ for each vertex $v \in V$. For a point s on $\mathbf{J}^{\mathbf{b}}$, we denote by $\mathcal{L}_v(s)$ the restriction of $\mathcal{L}_v$ to $X_v \cong X_v \times s$.

For each oriented edge $e = uv \in \mathbb{E}$, let

$$\mathcal{F}_e := \mathcal{L}_u|_{N_e\times\mathbf{J}^{\mathbf{b}}} \oplus \mathcal{L}_v|_{N_e\times\mathbf{J}^{\mathbf{b}}}.$$

It is a rank-two vector bundle over $\mathbf{J}^{\mathbf{b}}$, under the natural isomorphism $N_e \times \mathbf{J}^{\mathbf{b}} \to \mathbf{J}^{\mathbf{b}}$. Let $\mathbf{P}_{\mathbf{J}^{\mathbf{b}}}(\mathcal{F}_e)$ be the corresponding $\mathbf{P}^1$-bundle over $\mathbf{J}^{\mathbf{b}}$, and define

$$\mathbf{P}_{\mathbf{J}^{\mathbf{b}}} := \prod_{e\in E^{\mathfrak{o}}} \mathbf{P}_{\mathbf{J}^{\mathbf{b}}}(\mathcal{F}_e),$$

the product fibered over $\mathbf{J}^{\mathbf{b}}$. It is a $\mathbf{P}_0$-bundle over $\mathbf{J}^{\mathbf{b}}$, where

$$\mathbf{P}_0 := \prod_{e\in E^{\mathfrak{o}}} \mathbf{P}^1.$$

Fix now an element $\mathbf{c} \in C^1(G,\mathbb{Z})$. For each $v \in V$, let $\widetilde{\mathcal{L}}_v$ denote the pullback of $(\iota_v \times 1_{\mathbf{J}^{\mathbf{b}}})_*\mathcal{L}_v$ to $X \times \mathbf{P}_{\mathbf{J}^{\mathbf{b}}}$. We will also denote by $\widetilde{\mathcal{F}}_e$ the pullback of $\mathcal{F}_e$ to $N_e \times \mathbf{P}_{\mathbf{J}^{\mathbf{b}}}$ and by

$$q_e^{\mathbf{c}}\colon \widetilde{\mathcal{F}}_e \longrightarrow \mathcal{M}_e$$

the pullback of the universal quotient over $\mathbf{P}_{\mathbf{J}^{\mathbf{b}}}(\mathcal{F}_e)$ to $N_e \times \mathbf{P}_{\mathbf{J}^{\mathbf{b}}}$ for each $e \in E^{\mathfrak{o}}$. Notice that

$$\widetilde{\mathcal{F}}_e = \widetilde{\mathcal{L}}_{\mathrm{t}_e}|_{N_e\times\mathbf{P}_{\mathbf{J}^{\mathbf{b}}}} \oplus \widetilde{\mathcal{L}}_{\mathrm{h}_e}|_{N_e\times\mathbf{P}_{\mathbf{J}^{\mathbf{b}}}} \quad \text{for each } e \in E^{\mathfrak{o}}.$$

We use the quotients $q_e^{\mathbf{c}}$ to construct a natural relative torsion-free, rank-one, degree-d sheaf $\mathcal{I}^{\mathbf{c}}$ on $X \times \mathbf{P}_{\mathbf{J}^{\mathbf{b}}}/\mathbf{P}_{\mathbf{J}^{\mathbf{b}}}$, the kernel of the composition of surjections:

$$\bigoplus_{v\in V} \widetilde{\mathcal{L}}_v \longrightarrow \bigoplus_{v\in V} \bigoplus_{\substack{e\in E^{\mathfrak{o}}\\ e\ni v}} \widetilde{\mathcal{L}}_v|_{N_e\times\mathbf{P}_{\mathbf{J}^{\mathbf{b}}}} = \bigoplus_{e\in E^{\mathfrak{o}}} \Big(\widetilde{\mathcal{L}}_{\mathrm{t}_e}|_{N_e\times\mathbf{P}_{\mathbf{J}^{\mathbf{b}}}} \oplus \widetilde{\mathcal{L}}_{\mathrm{h}_e}|_{N_e\times\mathbf{P}_{\mathbf{J}^{\mathbf{b}}}}\Big) \longrightarrow \bigoplus_{e\in E^{\mathfrak{o}}} \mathcal{M}_e.$$

For a point t on $\mathbf{P}_{\mathbf{J}^{\mathbf{b}}}$, denote by $\mathcal{I}^{\mathbf{c}}(t)$ the induced torsion-free rank-one sheaf on $X \simeq X \times t$.

For each $c \in C^1(G,\mathbb{Z})$, we may modify the above construction as follows: Let

$$\mathcal{L}_v^c := \mathcal{L}_v\Big(\sum_{\substack{e\in\mathbb{E}\\ \mathrm{h}_e=v}} (\mathbf{c}_e - c_e)N_e \times \mathbf{J}^{\mathbf{b}}\Big),$$

and do the same construction as above but replacing the sheaves $\mathcal{L}_v$ with the sheaves $\mathcal{L}_v^c$. More precisely, for each $e = uv \in \mathbb{E}$, put

$$\mathcal{F}_e^{c_e} := \mathcal{L}_u^c|_{N_e\times\mathbf{J}^{\mathbf{b}}} \oplus \mathcal{L}_v^c|_{N_e\times\mathbf{J}^{\mathbf{b}}},$$

and set

$$\mathbf{P}^c_{\mathbf{J^b}} := \prod_{e\in E^{\mathfrak{o}}} \mathbf{P}_{\mathbf{J^b}}(\mathcal{F}_e^{c_e}),$$

the fibered product over $\mathbf{J^b}$. Instead of $\mathcal{I}^{\mathbf{c}}$, let $\mathcal{I}^c$ be the relative torsion-free, rank-one, degree-d sheaf on $X\times\mathbf{P}^c_{\mathbf{J^b}}/\mathbf{P}^c_{\mathbf{J^b}}$ obtained from the pullbacks q_e^c to $N_e\times\mathbf{P}^c_{\mathbf{J^b}}$ of the universal quotients on $\mathbf{P}_{\mathbf{J^b}}(\mathcal{F}_e^{c_e})$, instead of the $q_e^{\mathbf{c}}$. As above, for a point t on $\mathbf{P}^c_{\mathbf{J^b}}$, we denote by $\mathcal{I}^c(t)$ the induced torsion-free rank-one sheaf on X. We will also say that a torsion-free, rank-one sheaf I is *represented* by $t\in\mathbf{P}^c_{\mathbf{J^b}}$ if $\mathcal{I}^c(t)\cong I$.

Observe that for $t\in\mathbf{P}^c_{\mathbf{J^b}}$ lying over $s\in\mathbf{J^b}$, and for $e=uv\in E^{\mathfrak{o}}$, if $\mathcal{I}^c(t)$ is invertible at N_e, then it generates both $\mathcal{L}^c_u(s)$ and $\mathcal{L}^c_v(s)$ in a neighborhood of N_e. On the other hand, if $\mathcal{I}^c(t)$ fails to be invertible at N_e, then either $q_e^c(t)(\mathcal{L}^c_u(s)|_{N_e})=0$ or $q_e^c(t)(\mathcal{L}^c_v(s)|_{N_e})=0$. In the first case, in a neighborhood of N_e, the subsheaf of $\mathcal{L}^c_v(s)$ generated by $\mathcal{I}^c(t)$ is $\mathcal{L}^c_v(s)(-N_e)$, whereas that of $\mathcal{L}^c_u(s)$ is $\mathcal{L}^c_u(s)$ itself. In the second case, the reverse is true, that is, the same statement holds with u and v exchanged. Letting E_t denote the set of edges $e\in E$ for which $\mathcal{I}^c(t)$ fails to be invertible at N_e, we obtain an orientation $\mathfrak{o}^c_t\colon E_t\to\mathbb{E}$ by assigning to each $e\in E_t$ the oriented edge whose head v is such that $\mathcal{I}^c(t)$ generates $\mathcal{L}^c_v(s)(-N_e)$ in a neighborhood of N_e. Thus $\mathcal{I}^c(t)$ generates the subsheaf

$$\mathcal{L}^c_v(s)\Big(-\sum_{\substack{e\in E^c_t\\ \mathrm{h}_e=v}} N_e\Big)$$

for each $v\in V$, where, for simplicity, we put $E^c_t := E_t^{\mathfrak{o}^c_t}$.

2.2. **The atlas.** We shall view the sheaves $\mathcal{I}^c$ as restrictions of a sheaf defined over a larger base, containing all the schemes $\mathbf{P}^c_{\mathbf{J^b}}$ for all $c\in C^1(G,\mathbb{Z})$ as closed subschemes. This works as follows. For each vertex $v\in V$, and each natural number $n\in\mathbb{N}$, define the sheaf $\mathcal{L}_v^{(n)}$ as

$$\mathcal{L}_v^{(n)} := \mathcal{L}_v\Big(n\sum_{\substack{e\in E^{\mathfrak{o}}\\ e\ni v}} N_e\times\mathbf{J^b}\Big).$$

By means of the natural embeddings $\mathcal{O}_{X_v}\hookrightarrow\mathcal{O}_{X_v}(N_e)$, we may view all the sheaves $\mathcal{L}^c_v$, for bounded $c\in C^1(G,\mathbb{Z})$, more precisely for those c verifying $|c_e-\mathbf{c}_e|\leq n$ for every $e\in\mathbb{E}$, as subsheaves of the sheaf $\mathcal{L}_v^{(n)}$. We may thus view the schemes $\mathbf{P}^c_{\mathbf{J^b}}$ as closed subschemes of the relative Quot-scheme

$$\mathrm{Quot}_{X\times\mathbf{J^b}/\mathbf{J^b}}\Big(\bigoplus_{v\in V}(\iota_v\times 1_{\mathbf{J^b}})_*\mathcal{L}_v^{(n)}\Big),$$

more precisely, of the piece of the relative Quot-scheme parameterizing subsheaves of rank one and degree d or, equivalently, quotients of finite length equal to $(2n+1)|E|$.

Given two orientations $\mathfrak{o}_1$ and $\mathfrak{o}_2$ of a subset $A \subseteq E$, define $(\mathfrak{o}_2, \mathfrak{o}_1) \in C^1(G, \mathbb{Z})$ by

$$(\mathfrak{o}_2, \mathfrak{o}_1)_e := \begin{cases} 1 & \text{if } e \in A^{\mathfrak{o}_2} - A^{\mathfrak{o}_1}, \\ -1 & \text{if } e \in A^{\mathfrak{o}_1} - A^{\mathfrak{o}_2}, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.1. *Notations as above, let $c, c' \in C^1(G, \mathbb{Z})$ with $|c_e - \mathbf{c}_e| \le n$ and $|c'_e - \mathbf{c}_e| \le n$ for all $e \in \mathbb{E}$. Viewing $\mathbf{P}^c_{\mathbf{J^b}}$ and $\mathbf{P}^{c'}_{\mathbf{J^b}}$ in the Quot-scheme, the following statements are true:*

1. *$\mathbf{P}^c_{\mathbf{J^b}}$ intersects $\mathbf{P}^{c'}_{\mathbf{J^b}}$ if and only if $|c'_e - c_e| \le 1$ for every $e \in \mathbb{E}$.*
2. *More precisely, given $t \in \mathbf{P}^c_{\mathbf{J^b}}$, we have that $t \in \mathbf{P}^{c'}_{\mathbf{J^b}}$ if and only if $c - c' = (\mathfrak{o}', \mathfrak{o}^c_t)$ where $\mathfrak{o}' \colon E_t \to \mathbb{E}$ is an orientation of the set E_t of edges e for which $\mathcal{I}^c(t)$ fails to be invertible at N_e. In this case, $\mathfrak{o}' = \mathfrak{o}^{c'}_t$.*

Proof. The first statement follows from the second. Indeed, if $t \in \mathbf{P}^c_{\mathbf{J^b}} \cap \mathbf{P}^{c'}_{\mathbf{J^b}}$, then Statement (2) yields that $|c'_e - c_e| = |(\mathfrak{o}', \mathfrak{o}^c_t)_e| \le 1$ for each $e \in \mathbb{E}$. Conversely, if $|c'_e - c_e| \le 1$ for every $e \in \mathbb{E}$, we may choose a point $t \in \mathbf{P}^c_{\mathbf{J^b}}$ such that E_t is the subset of edges $e \in E$ in the support of $c - c'$ and E^c_t is the collection of oriented edges e such that $c'_e - c_e = 1$. If $\mathfrak{o}'$ is the "opposite orientation," that is, $E^{\mathfrak{o}'}_t$ is the collection of oriented edges e such that $c_e - c'_e = 1$, then $c - c' = (\mathfrak{o}', \mathfrak{o}_t)$. Thus, $t \in \mathbf{P}^{c'}_{\mathbf{J^b}}$ by Statement (2).

As for Statement (2), observe first that $\mathcal{I}^c(t)$ is a subsheaf of $\bigoplus \iota_{v*}\mathcal{L}^{(n)}_v(s)$ such that

$$\mathcal{I}^c(t) \subseteq \bigoplus_{v \in V} \iota_{v*}\mathcal{L}^c_v(s)\Big(-\sum_{\substack{e \in E^c_t \\ \mathrm{h}_e = v}} N_e\Big),$$

where $s \in \mathbf{J^b}$ lies under t. Thus, for each other orientation $\mathfrak{o}'$ of E_t, we have

$$\mathcal{I}^c(t) \subseteq \bigoplus_{v \in V} \iota_{v*}\mathcal{L}^c_v(s)\Big(\sum_{\substack{e \in E^{\mathfrak{o}'}_t \\ \mathrm{h}_e = v}} N_e - \sum_{\substack{e \in E^c_t \\ \mathrm{h}_e = v}} N_e\Big) = \bigoplus_{v \in V} \iota_{v*}\mathcal{L}^{c'}_v(s)$$

if $c' = c - (\mathfrak{o}', \mathfrak{o}^c_t)$, and thus $\mathcal{I}^c(t) = \mathcal{I}^{c'}(t')$ as subsheaves of $\bigoplus \iota_{v*}\mathcal{L}^{(n)}_v(s)$ for a certain $t' \in \mathbf{P}^{c'}_{\mathbf{J^b}}$ such that $E_{t'} = E_t$ and $\mathfrak{o}^{c'}_{t'} = \mathfrak{o}'$. So $t = t'$ in the Quot-scheme, and thus $t \in \mathbf{P}^{c'}_{\mathbf{J^b}}$.

Conversely, if $t = t' \in \mathbf{P}^{c'}_{\mathbf{J^b}}$ as well, then

$$\mathcal{I}^c(t) \subseteq \bigoplus_{v \in V} \iota_{v*}\mathcal{L}^c_v(s)\Big(-\sum_{\substack{e \in A \\ \mathrm{h}_e = v}} (c'_e - c_e)N_e\Big) \tag{2.1}$$

where $A := \{e \in \mathbb{E} \,|\, c'_e > c_e\}$. Since $\mathcal{I}^c(t)$ generates $\mathcal{L}^c_u(s)$ and $\mathcal{L}^c_v(s)(-N_e)$ in a neighborhood of N_e for each $e = uv \in E^c_t$, it follows that $|c'_e - c_e| \le 1$ for every $e \in \mathbb{E}$. Also, $c'_e - c_e = 1$ only if $e \in E^c_t$, that is $A \subseteq E^c_t$.

Similarly,

$$\mathcal{I}^{c'}(t') \subseteq \bigoplus_{v\in V} \iota_{v*}\mathcal{L}_v^{c'}(s)\Big(-\sum_{\substack{e\in A'\\ \mathrm{h}_e=v}}(c_e-c'_e)N_e\Big) \tag{2.2}$$

where $A' := \{e \in \mathbb{E} \,|\, c_e > c'_e\}$. Then $A' \subseteq E_{t'}^{c'}$.

Note that $E_t = E_{t'}$, since $\mathcal{I}^c(t) = \mathcal{I}^{c'}(t')$. Thus $\mathfrak{o}_{t'}^{c'}$ is another orientation for E_t. Also, if $e \in A$ then $\overline{e} \in A'$, and hence $e \in E_t^c$ and $\overline{e} \in E_t^{c'}$. Thus $A \subseteq E_t^c - E_{t'}^{c'}$.

On the other hand, let $e \in \mathbb{E}$ such that $e \in E_t^c - E_{t'}^{c'}$. As subsheaves of $\bigoplus \iota_{v*}\mathcal{L}_v^{(n)}(s)$, the two sheaves in (2.1) are equal to the corresponding ones in (2.2). Suppose by contradiction that $e \notin A$. Then $c'_e \leq c_e$. Since $e \in E_t^c$, we have that $\mathcal{I}^c(t)$ is contained in

$$\bigoplus_{v\in V-\{\mathrm{h}_e\}} \iota_{v*}\mathcal{L}_v^c(s)\Big(-\sum_{\substack{f\in A\\ \mathrm{h}_f=v}}(c'_f-c_f)N_f\Big) \bigoplus \iota_{\mathrm{h}_e*}\mathcal{L}_{\mathrm{h}_e}^c(s)\Big(-\sum_{\substack{f\in A\\ \mathrm{h}_f=\mathrm{h}_e}}(c'_f-c_f)N_f - N_e\Big).$$

But, because of the equality of (2.1) and (2.2), also $\mathcal{I}^{c'}(t')$ is contained in

$$\bigoplus_{v\in V-\{\mathrm{h}_e\}} \iota_{v*}\mathcal{L}_v^{c'}(s)\Big(-\sum_{\substack{f\in A'\\ \mathrm{h}_f=v}}(c_f-c'_f)N_f\Big) \bigoplus \iota_{\mathrm{h}_e*}\mathcal{L}_{\mathrm{h}_e}^{c'}(s)\Big(-\sum_{\substack{f\in A'\\ \mathrm{h}_f=\mathrm{h}_e}}(c_f-c'_f)N_f - N_e\Big).$$

But then $e \in E_{t'}^{c'}$, an absurd.

It follows that $A = E_t^c - E_{t'}^{c'}$. Then $c - c' = (\mathfrak{o}_{t'}^{c'}, \mathfrak{o}_t^c)$. □

It follows from Proposition 2.1 that we may let n tend to ∞, and consider the union of the $\mathbf{P}_{\mathbf{J^b}}^c$ for all $c \in C^1(G,\mathbb{Z})$. We will denote this union by

$$\mathbf{R}_{\mathbf{J^b}} := \bigcup_{c\in C^1(G,\mathbb{Z})} \mathbf{P}_{\mathbf{J^b}}^c.$$

It is a scheme locally of finite type over $\mathbf{J^b}$. In fact, there is another way of describing $\mathbf{R}_{\mathbf{J^b}}$, which shows that $\mathbf{R}_{\mathbf{J^b}}$ is naturally fibered over $\mathbf{J^b}$ with fibers equal to $\mathbf{R}$, for the scheme $\mathbf{R}$ defined in [AE20b], Subsection 3.2, and recalled below.

More precisely, for each $e = uv \in E^{\mathfrak{o}}$ and $i \in \mathbb{Z}$, let

$$\mathcal{F}_e^i := \mathcal{L}_u(-(\mathbf{c}_e - i)N_e \times \mathbf{J^b})|_{N_e\times\mathbf{J^b}} \oplus \mathcal{L}_v((\mathbf{c}_e - i)N_e \times \mathbf{J^b})|_{N_e\times\mathbf{J^b}}.$$

Note that this definition is compatible with that of $\mathcal{F}_e^{c_e}$, given previously, as the two sheaves coincide if $i = c_e$.

As before, we may view the $\mathbf{P}_{\mathbf{J^b}}(\mathcal{F}_e^i)$ for integers i with $|i - \mathbf{c}_e| \leq n$ as closed subschemes of the relative Quot-scheme

$$\mathrm{Quot}_{X\times\mathbf{J^b}/\mathbf{J^b}}\Big((\iota_u \times 1_{\mathbf{J^b}})_*\big(\mathcal{L}_u(nN_e \times \mathbf{J^b})\big) \oplus (\iota_v \times 1_{\mathbf{J^b}})_*\big(\mathcal{L}_v(nN_e \times \mathbf{J^b})\big)\Big),$$

more precisely, of the component of the relative Quot-scheme parameterizing quotients of finite length equal to $2n+1$. Letting n tend to ∞, we may consider the union of all the

$\mathbf{P}_{\mathbf{J^b}}(\mathcal{F}_e^i)$ for all $i \in \mathbb{Z}$. We will denote this union by

$$\mathbf{R}^e_{\mathbf{J^b}} := \bigcup_{i\in\mathbb{Z}} \mathbf{P}_{\mathbf{J^b}}(\mathcal{F}_e^i).$$

It is an $\mathbf{R}_e$-bundle over $\mathbf{J^b}$, where $\mathbf{R}_e$ is the doubly infinite chain of smooth rational curves over κ, as in [AE20b], Subsection 3.2. In addition, $\mathbf{R}_{\mathbf{J^b}}$ can be naturally identified with the fibered product of the $\mathbf{R}^e_{\mathbf{J^b}}$ for all $e \in E^{\mathfrak{o}}$ over $\mathbf{J^b}$, that is,

$$\mathbf{R}_{\mathbf{J^b}} = \prod_{e\in E^{\mathfrak{o}}} \mathbf{R}^e_{\mathbf{J^b}}.$$

It is an $\mathbf{R}$-bundle over $\mathbf{J^b}$, where $\mathbf{R} := \prod_{e\in E^{\mathfrak{o}}} \mathbf{R}_e$.

Yet more precisely, we identify a fiber of $\mathbf{R}_{\mathbf{J^b}}/\mathbf{J^b}$ with $\mathbf{R}$ in the following way: A point $s \in \mathbf{J^b}$ corresponds to a collection of torsion-free, rank-one sheaves $(L_v\,;\,v \in V)$ with L_v of degree $\mathbf{b}_v$. We fix trivializations $L_v|_{N_e} \cong \kappa$ for each $e \in E$ and each $v \in e$. We fix as well trivializations $\mathcal{O}_{X_v}(N_e)|_{N_e} \cong \kappa$ for each $e \in E$ and each $v \in e$, and consider the induced trivializations $\mathcal{O}_{X_v}(mN_e + D)|_{N_e} \cong \kappa$ for each $m \in \mathbb{Z}$ and each Cartier divisor D of X_v not containing N_e in its support. These trivializations induce trivializations

$$L_v\Big(\sum_{\substack{e\in\mathbb{E}\\ \mathrm{h}_e=v}} (\mathbf{c}_e - c_e)N_e\Big)|_{N_f} \cong \kappa$$

for each $f \in E$, each $v \in f$ and each $c \in C^1(G,\mathbb{Z})$. These trivializations give rise to an isomorphism between the fiber of $\mathbf{R}^E_{\mathbf{J^b}}/\mathbf{J^b}$ over s and $\mathbf{R}$.

The universal subsheaf on

$$X \times \mathrm{Quot}_{X\times\mathbf{J^b}/\mathbf{J^b}}\Big(\bigoplus_{v\in V}\big(\iota_v \times 1_{\mathbf{J^b}}\big)_*\mathcal{L}_v^{(n)}\Big)$$

for $n \to \infty$ restricts to a relative torsion-free, rank-one, degree-d sheaf on $X \times \mathbf{R}_{\mathbf{J^b}}/\mathbf{R}_{\mathbf{J^b}}$, which shall be denoted by $\mathcal{I}$. Its restriction to the subscheme $X \times \mathbf{P}^c_{\mathbf{J^b}}$ is $\mathcal{I}^c$ for each $c \in C^1(G,\mathbb{Z})$.

For each $c \in C^1(G,\mathbb{Z})$, an open dense subscheme of $\mathbf{P}^c_{\mathbf{J^b}}$ parameterizes invertible sheaves of multidegree $\mathbf{b}+d^*_{\mathbb{Z}}(\mathbf{c}-c)$. Since deg is a cokernel for $d^*_{\mathbb{Z}}$, it follows that, up to translation, we could have changed $\mathbf{c}$ for any other 1-cochain and $\mathbf{b}$ for any other 0-cochain of degree d. Moreover, the orientation $\mathfrak{o}$ is just a convenient means of ordering the curves in the chain $\mathbf{R}_e$ for each $e \in E$; a different orientation would lead to the same fibration $\mathbf{R}_{\mathbf{J^b}}/\mathbf{J^b}$.

Another way of interpreting Proposition 2.1 is through the following definition and proposition: Given $t \in \mathbf{R}_{\mathbf{J^b}}$, let $c \in C^1(G,\mathbb{Z})$ such that $t \in \mathbf{P}^c_{\mathbf{J^b}}$. Define $c(t) \in C^1(G,\mathbb{Q})$ by:

$$c(t)_e := c_e + \begin{cases} +\frac{1}{2} & \text{if } e \in E^c_t, \\ -\frac{1}{2} & \text{if } \overline{e} \in E^c_t, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\mathbb{E}_t := \{e \in \mathbb{E} \,|\, c(t)_e \notin \mathbb{Z}\}$ is precisely the set of oriented edges supported in E_t.

Define as well, for each $c \in C^1(G, \frac{1}{2}\mathbb{Z})$,

$$\mathcal{L}_v^c := \mathcal{L}_v\Big(\sum_{\substack{e\in\mathbb{E}\\ \mathrm{h}_e=v}} \lfloor \mathbf{c}_e - c_e \rfloor N_e \times \mathbf{J}^{\mathbf{b}}\Big)$$

and

$$\mathbf{P}^c_{\mathbf{J}^{\mathbf{b}}} := \bigcap_{\substack{c'\in C^1(G,\mathbb{Z})\\ |c'_e - c_e|\leq\frac{1}{2}\forall e\in\mathbb{E}}} \mathbf{P}^{c'}_{\mathbf{J}^{\mathbf{b}}}.$$

Proposition 2.2. *Let $t \in \mathbf{R}_{\mathbf{J}^{\mathbf{b}}}$. Then the following statements are true:*

(1) *$c(t)$ is well-defined.*

(2) *$t \in \mathbf{P}^c_{\mathbf{J}^{\mathbf{b}}}$ for $c \in C^1(G, \mathbb{Z})$ if and only if $|c_e - c(t)_e| \leq \frac{1}{2}$ for every $e \in \mathbb{E}$. In particular, $t \in \mathbf{P}^{c(t)}_{\mathbf{J}^{\mathbf{b}}}$.*

(3) *The torsion-free sheaf $\mathcal{I}(t)_v$ generated by $\mathcal{I}(t)$ on X_v for each $v \in V$ is isomorphic to $\mathcal{L}_v^{c(t)}(s)$, where $s \in \mathbf{J}^{\mathbf{b}}$ is the point lying under t.*

Proof. Suppose t lies on $\mathbf{P}^c_{\mathbf{J}^{\mathbf{b}}} \cap \mathbf{P}^{c'}_{\mathbf{J}^{\mathbf{b}}}$. It follows from Proposition 2.1 that $c' - c = (\mathfrak{o}^c_t, \mathfrak{o}^{c'}_t)$. Then

$$c'_e := c_e + \begin{cases} +1 & \text{if } e \in E^c_t - E^{c'}_t \\ -1 & \text{if } e \in E^{c'}_t - E^c_t \\ 0 & \text{otherwise} \end{cases} = c(t)_e - \begin{cases} -\frac{1}{2} & \text{if } \overline{e} \in E^{c'}_t \\ +\frac{1}{2} & \text{if } e \in E^{c'}_t \\ 0 & \text{otherwise,} \end{cases}$$

where we used the fact that E^c_t and $E^{c'}_t$ are orientations of the same set, E_t. Thus, the definition of $c(t)$ does not change if c' were chosen instead of c.

Furthermore, from the above argument it follows that $|c_e - c(t)_e| \leq \frac{1}{2}$ for each $e \in \mathbb{E}$ and $c \in C^1(G, \mathbb{Z})$ such that $t \in \mathbf{P}^c_{\mathbf{J}^{\mathbf{b}}}$. Conversely, let $c \in C^1(G, \mathbb{Z})$ such that $t \in \mathbf{P}^c_{\mathbf{J}^{\mathbf{b}}}$. If $c' \in C^1(G, \mathbb{Z})$ is such that $|c'_e - c(t)_e| \leq \frac{1}{2}$ for every $e \in \mathbb{E}$, then $c'_e = c(t)_e = c_e$ for each $e \in \mathbb{E}$ such that $c(t)_e \in \mathbb{Z}$, that is, such that $e \notin \mathbb{E}_t$. On the other hand, if $e \in \mathbb{E}_t$ then $|c'_e - c(t)_e| = \frac{1}{2}$. Let $\mathfrak{o}'$ be the orientation of E_t such that $e \in E^{\mathfrak{o}'}_t$ if and only if $c(t)_e - c'_e = \frac{1}{2}$. Then $c - c' = (\mathfrak{o}', \mathfrak{o}^c_t)$, and it follows from Proposition 2.1 that $t \in \mathbf{P}^{c'}_{\mathbf{J}^{\mathbf{b}}}$.

Finally, let $s \in \mathbf{J}^{\mathbf{b}}$ be the point lying under t. Let $c \in C^1(G, \mathbb{Z})$ such that $t \in \mathbf{P}^c_{\mathbf{J}^{\mathbf{b}}}$. Then $\mathcal{I}^c(t)$ generates the subsheaf

$$\mathcal{L}^c_v(s)\Big(-\sum_{\substack{e\in E^c_t\\ \mathrm{h}_e=v}} N_e\Big)$$

of $\mathcal{L}^c_v(s)$ for each $v \in V$. But this subsheaf is $\mathcal{L}^{c(t)}_v(s)$. ∎

2.3. **Group action.** There is a natural action on $\mathbf{R}_{\mathbf{J}^{\mathbf{b}}}$ by $H^1(G, \mathbb{Z})$. Indeed, given $\gamma \in H^1(G, \mathbb{Z})$ and a point s of $\mathbf{J}^{\mathbf{b}}$, corresponding to a collection of torsion-free, rank-one sheaves

$(L_v\,;\,v\in V)$, we associate $s'\in \mathbf{J^b}$, corresponding to the tuple of sheaves $(L_v^{-\gamma}\,;\,v\in V)$, where

$$L_v^{-\gamma} := L_v\big(-\sum_{\substack{e\in\mathbb{E}\\ \mathrm{h}_e=v}} \gamma_e N_e\big).$$

This association gives a map $\tau^\gamma\colon \mathbf{J^b}\to\mathbf{J^b}$ which sends s to s'. Clearly, $\gamma\mapsto\tau^\gamma$ gives a group homomorphism $H^1(G,\mathbb{Z})\to \mathrm{Aut}(\mathbf{J^b})$. Also,

$$\mathcal{L}_v^c(\tau^\gamma(s)) = \mathcal{L}_v^{c+\gamma}(s) \quad \text{for each } s\in\mathbf{J^b},\ c\in C^1(G,\mathbb{Z}) \text{ and } v\in V.$$

By construction, for each point $s\in\mathbf{J^b}$, the restrictions of $\mathcal{I}^{c+\gamma}$ over points on the fiber of $\mathbf{P}^{c+\gamma}_{\mathbf{J^b}}$ over s are exactly the same restrictions of $\mathcal{I}^c$ over points on the fiber of $\mathbf{P}^c_{\mathbf{J^b}}$ over $\tau^\gamma(s)$. Thus, the translation $\tau^\gamma\colon\mathbf{J^b}\to\mathbf{J^b}$ lifts to an automorphism $\widetilde{\tau}^\gamma\colon \mathbf{R}_{\mathbf{J^b}}\to\mathbf{R}_{\mathbf{J^b}}$ sending $\mathbf{P}^{c+\gamma}_{\mathbf{J^b}}$ to $\mathbf{P}^c_{\mathbf{J^b}}$ for each $c\in C^1(G,\mathbb{Z})$. More precisely, given $t\in\mathbf{P}^{c+\gamma}_{\mathbf{J^b}}$ over $s\in\mathbf{J^b}$, corresponding to the subsheaf

$$\mathcal{I}^{c+\gamma}(t)\subseteq\bigoplus_v \mathcal{L}_v^{c+\gamma}(s),$$

τ^γ sends s to the point s' corresponding to the tuple $(\mathcal{L}_v(s)^{-\gamma}\,;\,v\in V)$ and $\widetilde{\tau}^\gamma$ sends t to the point on $\mathbf{P}^c_{\mathbf{J^b}}$ over s' which corresponds to the same subsheaf $\mathcal{I}^{c+\gamma}(t)$. In other words,

$$\mathcal{I}^c(\widetilde{\tau}^\gamma(t)) = \mathcal{I}^{c+\gamma}(t) \quad \text{as subsheaves of} \quad \bigoplus_v \mathcal{L}_v^c(\tau^\gamma(s)) = \bigoplus_v \mathcal{L}_v^{c+\gamma}(s).$$

Proposition 2.3. *Notations as above, the assignment $\gamma\mapsto\widetilde{\tau}^\gamma$ defines an injective group homomorphism $H^1(G,\mathbb{Z})\to\mathrm{Aut}(\mathbf{R}_{\mathbf{J^b}})$. Moreover, $\widetilde{\tau}^\gamma$ has fixed points only if $\gamma=0$.*

Proof. By construction, the assignment $\widetilde{\tau}^\bullet : H^1(G,\mathbb{Z})\to\mathrm{Aut}(\mathbf{R}_{\mathbf{J^b}})$ is clearly a group homomorphism. We prove the second statement, from which the injectivity of $\widetilde{\tau}^\bullet$ follows.

Let $\gamma\in H^1(G,\mathbb{Z})$, and suppose $\widetilde{\tau}^\gamma$ has a fixed point $t\in\mathbf{P}^c_{\mathbf{J^b}}$ for a certain $c\in C^1(G,\mathbb{Z})$. Since $\widetilde{\tau}^\gamma$ sends $\mathbf{P}^{c'+\gamma}_{\mathbf{J^b}}$ to $\mathbf{P}^{c'}_{\mathbf{J^b}}$ for each $c'\in C^1(G,\mathbb{Z})$, it follows that t lies on $\mathbf{P}^{c-n\gamma}_{\mathbf{J^b}}$ for every integer $n\geq 0$. This is possible, by Proposition 2.1, only if $\gamma=0$. ☐

The action of $H^1(G,\mathbb{Z})$ on $\mathbf{R}_{\mathbf{J^b}}$ is thus free. We may consider the quotient stack:

$$\mathbf{S}^d := \left[\frac{\mathbf{R}_{\mathbf{J^b}}}{H^1(G,\mathbb{Z})}\right].$$

Since $\mathcal{I}(t)=\mathcal{I}(\widetilde{\tau}^\gamma(t))$ for each $t\in\mathbf{R}^E_{\mathbf{J^b}}$ and $\gamma\in H^1(G,\mathbb{Z})$, the sheaf $\mathcal{I}$ descends to a torsion-free, rank-one, degree-d sheaf on $X\times\mathbf{S}^d/\mathbf{S}^d$, which we shall also denote by $\mathcal{I}$.

We call $\mathbf{S}^d$ the *moduli of embedded sheaves*. We have the following moduli description.

Proposition 2.4. *The stack $\mathbf{S}^d$ parameterizes the data of torsion-free, rank-one sheaves M_v on X_v for each $v\in V$, and subsheaves $I\subseteq\oplus M_v$ of degree d such that all the induced maps $h_v\colon I\to M_v$ are surjective.*

Proof. For each such data, let E_I be the collection of edges $e \in E$ for which I fails to be invertible at N_e. Let $\mathfrak{u}\colon E_I \to \mathbb{E}$ be an orientation. For each $v \in V$, let

$$\widetilde{M}_v := M_v\Big(\sum_{\substack{e\in E_I^{\mathfrak{u}}\\ \mathrm{h}_e=v}} N_e\Big).$$

Then $\sum_{v\in V} \deg(\widetilde{M}_v) = d$, whence there are $c \in C^1(G,\mathbb{Z})$ and $s \in \mathbf{J}^{\mathbf{b}}$ such that $\widetilde{M}_v \cong \mathcal{L}_v^c(s)$ for each $v \in V$. (Given c, the point s is unique.) Now, since

$$I \subseteq \bigoplus_{v\in V} M_v \subseteq \bigoplus_{v\in V} \mathcal{L}_v^c(s),$$

it follows that the data corresponds to a unique point t on $\mathbf{P}^c_{\mathbf{J}^{\mathbf{b}}}$ over s. Since the h_v are surjective, $\mathfrak{o}_t^c = \mathfrak{u}$.

A choice $\mathfrak{u}'\colon E_I \to \mathbb{E}$ different from $\mathfrak{u}$ would correspond to a point t' on $\mathbf{P}^{c'}_{\mathbf{J}^{\mathbf{b}}}$ over the same s, for $c' := c - (\mathfrak{u}', \mathfrak{u})$. Note however that $t' = t$ on $\mathbf{R}_{\mathbf{J}^{\mathbf{b}}}$ by Proposition 2.1.

Furthermore, for a fixed $\mathfrak{u}$, any other choice of c differs from the one above by an element of $H^1(G,\mathbb{Z})$. More precisely, if $c' \in C^1(G,\mathbb{Z})$ and $s' \in \mathbf{J}^{\mathbf{b}}$ are such that $\widetilde{M}_v \cong \mathcal{L}_v^{c'}(s')$ for every $v \in V$, then $\gamma := c - c' \in H^1(G,\mathbb{Z})$ and $s' = \tau_\gamma(s)$. As the construction yields the same subsheaf I, we have that the corresponding point t' on $\mathbf{P}^{c'}_{\mathbf{J}^{\mathbf{b}}}$ over s' satisfies $t' = \widetilde{\tau}_\gamma(t)$. Then the images of t and t' on $\mathbf{S}^d$ are the same.

Conversely, given a point on $\mathbf{S}^d$, let $t \in \mathbf{R}_{\mathbf{J}^{\mathbf{b}}}$ be a lifting, and $s \in \mathbf{J}^{\mathbf{b}}$ its image. Let c be an element of $C^1(G,\mathbb{Z})$ such that $t \in \mathbf{P}^c_{\mathbf{J}^{\mathbf{b}}}$. Then s corresponds to the sheaves $\mathcal{L}_v(s)$ for $v \in V$ and t to the subsheaf

$$\mathcal{I}^c(t) \subseteq \bigoplus_{v\in V} \mathcal{L}_v^c(s). \tag{2.3}$$

For each $v \in V$, the sheaf $\mathcal{I}^c(t)$ generates the subsheaf

$$M_v := \mathcal{L}_v^c(s)\Big(-\sum_{\substack{e\in E_t^c\\ \mathrm{h}_e=v}} N_e\Big),$$

Then

$$I := \mathcal{I}^c(t) \subseteq \bigoplus_{v\in V} M_v$$

with surjective induced maps $I \to M_v$.

Picking a different c will not change the data of the M_v and $I \subseteq \oplus M_v$, since replacing $\oplus\mathcal{L}_v^c(s)$ by the larger sheaf $\oplus_v \mathcal{L}_v^{(n)}(s)$ for large n does not change the sheaves M_v obtained.

If $t' \in \mathbf{R}_{\mathbf{J}^{\mathbf{b}}}$ is another lifting, lying over $s' \in \mathbf{J}^{\mathbf{b}}$, then $s' = \tau_\gamma(s)$ and $t' = \widetilde{\tau}_\gamma(t)$ for a certain $\gamma \in H^1(G,\mathbb{Z})$. Since $t \in \mathbf{P}^c_{\mathbf{J}^{\mathbf{b}}}$ we have $t' \in \mathbf{P}^{c-\gamma}_{\mathbf{J}^{\mathbf{b}}}$. Also, the inclusion in (2.3) is the same as

$$\mathcal{I}^{c-\gamma}(t') \subseteq \bigoplus_{v\in V} \mathcal{L}_v^{c-\gamma}(s').$$

Thus we obtain the same data of sheaves M_v on X_v and subsheaf of $\oplus M_v$.

We omit the simple verification that the two procedures described above are inverse to each other. ☐

Observe that a surjection $h_v\colon I \to M_v$ from a torsion-free, rank-one sheaf on X to one on X_v induces an isomorphism $\widetilde{h}_v\colon I_v \to M_v$, where I_v is the restriction of I to X_v modulo torsion. So we may view the data defining a point on $\mathbf{S}^d$ as the equivalence class of the data of an abstract torsion-free, rank-one degree-d sheaf I on X together with elements $z_v \in \mathbf{G_m}(\kappa)$ for each $v \in V$; these data give rise to a map

$$I \hookrightarrow \oplus I_v \xrightarrow{(z_v\,;\,v\in V)} \oplus I_v,$$

and the equivalence identifies data with the same image.

Another way to put this is as follows. Observe that there is a natural action of the character group of $C^0(G,\mathbb{Z})$, henceforth denoted $\mathbf{G}_\mathbf{m}^V$, on $\mathbf{R}_{\mathbf{J^b}}$. In fact, the action of each $z \in \mathbf{G}_\mathbf{m}^V(\kappa)$ is the restriction of the one on $\mathrm{Quot}_{X\times\mathbf{J^b}/\mathbf{J^b}}\Big(\bigoplus_{v\in V}(\iota_v \times 1_{\mathbf{J^b}})_*\mathcal{L}_v^{(n)}\Big)$ induced by the isomorphisms

$$\mathcal{L}_v^{(n)} \xrightarrow{z_v} \mathcal{L}_v^{(n)} \text{ for } v \in V.$$

If all the z_v are equal, the action is the identity. We have thus defined an action of $\mathbf{G}_\mathbf{m}^V/\mathbf{G_m}$, where $\mathbf{G_m}$ is embedded diagonally in $\mathbf{G}_\mathbf{m}^V$, on $\mathbf{R}_{\mathbf{J^b}}$ that sends a fiber over $\mathbf{J^b}$ to itself. Moreover, the isomorphism between a fiber of $\mathbf{R}^E_{\mathbf{J^b}}$ over $\mathbf{J^b}$ and $\mathbf{R}$ is equivariant with respect to the action of $\mathbf{G}_\mathbf{m}^V/\mathbf{G_m}$ on $\mathbf{R}$ defined in [AE20b], Subsection 5.1, and recalled below in Section 4. Also, the action on $\mathbf{R}_{\mathbf{J^b}}$ restricts to an action on $\mathbf{P}^c_{\mathbf{J^b}}$ for each $c \in C^1(G,\mathbb{Z})$. Finally, this action commutes with the action of $H^1(G,\mathbb{Z})$ and thus induces an action on $\mathbf{S}^d$.

The (abstract) torsion-free, rank-one sheaves I of degree d on X correspond to orbits of $\mathbf{S}^d$ under $\mathbf{G}_\mathbf{m}^V/\mathbf{G_m}$. The specific point on the orbit tells us how I is viewed inside $\oplus I_v$. So, we may view the space of orbits, the quotient stack

$$\mathbf{J}^d := \left[\frac{\mathbf{S}^d}{\mathbf{G}_\mathbf{m}^V/\mathbf{G_m}}\right],$$

as a parameter space for torsion-free, rank-one sheaves of degree d on X. The sheaf $\mathcal{I}$ on $X\times\mathbf{S}^d$ is $\mathbf{G}_\mathfrak{m}^V/\mathbf{G}_\mathfrak{m}$-invariant, and thus descends to a relative torsion-free, rank-one, degree-d sheaf on $X\times\mathbf{J}^d/\mathbf{J}^d$, a universal torsion-free, rank-one, degree-d sheaf.

However, the quotient $\mathbf{J}^d$ is not well-behaved because the orbits of the action have variable dimension.

Proposition 2.5. *The orbits of maximum dimension of* $\mathbf{S}^d$ *under the action of* $\mathbf{G}_\mathbf{m}^V/\mathbf{G_m}$ *have dimension* $|V|-1$ *and correspond to simple torsion-free, rank-one sheaves of degree* d*.*

Proof. Simple, torsion-free, rank-one sheaves on X are direct images of invertible sheaves on *connected* partial desingularizations of X. So the proof reduces to considering invertible sheaves on X, and to showing that the subscheme of the Picard scheme of X parameterizing invertible sheaves with given restrictions to the components X_v is isomorphic to $\mathbf{G}_\mathbf{m}^{|E|-|V|+1}$. This has been shown by a number of authors — see e.g. [OS79, 10.2] — and follows formally from the exact sequence

$$1 \to H^1(G, \mathbf{G_m}) \to \mathrm{Pic}^0(X) \to \prod_{v \in V} \mathrm{Pic}^0(X_v) \to 1.$$

□

Thus, restricting to orbits of dimension $|V|-1$, and considering the orbit space, we get the moduli space of simple, torsion-free, rank-one sheaves of degree d on X. This space has been constructed by Altman and Kleiman as an algebraic space [AK80, AK79]. (Actually, they prove the representability by an algebraic space of a functor parameterizing far more general objects.) The second author of the present article has shown later that this space is actually a scheme locally of finite type and universally closed over the field, though not separated; see [Est01].

The second author has also considered certain open subspaces of the moduli space of simple sheaves which are actually proper over the field, actually projective [Est09], thus producing various compactifications of the Jacobian, even for general families of curves satisfying certain mild conditions [Est01]. Instead of doing this, however, we will consider clusters of orbits. The following proposition, which collects statements proved along the section and states new ones, will serve us in describing these clusters.

Proposition 2.6. *Let I be a torsion-free, rank-one, degree-d sheaf on X. Let E_I be the collection of edges $e \in E$ for which I fails to be invertible at N_e. For each $v \in V$, let I_v denote the maximum torsion-free quotient of $I|_{X_v}$. Then the following statements hold:*

(1) For each orientation $\mathfrak{u}\colon E_I \to \mathbb{E}$, there are $c \in C^1(G, \mathbb{Z})$ and $s \in \mathbf{J^b}$ such that

$$\mathcal{L}_v^c(s) \cong I_v\Big(\sum_{\substack{e \in E_I^{\mathfrak{u}} \\ \mathrm{h}_e = v}} N_e\Big) \quad \text{for each } v \in V. \tag{2.4}$$

(2) For each $\mathfrak{u}$, c and s satisfying Equations (2.4), there is a point $t \in \mathbf{P}^c_{\mathbf{J^b}}$ over s representing I with $\mathfrak{o}^c_t = \mathfrak{u}$.

(3) For each point $s \in \mathbf{J^b}$ and each element $\mathfrak{c} \in C^1(G, \frac{1}{2}\mathbb{Z})$ satisfying $e \in E_I$ if and only if $\mathfrak{c}_e \notin \mathbb{Z}$, and such that

$$\mathcal{L}_v^{\mathfrak{c}}(s) \cong I_v \quad \text{for each } v \in V, \tag{2.5}$$

there is a point $t \in \mathbf{P}^{\mathfrak{c}}_{\mathbf{J^b}}$ over s representing I with $c(t) = \mathfrak{c}$.

(4) Conversely, given $c \in C^1(G, \mathbb{Z})$ and $s \in \mathbf{J^b}$, and a point t on the fiber of $\mathbf{P}^c_{\mathbf{J^b}}$ over s representing I, Equations (2.4) hold for $\mathfrak{u} := \mathfrak{o}^c_t$ and Equations (2.5) hold for $\mathfrak{c} := c(t)$.

(5) *For each two sets of data* $(\mathfrak{u}_1, c_1, s_1)$ *and* $(\mathfrak{u}_2, c_2, s_2)$ *satisfying* (2.4), *and each two points* t_1 *and* t_2 *representing* I *on the fibers of* $\mathbf{P}^{c_1}_{\mathbf{J^b}}$ *and* $\mathbf{P}^{c_2}_{\mathbf{J^b}}$ *over* s_1 *and* s_2*, respectively, with* $\mathfrak{o}^{c_1}_{t_1} = \mathfrak{u}_1$ *and* $\mathfrak{o}^{c_2}_{t_2} = \mathfrak{u}_2$*, we have that*

$$\gamma := c_1 - c_2 + (\mathfrak{u}_1, \mathfrak{u}_2) = c(t_1) - c(t_2)$$

is in $H^1(G, \mathbb{Z})$*, satisfies* $\tau^\gamma(s_1) = s_2$ *and is such that* t_2 *differs from* $\widetilde{\tau}^\gamma(t_1)$ *by the action of an element of* $\mathbf{G}^V_{\mathbf{m}}/\mathbf{G}_{\mathbf{m}}(\kappa)$.

Proof. For each orientation $\mathfrak{u}\colon E_I \to \mathbb{E}$, let $f^{\mathfrak{u}} \in C^0(G, \mathbb{Z})$ be defined by

$$f^{\mathfrak{u}}(v) := \deg(I_v) + |E^{\mathfrak{u}}_I(v)| \text{ for each } v \in V, \text{ where } E^{\mathfrak{u}}_I(v) := \{e \in E^{\mathfrak{u}}_I \,|\, \mathrm{h}_e = v\}.$$

Then (2.4) is satisfied for $\mathfrak{u}$, c and s only if

$$f^{\mathfrak{u}} = d^*(\mathbf{c} - c) + \mathbf{b}. \tag{2.6}$$

Furthermore, if the latter holds for given $\mathfrak{u}$ and $c \in C^1(G, \mathbb{Z})$, then there is a unique $s \in \mathbf{J^b}$ such that (2.4) holds for $\mathfrak{u}$, c and s. Indeed, as the points $s \in \mathbf{J^b}$ parameterize all tuples of sheaves with degrees given by $\mathbf{b}$, the tuples $(\mathcal{L}^c_v(s) \,|\, v \in V)$ run through all tuples of sheaves with degrees given by $d^*(\mathbf{c} - c) + \mathbf{b}$.

Now, given $\mathfrak{u}$, since the union of all $E^{\mathfrak{u}}_I(v)$ for $v \in V$ is $E^{\mathfrak{u}}_I$, it follows that $f^{\mathfrak{u}}$ has degree d, the same as $\mathbf{b}$. So there is $c \in C^1(G, \mathbb{Z})$ such that $f^{\mathfrak{u}} = d^*(\mathbf{c} - c) + \mathbf{b}$. This finishes the proof of Statement (1).

Statement (2) and the first part of Statement (4) have already been addressed before, in the proof of Proposition 2.4. The remaining of Statement (4) follows from the definition of $c(t)$ for $t \in \mathbf{P}^c_{\mathbf{J^b}}$.

As for Statement (3), let $\mathfrak{u}\colon E_I \to \mathbb{E}$ be any orientation. And let $c \in C^1(G, \mathbb{Z})$ such that $c_e = \mathfrak{c}_e$ for each e such that $\mathfrak{c}_e \in \mathbb{Z}$ and $c_e = \lfloor \mathfrak{c}_e \rfloor$ for each $e \in E^{\mathfrak{u}}_I$. Then

$$\mathcal{L}^c_v(s) \cong \mathcal{L}^{\mathfrak{c}}_v(s)\Big(\sum_{\substack{e \in E^{\mathfrak{u}}_I \\ \mathrm{h}_e = v}} N_e\Big) \quad \text{for each } v \in V,$$

from which it follows that $\mathfrak{u}$, c and s satisfy Equations (2.4). By Statement (2), there is a point $t \in \mathbf{P}^c_{\mathbf{J^b}}$ over s representing I with $\mathfrak{o}^c_t = \mathfrak{u}$. From the definition of $c(t)$, it follows that $c(t) = \mathfrak{c}$.

Finally, we prove Statement (5). First observe that

$$f^{\mathfrak{u}_1}(v) - f^{\mathfrak{u}_2}(v) = |E^{\mathfrak{u}_1}_I(v)| - |E^{\mathfrak{u}_2}_I(v)| = d^*((\mathfrak{u}_1, \mathfrak{u}_2))(v) \text{ for each } v \in V.$$

Since (2.4) holds for the two sets of data, $(\mathfrak{u}_1, c_1, s_1)$ and $(\mathfrak{u}_2, c_2, s_2)$, instead of $(\mathfrak{u}, c, s)$, Equations (2.6) hold, that is,

$$f^{\mathfrak{u}_1} = d^*(\mathbf{c} - c_1) + \mathbf{b} \quad \text{and} \quad f^{\mathfrak{u}_2} = d^*(\mathbf{c} - c_2) + \mathbf{b}.$$

Thus

$$d^*(c_2 - c_1) = f^{\mathfrak{u}_1} - f^{\mathfrak{u}_2} = d^*((\mathfrak{u}_1, \mathfrak{u}_2)),$$

and hence γ, as defined, is in $H^1(G,\mathbb{Z})$. The equality

$$c_1 - c_2 + (\mathfrak{u}_1, \mathfrak{u}_2) = c(t_1) - c(t_2)$$

follows from the definition of $c(t_1)$ and $c(t_2)$, using that $\mathfrak{o}_{t_1}^{c_1} = \mathfrak{u}_1$ and $\mathfrak{o}_{t_2}^{c_2} = \mathfrak{u}_2$.

Now, since (2.4) holds for $(\mathfrak{u}_1, c_1, s_1)$ and for $(\mathfrak{u}_2, c_2, s_2)$, instead of $(\mathfrak{u}, c, s)$, we have

$$\begin{aligned}
\mathcal{L}_v^{c_2}(\tau^\gamma(s_1)) \cong \mathcal{L}_v^{c_2+\gamma}(s_1) &= \mathcal{L}_v^{c_1-(\mathfrak{u}_2,\mathfrak{u}_1)}(s_1)\\
&\cong \mathcal{L}_v^{c_1}(s_1)\Big(\sum_{\substack{e\in\mathbb{E}\\ \mathrm{h}_e=v}}(\mathfrak{u}_2,\mathfrak{u}_1)_e N_e\Big)\\
&\cong I_v\Big(\sum_{e\in E_I^{\mathfrak{u}_1}(v)} N_e + \sum_{e\in E_I^{\mathfrak{u}_2}(v)} N_e - \sum_{e\in E_I^{\mathfrak{u}_1}(v)} N_e\Big)\\
&\cong I_v\Big(\sum_{e\in E_I^{\mathfrak{u}_2}(v)} N_e\Big) \cong \mathcal{L}_v^{c_2}(s_2)
\end{aligned}$$

for each $v \in V$, whence $\tau^\gamma(s_1) = s_2$.

Also, $\widetilde{\tau}^\gamma(t_1)$ represents the same sheaf as t_1, and $\widetilde{\tau}^\gamma(t_1)$ lies on $\mathbf{P}_{\mathbf{J^b}}^{c_1-\gamma}$, on the same fiber of $\mathbf{R}_{\mathbf{J^b}}$ over $\mathbf{J^b}$ as t_2, that over s_2. We may thus assume $s_1 = s_2$ and $\gamma = 0$.

Now, $\mathfrak{o}_{t_1}^{c_1} = \mathfrak{u}_1$ and $\mathfrak{o}_{t_2}^{c_2} = \mathfrak{u}_2$. Since $c_2 = c_1 + (\mathfrak{u}_1, \mathfrak{u}_2)$, it follows from Proposition 2.1 that $t_1 \in \mathbf{P}_{\mathbf{J^b}}^{c_2}$ as well. We may thus assume $c_1 = c_2$, whence $\mathfrak{u}_1 = \mathfrak{u}_2$. Set $c := c_1$.

Since t_1 and t_2 represent I, there is an isomorphism $\mathcal{I}^c(t_1) \to \mathcal{I}^c(t_2)$. This isomorphism induces isomorphisms $\mathcal{I}^c(t_1)_v \to \mathcal{I}^c(t_2)_v$ for each $v \in V$ making the diagram commute:

$$\begin{array}{ccc}
\mathcal{I}^c(t_1) & \longrightarrow & \bigoplus \mathcal{I}^c(t_1)_v\\
\downarrow & & \downarrow\\
\mathcal{I}^c(t_2) & \longrightarrow & \bigoplus \mathcal{I}^c(t_2)_v.
\end{array}$$

Since $\mathfrak{o}_{t_1}^c = \mathfrak{u}_1 = \mathfrak{u}_2 = \mathfrak{o}_{t_2}^c$, it follows from (2.4) that the isomorphisms $\mathcal{I}^c(t_1)_v \to \mathcal{I}^c(t_2)_v$ extend to isomorphisms $\mathcal{L}_v^c(s) \to \mathcal{L}_v^c(s)$ making the extended diagram commute:

$$\begin{array}{ccccc}
\mathcal{I}^c(t_1) & \longrightarrow & \bigoplus_{v\in V}\mathcal{I}^c(t_1)_v & \longrightarrow & \bigoplus_{v\in V}\mathcal{L}_v^c(s)\\
\downarrow & & \downarrow & & \downarrow\\
\mathcal{I}^c(t_2) & \longrightarrow & \bigoplus_{v\in V}\mathcal{I}^c(t_2)_v & \longrightarrow & \bigoplus_{v\in V}\mathcal{L}_v^c(s).
\end{array}$$

But an automorphism of $\mathcal{L}_v^c(s)$ is multiplication by a certain $z_v \in \mathbf{G_m}(\kappa)$. It follows that $\mathcal{I}^c(t_2) = (z_v \,|\, v \in V)\,\mathcal{I}^c(t_1)$ as subsheaves of $\oplus\mathcal{L}_v^c(s)$, and hence that t_1 and t_2 differ by the action of a κ-point on $\mathbf{G_m}^V/\mathbf{G_m}$. □

3. Degenerations of line bundles I

3.1. **Admissible extensions.** Let B be the spectrum of the power series ring $\kappa[[t]]$. Let $\pi\colon \mathcal{X} \to B$ be a flat, projective map with smooth generic fiber and special fiber isomorphic to X. We say π is a *smoothing* of X. We will identify the special fiber with X, through a fixed isomorphism. For each $e \in \mathbb{E}$, let ℓ_e be the *singularity degree* of $\mathcal{X}$ at N_e. More precisely, ℓ_e is the unique positive integer such that the completion of the local ring of $\mathcal{X}$ at N_e is $\kappa[[t]]$-isomorphic to $\kappa[[x,y,t]]/(xy-t^{\ell_e})$. Let $\ell\colon E \to \mathbb{N}$ be the function assigning ℓ_e to e for each $e \in E$. For each $v \in V$, the irreducible component X_v of X fails to be a Cartier divisor of $\mathcal{X}$ at the nodes N_e for which $\ell_e > 1$.

By a sequence of blowups supported over the nodes N_e, we obtain a map $\sigma\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ for which the strict transform of X_v is Cartier in $\widetilde{\mathcal{X}}$ for each $v \in V$. Then all irreducible components of $\sigma^{-1}(X)$ are Cartier in $\widetilde{\mathcal{X}}$. We choose $\widetilde{\mathcal{X}}$ minimal with this property. (Observe that σ is an isomorphism over any node of X other than the N_e, so $\widetilde{\mathcal{X}}$ is not regular if the singularity degree of any such node is greater than 1.) We call σ the *Cartier reduction* of π. It is the semistable reduction if $\widetilde{\mathcal{X}}$ fails to be regular only at the N_e.

Put $\widetilde{\pi} := \pi\sigma$ and let X^ℓ be the special fiber of $\widetilde{\pi}$. It is obtained by replacing each N_e by a chain $Z^e := \sigma^{-1}(N_e)$ of rational smooth curves of length $\ell_e - 1$. For each $e = uv$ in $\mathbb{E}$ we will order the components of Z^e as $Z^e_1, \dots, Z^e_{\ell_e-1}$, where Z^e_1 intersects X_u and $Z^e_{\ell_e-1}$ intersects X_v. Sometimes it will be convenient to set $Z^e_0 := X_u$ and $Z^e_{\ell_e} := X_v$. We will identify the remaining components of X^ℓ with their corresponding images on X. The curve X^ℓ and the restriction $\sigma^\ell := \sigma|_{X^\ell}\colon X^\ell \to X$ depend only on ℓ.

Let H be the graph obtained as a subdivision of G by inserting $\ell_e - 1$ new vertices in the middle of each edge, that is, by replacing each $e = uv$ in $\mathbb{E}$ by a path $P_e = u z^e_1 z^e_2 \dots z^e_{\ell_e-1} v$, for new vertices $z^e_1, \dots, z^e_{\ell_e-1}$. Sometimes it will be convenient to set $z^e_0 := u$ and $z^e_{\ell_e} := v$. The graph H coincides precisely with the graph obtained from the dual graph of X^ℓ by removing all loop edges: the new components Z^e_i correspond to the vertices z^e_i and the nodes $Z^e_i \cap Z^e_{i+1}$ to the edges $\{z^e_i, z^e_{i+1}\}$ on the path P_e. Abusing the notation, given a vertex v (resp. edge e) of H, we will also denote by X_v (resp. N_e) the corresponding component (resp. node) of X^ℓ. To avoid confusion, we will sometimes denote by $V(G)$ and $E(G)$ the vertex and edge sets of G, likewise for H. We will also view $V(G)$ as a subset of $V(H)$ in the natural way.

To an *almost invertible* sheaf L on X, that is, a torsion-free, rank-one sheaf on X that is invertible at all the nodes N_e for $e \in E(G)$, we may associate a divisor $D \in \mathrm{Div}(G)$, namely, $D := \sum \deg(L|_{X_v})v$. Likewise for sheaves on X^ℓ and divisors on H. Recall from [AE20a, Subsection 2.3] that a divisor D on H is called *G-admissible* if D contains in its support at most one vertex among the new vertices $z^e_1, \dots, z^e_{\ell_e-1}$ for each edge $e \in \mathbb{E}(G)$, and if so, with coefficient equal to 1. We set

$$t^D_e := \sum_{j=1}^{\ell_e} (\ell_e - j) D(z^e_j) \quad \text{for each } D \in \mathrm{Div}(H) \text{ and } e \in \mathbb{E}(G)$$

to identify the vertex. Similarly, we say an almost invertible sheaf L on X^ℓ is *σ^ℓ-admissible* if its associated divisor is G-admissible. If L is a σ^ℓ-admissible almost invertible sheaf of degree d, then the push-forward $\sigma^\ell_* L$ is a torsion-free, rank-one sheaf on X of degree d; see e.g. [EP16], Thm. 3.1, p. 63 (where a more general notion of admissibility, for which the statement still holds, is considered).

Let now L_η be an invertible sheaf on the generic fiber of π, which is the same as that of $\widetilde{\pi}$. Let $d := \deg L_\eta$. Since σ is the Cartier reduction of π, the sheaf L_η extends to an *almost invertible* sheaf on $\widetilde{\mathcal{X}}$, that is, a relative torsion-free, rank-one sheaf on $\widetilde{\mathcal{X}}/B$ whose restriction to the special fiber X^ℓ is almost invertible. The extension is not unique though: Given an almost invertible sheaf $\mathcal{L}$ on $\widetilde{\mathcal{X}}$, all the sheaves of the form

$$\mathcal{L}(f) := \mathcal{L} \otimes \mathcal{O}_{\widetilde{\mathcal{X}}}\Big(\sum_{v \in V(H)} f(v) X_v\Big)$$

for $f \in C^0(H, \mathbb{Z})$ have the same restriction to X_η as $\mathcal{L}$, and these are the only almost invertible sheaves with this property. The invertible sheaves

$$\mathcal{T}_f := \mathcal{O}_{\widetilde{\mathcal{X}}}\Big(\sum_{v \in V(H)} f(v) X_v\Big)$$

are called *twisters*. Given $f \in C^0(H, \mathbb{Z})$, the divisor on H associated to $\mathcal{T}_f|_{X^\ell}$ does not depend on the choice of the smoothing π. In addition, two twisters $\mathcal{T}_f$ and $\mathcal{T}_h$ yield the same divisor if and only if $f - h$ is constant.

The divisor on H associated to $\mathcal{T}_f|_{X^\ell}$ is the principal divisor defined by

$$\operatorname{div}(f) := \sum_{e \in \mathbb{E}(H)} (f(\mathrm{t}_e) - f(\mathrm{h}_e))\mathrm{h}_e$$

for each $f \in C^0(H, \mathbb{Z})$. The assignment defines a group homomorphism

$$\operatorname{div}\colon C^0(H, \mathbb{Z}) \to \operatorname{Div}(H)$$

whose kernel is the subgroup of constant functions and whose image defines an equivalence relation on $\operatorname{Div}(H)$, called *linear equivalence* [BLN97, BN07]. It follows that the divisors in $\operatorname{Div}(H)$ associated to the $\mathcal{L}|_{X^\ell}$ for all almost invertible extensions $\mathcal{L}$ of L_η to $\widetilde{\mathcal{X}}$ are those in a certain linear equivalence class.

We are interested in special extensions of L_η to $\widetilde{\mathcal{X}}$, those that yield meaningful extensions of L_η to $\mathcal{X}$ as well, so we make the following definition.

Definition 3.1. An almost invertible sheaf $\mathcal{L}$ on $\widetilde{\mathcal{X}}$ is called σ-*admissible* if $\mathcal{L}|_{X^\ell}$ is σ^ℓ-admissible.

The admissible sheaves we consider are special in the sense that the last statement of the following proposition holds.

Proposition 3.2. *Let $\mathcal{L}$ be a σ-admissible almost invertible sheaf on $\widetilde{\mathcal{X}}$. Then $R^1\sigma_*\mathcal{L} = 0$ and $\mathcal{I} := \sigma_*\mathcal{L}$ is a (relative) torsion-free, rank-one sheaf on $\mathcal{X}/B$, with formation commuting with base change. In particular, the restriction of $\mathcal{I}$ to X_η is the same as that of $\mathcal{L}$. Furthermore, $I_v \cong \mathcal{L}|_{X_v}$ for each $v \in V(G)$, where I_v is the restriction of $\mathcal{I}$ to X_v modulo torsion.*

Proof. The first statement follows from [EP16], Thm. 3.1, p. 63. As for the last statement, set $L := \mathcal{L}|_{X^\ell}$ and $I := \mathcal{I}|_X$. By the first statement, $I = \sigma^\ell_* L$. Let $Z \subseteq X^\ell$ be the union of the chains Z^e of exceptional components over which L has total degree 1, and $W \subseteq X^\ell$ the union of the remaining components. Let $\mu\colon X' \to X$ be the partial normalization of X along the nodes N_e at which I fails to be invertible. Since $I = \sigma^\ell_* L$, it follows from Thm. 3.1 in loc. cit. that σ^ℓ induces upon restriction a map $\nu\colon W \to X'$. It follows from Lemma 2.1 in loc. cit. that applying σ^ℓ_* to the short exact sequence

$$0 \longrightarrow L|_Z\Big(-\sum_{P\in Z\cap W} P\Big) \longrightarrow L \longrightarrow L|_W \longrightarrow 0$$

yields an isomorphism $I \cong \mu_*\nu_*(L|_W)$. As W is a semistable model of X', and $L|_W$ has degree 0 on every exceptional component of W, it follows again from Thm. 3.1 in loc. cit. that $\nu_*(L|_W)$ is almost invertible and $\nu^*\nu_*(L|_W) = L|_W$. In particular, $\nu_*(L|_W)|_{X_v} \cong L|_{X_v}$ for each $v \in V$. Finally, since μ is finite, the natural map $\mu^*\mu_*\nu_*(L|_W) \to \nu_*(L|_W)$ is surjective. Thus, restricting it to X_v we get a surjection $\mu_*\nu_*(L|_W)|_{X_v} \to L|_{X_v}$. Since L is almost invertible, whence $L|_{X_v}$ is torsion-free, and $I \cong \mu_*\nu_*(L|_W)$, it follows that $I_v \cong L|_{X_v}$, finishing the proof. ☐

Admissible extensions exist, according to the following proposition.

Proposition 3.3. *There is a σ-admissible almost invertible extension $\mathcal{L}$ of L_η to $\widetilde{\mathcal{X}}$. Furthermore, if L_η admits a nonzero section, there is such an extension whose restriction to X^ℓ has an effective associated divisor in* $\mathrm{Div}(H)$.

Proof. The first statement is essentially a restatement of the results proved in [AE20a, Section 2]. Indeed, it is enough to see that any divisor on H is linearly equivalent to a G-admissible divisor. This is proved in [AE20a, Thm. 2.10]. As for the second statement, first observe that for any almost invertible extension $\mathcal{L}$ of L_η, any nonzero section of L_η extends to a section of $\mathcal{L}$ with a nonzero restriction to $\mathcal{L}|_{X^\ell}$. For each component X_v of X^ℓ, this section vanishes to a certain finite order, say $-f(v)$, on X_v. So it induces a section of $\mathcal{M} := \mathcal{L} \otimes \mathcal{T}_f$ whose zero scheme is finite over B. In particular, the divisor $D \in \mathrm{Div}(H)$ associated to $\mathcal{M}|_{X^\ell}$ is effective. Hence $t^D_e \geq 0$ and thus $\delta_e(0; t^D) := \lfloor \frac{t^D_e}{\ell_e} \rfloor \geq 0$ for each $e \in \mathbb{E}(G)$. The statement follows now from [AE20a, Prop. 2.9], which yields that the G-admissible divisor $D' = D + \mathrm{div}(h)$ is effective, where $h \in C^0(H, \mathbb{Z})$ is the canonical extension of the zero function in $C^0(G, \mathbb{Z})$ with respect to D. ☐

Special twisters allow us to pass from one σ-admissible almost invertible sheaf on $\widetilde{\mathcal{X}}$ to any other with the same restriction to X_η.

First, we introduce notation. By [AE20a], Prop. 2.7, for each divisor $D \in \mathrm{Div}(H)$ and each $f \in C^0(G, \mathbb{Z})$, there is a unique extension $\widetilde{f} \in C^0(H, \mathbb{Z})$ of f such that $D + \mathrm{div}(\widetilde{f})$ is G-admissible. The function $\widetilde{f}$ is called the *canonical extension* of f with respect to D. Thus, for each almost invertible sheaf $\mathcal{L}$ on $\widetilde{\mathcal{X}}$ with D as the associated divisor to $\mathcal{L}|_{X^\ell}$, the sheaf $\mathcal{L}(\widetilde{f})$ is σ-admissible. As $\widetilde{f}$ depends in this case on $\mathcal{L}$ and f, we abuse the notation by setting $\mathcal{L}(f) := \mathcal{L}(\widetilde{f})$.

We will often deal with the special case where $D \in \mathrm{Div}(H)$ is G-admissible and $f = \chi_v \in C^0(G, \mathbb{Z})$ for a vertex $v \in V$. In this case we denote $\widetilde{f}$ by $f_{D,v}$. We have that $f_{D,v}(z_i^e) = 1$ for each $e \in \mathbb{E}$ with $\mathrm{h}_e = v$ and each $i = \ell_e - t_e^D, \dots, \ell_e$, whereas $f_{D,v}(w) = 0$ for all other vertices $w \in V(H)$. Also, for a σ-admissible almost invertible sheaf $\mathcal{L}$ on $\widetilde{\mathcal{X}}$ with D as the associated divisor to $\mathcal{L}|_{X^\ell}$, we denote $\mathcal{L}(f)$ by $\mathcal{M}_v(\mathcal{L})$. The divisor on H associated to $\mathcal{M}_v(\mathcal{L})|_{X^\ell}$ is the G-admissible chip firing move of D at v, denoted $M_v(D)$ in [AE20a, Subsection 2.6]. So the notations are coherent.

Proposition 3.4. *Let $\mathcal{L}$ be a σ-admissible almost invertible sheaf on $\widetilde{\mathcal{X}}$. Then:*

1. *For each pair of vertices $v, w \in V(G)$, we have $\mathcal{M}_v(\mathcal{M}_w(\mathcal{L})) = \mathcal{M}_w(\mathcal{M}_v(\mathcal{L}))$.*
2. *Enumerating the vertices of G as $v_1, \dots, v_n$, we have*
$$\mathcal{M}_{v_1}(\mathcal{M}_{v_2}(\cdots \mathcal{M}_{v_n}(\mathcal{L}) \cdots)) \cong \mathcal{L}.$$
3. *For each σ-admissible almost invertible $\mathcal{M}$ on $\widetilde{\mathcal{X}}$ with the same restriction to X_η as $\mathcal{L}$, there exists a sequence $v_1, \dots, v_m$ of vertices of G such that*
$$\mathcal{M} \cong \mathcal{M}_{v_1}(\mathcal{M}_{v_2}(\cdots \mathcal{M}_{v_m}(\mathcal{L}) \cdots)).$$
Furthermore, the sequence is unique up to reordering the vertices and adding or subtracting all the vertices of G.

Proof. Statements (1) and (2) are easy to check, and follow from [AE20a], Prop. 2.11. The existence part in Statement (3) follows from Prop. 2.14 in loc. cit.

As for the uniqueness part in Statement (3), the first two statements yield that the mentioned operations to a sequence $v_1, \dots, v_m$ do not change the resulting sheaf. In addition, the proof of Prop. 2.14 in loc. cit. shows that $D' = D + \mathrm{div}(h)$, where D (resp. D') is the divisor in H associated to $\mathcal{L}|_{X^\ell}$ (resp. $\mathcal{M}|_{X^\ell}$), and $h \in C^0(H, \mathbb{Z})$ is the canonical extension of $f := \chi_{v_1} + \cdots + \chi_{v_m}$ with respect to D. Thus, for any other sequence $v'_1, \dots, v'_p$ of vertices of G with the same property as $v_1, \dots, v_m$, we have $\mathrm{div}(h) = \mathrm{div}(h')$, where h' is the canonical extension of $f' := \chi_{v'_1} + \cdots + \chi_{v'_p}$ with respect to D. Then $h' - h$ is constant. Since $h' - h$ extends $f' - f$, it follows that $f' - f$ is constant, finishing the proof. ☐

Let $\mathcal{L}$ be a σ-admissible almost invertible sheaf on $\widetilde{\mathcal{X}}$. Set $\mathcal{I}(f) := \sigma_*\mathcal{L}(f)$ for each $f \in C^0(G, \mathbb{Z})$. It will be useful to have a direct interpretation of the $\mathcal{I}(f)$. It is given by Proposition 3.5 below, from which we see how to obtain the $\mathcal{I}(f)$ recursively from $\mathcal{I} := \sigma_*\mathcal{L}$.

Proposition 3.5. *Notations as above, for each $v \in V(G)$ and $f \in C^0(G, \mathbb{Z})$, the sheaf $\mathcal{I}(f)$ is the kernel of the surjection*

$$\mathcal{I}(f + \chi_v) \longrightarrow \mathcal{I}(f + \chi_v)_v, \tag{3.1}$$

where the sheaf to the right is the torsion-free sheaf generated on X_v by $\mathcal{I}(f + \chi_v)$.

Proof. The kernel of the surjection (3.1) is a relatively torsion-free, rank-1 sheaf on $\mathcal{X}/B$ by an argument analogous to the one found in [Lan75], Prop. 6, p. 100; see [Est01], Section 3. On the other hand, there is a natural exact sequence on $\widetilde{\mathcal{X}}$,

$$0 \longrightarrow \mathcal{L}(f) \longrightarrow \mathcal{L}(f+\chi_v) \longrightarrow \mathcal{L}(f+\chi_v)|_Y \longrightarrow 0, \tag{3.2}$$

where Y is the subcurve of X^ℓ which is the union of X_v and the components Z_j^e for all $e \in \mathbb{E}$ with $\mathrm{h}_e = v$ and $j = \ell_e - t_e^D + 1, \dots, \ell_e - 1$, where $D \in \mathrm{Div}(H)$ is the divisor associated to $\mathcal{L}(f+\chi_v)|_{X^\ell}$. Since $\mathcal{L}(f+\chi_v)$ has degree 0 on these Z_j^e, we have

$$\sigma_*(\mathcal{L}(f+\chi_v)|_Y) = \mathcal{L}(f+\chi_v)|_{X_v} = \mathcal{I}(f+\chi_v)_v,$$

and the surjection in (3.2) restricts to the surjection (3.1). It follows that $\sigma_*\mathcal{L}(f)$ is the kernel of (3.1), as claimed. □

3.2. **Generalized enriched structures.** We apply the above recursive interpretation to describe the $\mathcal{O}_{\mathcal{X}}(f)$. (Here $\mathcal{O}_{\mathcal{X}}(f) := \sigma_*\mathcal{O}_{\widetilde{\mathcal{X}}}(f)$ for each $f \in C^0(G,\mathbb{Z})$.) That interpretation allows us to view $\mathcal{O}_{\mathcal{X}}(f)$ as a sheaf of fractional ideals of $\mathcal{X}$, which we will describe below.

In her thesis [Mai98], Mainò defined the notion of enriched structures over X, and constructed the moduli space of enriched curves, that is, stable curves with enriched structures. Under the interpretation given by the second author and Medeiros [EM02], an enriched structure on X arises from a smoothing $\mathcal{X} \to B$ with regular total space $\mathcal{X}$, thus $\ell_e = 1$ for each $e \in E(G)$, as the group homomorphism $L\colon C^0(G,\mathbb{Z}) \to \mathrm{Pic}^0(X)$ given by $L(f) := \mathcal{O}_{\mathcal{X}}(f)|_X$ for each $f \in C^0(G,\mathbb{Z})$.

In the general case of higher singularity degrees, we may thus see the $\mathcal{O}_{\mathcal{X}}(f)|_X$ as part of a generalized enriched structure over X; the precise definition is given in Definition 4.1. Mainò constructed in loc. cit. a quasiprojective variety parameterizing enriched curves over the moduli space of stable curves. One of our goals, to be pursued in a subsequent work with the contributions given here, is to compactify this variety in a meaningful way. (We refer to recent work by Biesel and Holmes [BH16] for a compactification of Mainò's moduli space following a different approach.)

Let V/M be the versal deformation of X. As explained in [DM69], pp. 79–81 and reviewed in [EM02], pp. 288–9, we have $M = \mathrm{Spec}(R)$, where R is the power series ring over κ in the variables t_e, for $e \in E^\circ$, and $s_1, \dots, s_p$, for a certain integer p. Furthermore, the variables can be chosen so that for each $e \in E^\circ$ we have an isomorphism of R-algebras,

$$\psi_e\colon \widehat{\mathcal{O}}_{V,N_e} \to R[[z_e,w_e]]/(z_e w_e - t_e).$$

The versal deformation comes with an identification of the special fiber of V/M with X, thus we may view $X \subseteq V$. We may assume that $z_e = 0$ corresponds to the component X_v and $w_e = 0$ to X_u, where $v := \mathrm{h}_e$ and $u := \mathrm{t}_e$. Letting $\hat{z}_e$ and $\hat{w}_e$ denote the elements of $\widehat{\mathcal{O}}_{V,N_e}$ corresponding to z_e and w_e, we have that $\hat{z}_e$ restricts to a local parameter z_e of $\widehat{\mathcal{O}}_{X_u,N_e}$, whereas $\hat{w}_e$ restricts to a local parameter w_e of $\widehat{\mathcal{O}}_{X_v,N_e}$.

As $\mathcal{X}/B$ is a deformation of X, there is a natural Cartesian diagram factoring the inclusion of X in V:

$$\begin{array}{ccccc} X & \longrightarrow & \mathcal{X} & \longrightarrow & V \\ \big\downarrow & & {\scriptstyle\pi}\big\downarrow & & \big\downarrow \\ \mathrm{Spec}(\kappa) & \longrightarrow & B & \longrightarrow & M \end{array}$$

The map $B \to M$ sends via pullback t_e to $a_e t^{\ell_e}\xi_e$, for certain $a_e \in \kappa$ and $\xi_e \in \kappa[[t]]$ with $\xi_e(0) = 1$. If we denote by $\tilde{z}_e$ and $\tilde{w}_e$ the pullbacks under the map $\mathcal{X} \to V$ of $\hat{z}_e$ and $\hat{w}_e$, respectively, we have $\tilde{z}_e\tilde{w}_e = a_e t^{\ell_e}\xi_e$ in $\widehat{\mathcal{O}}_{\mathcal{X},N_e}$ for each $e \in E^{\mathrm{o}}$.

Now, for each $f \in C^0(G,\mathbb{Z})$ define

$$\delta_e(f) := \left\lfloor \frac{f(v)-f(u)}{\ell_e} \right\rfloor \quad \text{for each } e = uv \in \mathbb{E}.$$

We claim that, for each $e = uv \in E^{\mathrm{o}}$, the sheaf of fractional ideals $\mathcal{O}_{\mathcal{X}}(f)$ is generated locally analytically at N_e by

$$(t^{-f(u)}\tilde{z}_e^{-\delta_e(f)}, t^{-f(v)}\tilde{w}_e^{-\delta_{\overline{e}}(f)}). \tag{3.3}$$

Indeed, suppose $f(v) \geq f(u)$. As $t = 0$ gives X, we have that

$$\mathcal{O}_{\mathcal{X}}(f) = t^{-f(v)}\mathcal{O}_{\mathcal{X}}\big(f - f(v)\chi_V\big), \text{ where } \chi_V := \sum \chi_v.$$

And, as shown in [CEG08], p. 14, locally analytically at N_e the sheaf $\mathcal{O}_{\mathcal{X}}(f - f(v)\chi_V)$ is the ideal generated by $(\tilde{w}_e^{q+1}, \tilde{w}_e^q t^r)$, where q and r are the quotient and the remainder of the Euclidean division of $f(v)-f(u)$ by ℓ_e. In other words, $q = \delta_e(f)$ and $r = f(v)-f(u)-\ell_e q$. It is now easy to check, using $\tilde{z}_e\tilde{w}_e = a_e t^{\ell_e}\xi_e$, that $t^{-f(v)}(\tilde{w}_e^{q+1}, \tilde{w}_e^q t^r)$ is the fractional ideal (3.3). An analogous argument works when $f(v) \leq f(u)$. Notice that

$$t^{-f(v)}\tilde{w}_e^{-\delta_{\overline{e}}(f)} = (a_e\xi_e)^{\delta_e(f)} t^{-f(u)}\tilde{z}_e^{-\delta_e(f)}$$

if ℓ_e divides $f(v) - f(u)$; in particular, $\mathcal{O}_{\mathcal{X}}(f)$ is principal at N_e in this case.

For each $f \in C^0(G,\mathbb{Z})$, the sheaf of fractional ideals $t^{f(v)}\mathcal{O}_{\mathcal{X}}(f)$ of $\mathcal{X}$ generates a sheaf of fractional ideals $\mathcal{J}_v(f)$ of X_v for each $v \in V(G)$. Given the above description, for each $e = uv \in E^{\mathrm{o}}$ we have that, locally analytically at N_e, the element $\tilde{z}_e^{-\delta_e(f)}$ is mapped to $z_e^{-\delta_e(f)}$ in $\mathcal{J}_u(f)$, whereas $\tilde{w}_e^{-\delta_{\overline{e}}(f)}$ is mapped to $w_e^{-\delta_{\overline{e}}(f)}$ in $\mathcal{J}_v(f)$. On the other hand, if ℓ_e does not divide $f(v) - f(u)$, then $t^{f(u)-f(v)}\tilde{w}_e^{-\delta_{\overline{e}}(f)}$ and $t^{f(v)-f(u)}\tilde{z}_e^{-\delta_e(f)}$ are mapped to 0 in $\mathcal{J}_u(f)$ and $\mathcal{J}_v(f)$, respectively. It follows that

$$\mathcal{J}_v(f) = \mathcal{O}_{X_v}\Big(\sum_{\substack{e\in\mathbb{E}\\ \mathrm{h}_e=v}} \delta_{\overline{e}}(f)N_e\Big)$$

for each $v \in V(G)$.

As in [EM02], p. 292, we may consider the image $\mathcal{J}(f)$ of the composition

$$\mathcal{O}_{\mathcal{X}}(f) \longrightarrow \textstyle\bigoplus_{v\in V} t^{f(v)}\mathcal{O}_{\mathcal{X}}(f) \longrightarrow \bigoplus_{v\in V} \mathcal{J}_v(f).$$

Then $\mathcal{J}(f)$ is a sheaf of fractional ideals of X isomorphic to $\mathcal{O}_{\mathcal{X}}(f)|_X$. Clearly, $\mathcal{J}(f)$ generates $\mathcal{J}_v(f)$ for each $v \in V(G)$. As the $\mathcal{J}_v(f)$ are described above, to describe the subsheaf $\mathcal{J}(f)$ we need only describe it locally analytically at the N_e where $\mathcal{J}(f)$ is invertible, that is, for $e = uv \in E^{\mathbf{o}}$ such that ℓ_e divides $f(v) - f(u)$.

Let $K := \prod_{v \in V} \kappa(X_v)$, the product of the fields of rational functions of the irreducible components of X. And put $\mathcal{K} := \prod_{v \in V} \mathcal{K}_v$, the product of the constant sheaves of rational functions of the irreducible components of X. For each $e \in E$ and $v \in e$, let $\widehat{K}_{v,e}$ denote the field of fractions of $\widehat{\mathcal{O}}_{X_v,N_e}$; it contains the field of fractions of $\mathcal{O}_{X_v,N_e}$, which is $\kappa(X_v)$. We may thus view any local description of a sheaf of fractional ideals of X at N_e in $\widehat{K}_{u,e} \times \widehat{K}_{v,e}$, for each $e = uv \in E^{\mathbf{o}}$.

In particular, for each $f \in C^0(G, \mathbb{Z})$ and each $e = uv \in E^{\mathbf{o}}$, the sheaf of fractional ideals $\mathcal{J}(f)$ is generated at N_e locally analytically in $\widehat{K}_{u,e} \times \widehat{K}_{v,e}$ by $(a_e^{\delta_e(f)} z_e^{-\delta_e(f)}, w_e^{-\delta_{\overline{e}}(f)})$ if ℓ_e divides $f(v) - f(u)$ and by $((z_e^{-\delta_e(f)}, 0), (0, w_e^{-\delta_{\overline{e}}(f)}))$ otherwise.

For each $e = uv \in E^{\mathbf{o}}$, we may use z_e to establish isomorphisms $\alpha_{e,m} \colon \mathcal{O}_{X_u}(mN_e)|_{N_e} \cong \kappa$ for each $m \in \mathbb{Z}$, by analytically identifying $\mathcal{O}_{X_u}(mN_e)$ at N_e with the fractional ideal generated by z_e^{-m}, and taking z_e^{-m} to 1. Doing the same for w_e, we get isomorphisms $\beta_{e,m} \colon \mathcal{O}_{X_v}(mN_e)|_{N_e} \cong \kappa$. Under these isomorphisms the sheaf $\mathcal{J}(f)$ is the subsheaf of $\bigoplus_{v \in V} \mathcal{J}(f)_v$ given locally at N_e, for each $e = uv \in E^{\mathbf{o}}$ such that ℓ_e divides $f(v) - f(u)$, as the kernel of the surjection

$$\mathcal{O}_{X_u}(\delta_e(f)N_e)|_{N_e} \oplus \mathcal{O}_{X_v}(\delta_{\overline{e}}(f)N_e)|_{N_e} \xrightarrow[\cong]{(\alpha_{e,\delta_e(f)}, \beta_{e,\delta_{\overline{e}}(f)})} \kappa \oplus \kappa \xrightarrow{(1, -a_e^{\delta_e(f)})} \kappa.$$

Notice that as $\mathcal{X}/B$ varies among smoothings of X with singularity degree function ℓ, the a_e on which the $\mathcal{O}_{\mathcal{X}}(f)$ ultimately depends vary freely. The $\mathcal{O}_{\mathcal{X}}(f)$ depends thus on the free choice of a homomorphism $a \colon C^1(G, \mathbb{Z}) \to \mathbf{G_m}$. Also, by Nakayama Lemma, giving an isomorphism $\alpha_e \colon \mathcal{O}_{X_v}(N_e)|_{N_e} \cong \kappa$ for each $e \in E$ and $v \in e$ is the same as choosing an analytic local parameter for X_v at N_e. We have chosen above, as a result of considering a versal deformation of X, certain analytic local parameters z_e and w_e for X_u and X_v at N_e for each $e = uv \in E^{\mathbf{o}}$. Different choices z'_e and w'_e can be expressed as power series $z'_e = \tau_e z_e + \cdots$ and $w'_e = \sigma_e w_e + \cdots$ for $\tau_e, \sigma_e \in \kappa$, and we would obtain the same subsheaf $\mathcal{J}(f)$ of $\bigoplus_{v \in V} \mathcal{J}(f)_v$ by replacing the a_e by $\tau_e a_e \sigma_e$ for each $e \in E^{\mathbf{o}}$.

4. Degenerations of line bundles II

Let B be the spectrum of the power series ring $\kappa[[t]]$. Let $\pi \colon \mathcal{X} \to B$ be a smoothing of X. Let $\ell \colon E \to \mathbb{N}$ be the function assigning to e the singularity degree ℓ_e of $\mathcal{X}$ at N_e. Let $\sigma \colon \widetilde{\mathcal{X}} \to \mathcal{X}$ be the Cartier reduction of $\mathcal{X}$. Put $\widetilde{\pi} := \pi\sigma$. Let X^ℓ be the special fiber of $\widetilde{\pi}$. Keep the remaining notation of Section 3. As f runs in $C^0(G, \mathbb{Z})$, the sheaf $\mathcal{I}(f)|_X$ runs through what we call *limits* of L_η. These are not all of what we call *stable limits* though. To obtain all of them, we consider base changes $t \mapsto t^n$ for positive integers n, as explained below.

For each $n \in \mathbb{N}$, let $\mu_n \colon B \to B$ be the base change map given by $t \mapsto t^n$, and let $\pi^n \colon \mathcal{X}^n \to B$ be the base extension of π. Let $\sigma^n \colon \widetilde{\mathcal{X}}^n \to \mathcal{X}^n$ be the Cartier reduction. If $m \in \mathbb{N}$ divides n, then we have the following commutative diagram of maps:

$$\begin{array}{ccccc} \widetilde{\mathcal{X}}^n & \xrightarrow{\sigma^n} & \mathcal{X}^n & \xrightarrow{\pi^n} & B \\ \downarrow & & \downarrow & & \downarrow \mu_{n/m} \\ \widetilde{\mathcal{X}}^m & \xrightarrow{\sigma^m} & \mathcal{X}^m & \xrightarrow{\pi^m} & B. \end{array}$$

The square to the right is Cartesian, so the geometric fibers of $\mathcal{X}^n/B$ are the same as those of $\mathcal{X}/B$ but the one to the left is not. The singularity degree of $\mathcal{X}^n$ at N_e is now $n\ell_e$ for each $e \in E(G)$, so the special fiber of $\widetilde{\mathcal{X}}^n/B$ is $X^{n\ell}$. We have the same configuration as before. The only difference is that ℓ is replaced by $n\ell$.

Given an invertible sheaf L_η on the generic fiber of π^m, for given $m \in \mathbb{N}$, and given $n \in \mathbb{N}$ divisible by m, we may pull L_η back to an invertible sheaf L_η^n on the generic fiber of π^n, and consider its extensions to $\widetilde{\mathcal{X}}^n$, as we did before for the case $n = 1$. Given an extension $\mathcal{L}$ of L_η to $\widetilde{\mathcal{X}}^m$ we may pull it back to an extension $\mathcal{L}^n$ of L_η^n to $\widetilde{\mathcal{X}}^n$. If $\mathcal{L}$ is σ^m-admissible, then $\mathcal{L}^n$ is σ^n-admissible. We will also denote by $\mathcal{I}^n$ the pullback of a relative torsion-free rank-one sheaf $\mathcal{I}$ on $\mathcal{X}^m/B$; it is one on $\mathcal{X}^n/B$. If $\mathcal{I} = \sigma^m_*\mathcal{L}$ then $\mathcal{I}^n = \sigma^n_*\mathcal{L}^n$.

As before, to each $f \in C^0(G, \mathbb{Z})$ and each σ^m-admissible almost invertible extension $\mathcal{L}$ of L_η to $\widetilde{\mathcal{X}}^m$, we associate a σ^n-admissible extension $\mathcal{L}^n(f)$ of L_η^n. Again by [EP16], Thm. 3.1, p. 63, the pushforward

$$\mathcal{I}^n(f) := \sigma^n_*\mathcal{L}^n(f)$$

is a relative torsion-free rank-one sheaf on $\mathcal{X}^n/B$. The notation is consistent, as $\mathcal{I}^n(0) = \mathcal{I}^n$.

Definition 4.1. Let $m \in \mathbb{N}$ and L_η be an invertible sheaf on the generic fiber of $\mathcal{X}^m/B$. Let $\mathcal{L}$ be a σ^m-admissible almost invertible extension of L_η to $\widetilde{\mathcal{X}}^m$. For each $n \in \mathbb{N}$ divisible by m and $f \in C^0(G, \mathbb{Z})$, let

$$L^n(f) := \mathcal{L}^n(f)|_{X^{n\ell}} \quad \text{and} \quad I^n(f) := \mathcal{I}^n(f)|_X.$$

We call

$$\mathfrak{I} := \{I^n(f) \,|\, n \in m\mathbb{N},\ f \in C^0(G, \mathbb{Z})\}$$

the collection of *stable limits* of L_η. In case L_η is the structure sheaf of the generic fiber of $\mathcal{X}/B$, we call $\mathfrak{I}$ a *generalized enriched structure*.

If ℓ is the constant function 1, and L_η is the structure sheaf of the generic fiber of $\mathcal{X}/B$, Mainò called an enriched structure the subset $\{I^1(\chi_v) \,|\, v \in V(G)\}$. In this case, the full set $\mathfrak{I}$ is obtained from this subset by tensor products and degeneration.

In general, $\mathfrak{I}$ is a set of torsion-free, rank-one sheaves of degree equal to that of L_η. If κ has characteristic 0, then the field of Puiseux series is the algebraic closure of the field of Laurent series, the field of fractions of $\kappa[[t]]$, and we may thus think of $\mathfrak{I}$ as the collection of all the limits of the pullback of L_η to the geometric generic fiber of $\mathcal{X}/B$.

For each $d \in \mathbb{Z}$, denote by $\mathbf{J}^d$ the stack parameterizing torsion-free, rank-one sheaves of degree d on X; see Section 2. If L_η has degree d, we may view $\mathfrak{I}$ as a subset of $\mathbf{J}^d$. The main theorem of this section, Theorem 4.7, asserts that the collection $\mathfrak{I}$ is the support of a closed substack of $\mathbf{J}^d$, and describes thoroughly the structure of this substack.

We need a few preliminary results though. Recall the notation introduced in [AE20a] and [AE20b]: For each $n \in \mathbb{N}$, each $\mathfrak{m} \in C^1(G, \mathbb{Q})$ and $f \in C^0(G, \mathbb{Z})$, we set

$$\delta^{\mathfrak{m},n}_{\ell,e}(f) := \Big\lfloor \frac{f(\mathrm{h}_e) - f(\mathrm{t}_e) + n\mathfrak{m}_e}{n\ell_e} \Big\rfloor \quad \text{for each } e \in \mathbb{E},$$

and let $\mathfrak{d}^{\mathfrak{m},n}_{\ell,f} \in C^1(G, \frac{1}{2}\mathbb{Z})$ be defined by putting

$$\mathfrak{d}^{\mathfrak{m},n}_{\ell,f}(e) = \frac{1}{2}\Big(\delta^{\mathfrak{m},n}_{\ell,e}(f) - \delta^{\mathfrak{m},n}_{\ell,\overline{e}}(f)\Big)$$

for each $e \in \mathbb{E}$. If ℓ is fixed we drop the subscript ℓ.

In addition, let H^n denote the graph obtained from the dual graph of $X^{n\ell}$ by removing the self loops. For each $e \in E(G)$, we let $Z^{e,n} \subseteq X^{n\ell}$ denote the chain of rational smooth curves of length $n\ell_e - 1$ lying over the node N_e of X. For each $e = uv \in \mathbb{E}$, order the components $Z^{e,n}_1, \dots, Z^{e,n}_{n\ell_e - 1}$ of $Z^{e,n}$ by assuming that $Z^{e,n}_i$ intersects $Z^{e,n}_{i+1}$ for $i = 1, \dots, n\ell_e - 2$ and $Z^{e,n}_1$ intersects X_u at N_e (whence $Z^{e,n}_{n\ell_e - 1}$ intersects X_v at N_e). For convenience, we set $Z^{e,n}_0 := X_u$ and $Z^{e,n}_{n\ell_e} := X_v$. We denote by $z^{e,n}_j$ the vertex of H^n associated to $Z^{e,n}_j$ for each $j = 0, \dots, n\ell_e$. As with H we may view $V(G)$ as a subset of $V(H^n)$, and more generally $V(H^m)$ as a subset of $V(H^n)$ for each $m \in \mathbb{N}$ with $m|n$.

If $n = 1$, the overscript n is dropped throughout.

Proposition 4.2. *Let $n \in \mathbb{N}$ and D be a G-admissible divisor of H^n. For each $e \in E^{\mathfrak{o}}$, put*

$$\mathfrak{m}_e := \frac{1}{n} \sum_{i=1}^{n\ell_e - 1} iD(z^{\overline{e},n}_i),$$

and let $\mathfrak{m}$ be the element of $C^1(G, \mathbb{Q})$ satisfying this. Then, for each $f \in C^0(G, \mathbb{Z})$ and $v \in V$, we have

$$\mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{f}\big)|_{X_v} \cong \mathcal{O}_{X_v}\Big(\sum_{\substack{e \in \mathbb{E} \\ \mathrm{t}_e = v}} \lfloor \mathfrak{d}^{\mathfrak{m},n}_f(e) \rfloor N_e + \sum_{\substack{e \in \mathbb{E} \\ \mathrm{t}_e = v \\ \mathfrak{m}_e < 0}} N_e\Big),$$

where $\widetilde{f} \in C^0(H^n, \mathbb{Z})$ is the canonical extension of f with respect to D.

(Recall that $\widetilde{f}$ is the unique extension of f to $V(H^n)$ such that $D + \mathrm{div}(\widetilde{f})$ is G-admissible; see [AE20a, Subsection 2.5].)

Proof. Since D is G-admissible, 0 is the canonical extension of 0 with respect to D. Clearly, $\mathcal{O}_{\widetilde{\mathcal{X}}^n}(0)$ is trivial. On the other hand, if $e \in \mathbb{E}$ is such that $\mathrm{t}_e = v$, then $\mathfrak{d}^{\mathfrak{m},n}_0(e) = 0$ unless $\mathfrak{m}_e \neq 0$. In this case, since $|\mathfrak{m}_e| < \ell_e$, if $\mathfrak{m}_e > 0$ then $\mathfrak{d}^{\mathfrak{m},n}_0(e) = 1/2$. And if $\mathfrak{m}_e < 0$ then $\mathfrak{d}^{\mathfrak{m},n}_0(e) = -1/2$. In any case, the proposition holds for $f = 0$.

Observe that f can be replaced by $f+b\chi_V$ for any $b\in\mathbb{Z}$, where $\chi_V:=\sum\chi_v$. We may thus assume $f\geq 0$. We argue by induction on $\deg(f)$, the initial case, $\deg(f)=0$, having just been considered.

Suppose first that $f=\chi_w$ for a certain $w\in V(G)$. Then

$$\mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{f}\big)=\mathcal{O}_{\widetilde{\mathcal{X}}^n}\Big(X_w+\sum_{\substack{e\in\mathbb{E}\\ \mathrm{t}_e=w}}\sum_{i=1}^{r_e}Z_i^{e,n}\Big),$$

where $r_e=0$ if $\mathfrak{m}_e=0$; otherwise r_e is the only integer i such that $D(z_i^{e,n})=1$.

If $v\neq w$ then

$$\mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{f}\big)|_{X_v}\cong\mathcal{O}_{X_v}\Big(\sum N_e\Big),$$

where the sum is over the $e\in\mathbb{E}$ such that $e=wv$ and either $n\ell_e=1$ or $D(z_1^{\bar{e},n})=1$. For such e we have that $\mathfrak{d}_f^{\mathfrak{m},n}(\bar{e})=1$ and $\mathfrak{m}_{\bar{e}}=0$ if $n\ell_e=1$; and if $n\ell_e>1$ we have that $\mathfrak{m}_{\bar{e}}=-1/n$ and $\mathfrak{d}_f^{\mathfrak{m},n}(\bar{e})=0$ if $e\in E^{\mathfrak{o}}$, whereas $\mathfrak{m}_{\bar{e}}=(n\ell_e-1)/n$ and $\mathfrak{d}_f^{\mathfrak{m},n}(\bar{e})=1$ if $\bar{e}\in E^{\mathfrak{o}}$. In all cases, N_e is in the support of

$$U:=\sum_{\substack{e\in\mathbb{E}\\ \mathrm{t}_e=v}}\lfloor\mathfrak{d}_f^{\mathfrak{m},n}(e)\rfloor N_e+\sum_{\substack{e\in\mathbb{E}\\ \mathrm{t}_e=v\\ \mathfrak{m}_e<0}}N_e$$

with multiplicity 1. Furthermore, suppose N_e is in the support of U for a certain $e\in\mathbb{E}$. Then $v\in e$ and we may suppose $\mathrm{h}_e=v$. If $\mathrm{t}_e\neq w$, then, as before, $\lfloor\mathfrak{d}_f^{\mathfrak{m},n}(\bar{e})\rfloor=0$ if $\mathfrak{m}_{\bar{e}}\geq 0$ and $\lfloor\mathfrak{d}_f^{\mathfrak{m},n}(\bar{e})\rfloor=-1$ otherwise. Since N_e is in the support of U we must then have $e=wv$. Suppose $n\ell_e>1$. If $\mathfrak{m}_{\bar{e}}\geq 0$, since $|\mathfrak{m}_{\bar{e}}|<\ell_e$, we have $\lfloor\mathfrak{d}_f^{\mathfrak{m},n}(\bar{e})\rfloor=0$ unless $n\mathfrak{m}_{\bar{e}}=n\ell_e-1$, in which case $\lfloor\mathfrak{d}_f^{\mathfrak{m},n}(\bar{e})\rfloor=1$. But this can only happen if $\bar{e}\in E^{\mathfrak{o}}$ and $D(z_1^{\bar{e},n})=1$. On the other hand, if $\mathfrak{m}_{\bar{e}}<0$ then $\lfloor\mathfrak{d}_f^{\mathfrak{m},n}(\bar{e})\rfloor=-1$ unless $n\mathfrak{m}_{\bar{e}}=-1$, in which case $\lfloor\mathfrak{d}_f^{\mathfrak{m},n}(\bar{e})\rfloor=0$. But this can only happen if $e\in E^{\mathfrak{o}}$ and $D(z_1^{\bar{e},n})=1$. At any rate, the statement of the proposition holds.

If $v=w$ then

$$\mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{f}\big)|_{X_v}\cong\mathcal{O}_{X_v}\Big(-\sum N_e\Big),$$

where the sum is over the $e\in\mathbb{E}$ such that $\mathrm{t}_e=v$ and $\mathfrak{m}_e=0$. For such e we have that $\mathfrak{d}_f^{\mathfrak{m},n}(e)=-1$, thus N_e is in the support of U with multiplicity -1. Furthermore, suppose N_e is in the support of U for a certain $e\in\mathbb{E}$. As before, we may suppose $\mathrm{t}_e=v$. If $\mathfrak{m}_e>0$ then, since $|\mathfrak{m}_{\bar{e}}|<\ell_e$, we have $\lfloor\mathfrak{d}_f^{\mathfrak{m},n}(\bar{e})\rfloor=0$. Since N_e is in the support of U we must then have $\mathfrak{m}_e\leq 0$. But if $\mathfrak{m}_e<0$ then $\lfloor\mathfrak{d}_f^{\mathfrak{m},n}(\bar{e})\rfloor=-1$. Thus, since N_e is in the support of U we must have $\mathfrak{m}_e=0$. It follows that the statement of the proposition holds.

In the general case, we may assume that $\deg(f)>0$ and let $w\in V(G)$ such that $f(w)>0$. Let $g:=f-\chi_w$. Then

$$\mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{f}\big)\cong\mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{g}\big)\otimes\mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{h}\big)$$

where $\widetilde{g} \in C^0(H^n, \mathbb{Z})$ is the canonical extension of g with respect to D, where $D' = D + \mathrm{div}(\widetilde{g})$, and where $\widetilde{h} \in C^0(H^n, \mathbb{Z})$ is the canonical extension of χ_w with respect to D'. The isomorphism holds by [AE20a, Prop. 2.11]. Now, by induction,

$$\mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{g}\big)|_{X_v} \cong \mathcal{O}_{X_v}\Big(\sum_{\substack{e\in\mathbb{E}\\ \mathrm{t}_e=v}} \lfloor \mathfrak{d}_g^{\mathfrak{m},n}(e)\rfloor N_e + \sum_{\substack{e\in\mathbb{E}\\ \mathrm{t}_e=v\\ \mathfrak{m}_e<0}} N_e\Big).$$

Thus we need only prove that

$$\mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{h}\big)|_{X_v} \cong \mathcal{O}_{X_v}\Big(\sum_{\substack{e\in\mathbb{E}\\ \mathrm{t}_e=v}} \big(\lfloor \mathfrak{d}_f^{\mathfrak{m},n}(e)\rfloor N_e - \lfloor \mathfrak{d}_g^{\mathfrak{m},n}(e)\rfloor\big) N_e\Big),$$

or, using what we have proved above, that for each $e \in \mathbb{E}$ with $\mathrm{t}_e = v$ we have that

$$\lfloor \mathfrak{d}_f^{\mathfrak{m},n}(e)\rfloor - \lfloor \mathfrak{d}_g^{\mathfrak{m},n}(e)\rfloor = \begin{cases} 1 & \text{if } \mathrm{h}_e = w \text{ and either } n\ell_e = 1 \text{ or } D'(z_1^{e,n}) = 1; \\ -1 & \text{if } v = w \text{ and } \mathfrak{m}'_e = 0; \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where $\mathfrak{m}' \in C^1(G, \mathbb{Q})$ satisfies

$$\mathfrak{m}'_e := \frac{1}{n}\sum_{i=1}^{n\ell_e - 1} iD'(z_i^{\overline{e},n})$$

for each $e \in E^{\mathfrak{o}}$.

Indeed, if $v \neq w$, then both sides in (4.1) are zero unless $\mathrm{h}_e = w$. Suppose $e = vw$. Then the left-hand side in (4.1) is 1 if $\mathfrak{d}_f^{\mathfrak{m},n}(e) \in \mathbb{Z}$ and 0 otherwise. But $\mathfrak{d}_f^{\mathfrak{m},n}(e) \in \mathbb{Z}$ if and only if either $n\ell_e = 1$ or $D'(z_1^{e,n}) = 1$.

If $v = w$ then the left-hand side in (4.1) is -1 if $\mathfrak{d}_g^{\mathfrak{m},n}(e) \in \mathbb{Z}$ and 0 otherwise. But $\mathfrak{d}_g^{\mathfrak{m},n}(e) \in \mathbb{Z}$ if and only if $\mathfrak{m}'_e = 0$.

In any case, Equation (4.1) follows. ☐

Recall the definitions of $\mathbf{R}$ and the subschemes $Y_{\ell,\mathfrak{m}}^{a,b} \subseteq \mathbf{R}$ from [AE20b], which we review now.

First, as seen in Section 2,

$$\mathbf{R} := \prod_{e\in E^{\mathfrak{o}}} \mathbf{R}_e,$$

where $\mathbf{R}_e$ is the doubly infinite chain of smooth rational curves. We may order the rational curves in the chain, and denote them by $\mathbf{P}_{e,i}$ for $i \in \mathbb{Z}$. We may give them coordinates $(\mathrm{x}_{e,i} : \mathrm{x}_{\overline{e},i})$ such that the point $0_{e,i}$, given by $\mathrm{x}_{e,i} = 0$, is the point of intersection of $\mathbf{P}_{e,i}$ with $\mathbf{P}_{e,i+1}$, and the point $\infty_{e,i}$, given by $\mathrm{x}_{\overline{e},i} = 0$, is the point of intersection of $\mathbf{P}_{e,i}$ with $\mathbf{P}_{e,i-1}$. We can also define $\mathbf{P}_{e,i}$ for each $i \in \mathbb{R} - \mathbb{Z}$ as the point of intersection of $\mathbf{P}_{e,\lfloor i\rfloor}$ with $\mathbf{P}_{e,\lceil i\rceil}$. Then

$$\mathbf{R} = \bigcup_{\alpha\in C^1(G,\mathbb{Z})} \mathbf{P}_\alpha,$$

where, more generally,

$$\mathbf{P}_\alpha := \prod_{e\in E^{\mathfrak{o}}} \mathbf{P}_{e,\alpha_e} \quad \text{for each } \alpha \in C^1\big(G, \frac{1}{2}\mathbb{Z}\big).$$

We see that $\mathbf{P}_\alpha \supseteq \mathbf{P}_\beta$ if and only if $|\beta_e - \alpha_e| \leq \frac{1}{2}$ for each $e \in \mathbb{E}$, with $\alpha_e \in \mathbb{Z}$ whenever $\beta_e \in \mathbb{Z}$. Removing from each $\mathbf{P}_\alpha$ all those $\mathbf{P}_\beta$ contained in it, we obtain the *interior* of $\mathbf{P}_\alpha$, the open subscheme denoted $\mathbf{P}^*_\alpha$. The $\mathbf{P}^*_\alpha$ for $\alpha \in C^1(G, \frac{1}{2}\mathbb{Z})$ form a stratification of $\mathbf{R}$.

Let $a\colon C^1(G,\mathbb{Z}) \to \mathbf{G_m}(\kappa)$ and $b\colon C^1(G,\mathbb{Z}) \to \mathbf{G_m}(\kappa)$ be characters. Let $\ell\colon E \to \mathbb{N}$ be an edge length function and $\mathfrak{m} \in C^1(G,\mathbb{Z})$. To each $f \in C^0(G,\mathbb{Z})$ we associate the subvariety $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$ of $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$ given by the equations

$$\forall \text{ oriented cycle } \gamma \text{ in } G^{\mathfrak{m}}_f,$$

$$\prod_{e\in\bar\gamma\cap E^{\mathfrak{o}}} b_e a_e^{\mathfrak{d}^{\mathfrak{m}}_f(e)} \prod_{e\in\gamma\cap E^{\mathfrak{o}}} \mathrm{x}_{e,\mathfrak{d}^{\mathfrak{m}}_f(e)} \prod_{e\in\bar\gamma\cap E^{\mathfrak{o}}} \mathrm{x}_{\bar e,\mathfrak{d}^{\mathfrak{m}}_f(e)} = \prod_{e\in\gamma\cap E^{\mathfrak{o}}} b_e a_e^{\mathfrak{d}^{\mathfrak{m}}_f(e)} \prod_{e\in\bar\gamma\cap E^{\mathfrak{o}}} \mathrm{x}_{e,\mathfrak{d}^{\mathfrak{m}}_f(e)} \prod_{e\in\gamma\cap E^{\mathfrak{o}}} \mathrm{x}_{\bar e,\mathfrak{d}^{\mathfrak{m}}_f(e)},$$

where $G^{\mathfrak{m}}_f$ is the spanning subgraph of G whose edges are those of G for which $\mathfrak{d}^{\mathfrak{m}}_f$ is an integer. The equation corresponding to γ may also be written in the format:

$$\prod_{e\in\gamma} \mathrm{x}_{e,\mathfrak{d}^{\mathfrak{m}}_f(e^{\mathfrak{o}})} = \prod_{e\in\gamma} b_e a_e^{\mathfrak{d}^{\mathfrak{m}}_f(e^{\mathfrak{o}})} \prod_{e\in\bar\gamma} \mathrm{x}_{e,\mathfrak{d}^{\mathfrak{m}}_f(e^{\mathfrak{o}})}, \tag{4.2}$$

where $e^{\mathfrak{o}} := e$ if $e \in E^{\mathfrak{o}}$ and $e^{\mathfrak{o}} := \overline{e}$ otherwise.

It is clear from (4.2) that $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$ depends rather on the restriction of b to $H^1(G,\mathbb{Z})$. As any character of $H^1(G,\mathbb{Z})$ extends to one of $C^1(G,\mathbb{Z})$, we will assume later that b is rather a character of $H^1(G,\mathbb{Z})$.

We denote by $Y^{a,b}_{\ell,\mathfrak{m}}$ the union of the $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$ for all $f \in C^0(G,\mathbb{Z})$.

There is a natural action of the character group of $C^1(G,\mathbb{Z})$, which we denote by $\mathbf{G}^E_{\mathbf{m}}$, on $\mathbf{R}$: To a character $c\colon C^1(G,\mathbb{Z}) \to \mathbf{G_m}(\kappa)$ and a point p on $\mathbf{P}_\alpha$ with coordinates $(x_{e,\alpha_e}, x_{\bar e,\alpha_e})$, for each $e \in E^{\mathfrak{o}}$, we associate the point on the same $\mathbf{P}_\alpha$ with coordinates $(c_e x_{e,\alpha_e}, x_{\bar e,\alpha_e})$. The homomorphism d^* induces a homomorphism from the character group of $C^0(G,\mathbb{Z})$, which we denote $\mathbf{G}^V_{\mathbf{m}}$, to $\mathbf{G}^E_{\mathbf{m}}$. The induced action on $\mathbf{R}$ of $c \in \mathbf{G}^V_{\mathbf{m}}$ takes the point p to that on the same $\mathbf{P}_\alpha$ with coordinates $(c_v x_{e,\alpha_e}, c_u x_{\bar e,\alpha_e})$ for each $e = uv \in E^{\mathfrak{o}}$. Finally, the degree map induces a "diagonal" injective homomorphism $\mathbf{G_m} \hookrightarrow \mathbf{G}^V_{\mathbf{m}}$, and the induced action of $\mathbf{G_m}$ on $\mathbf{R}$ is trivial. We may thus speak of the action of the quotient $\mathbf{G}^V_{\mathbf{m}}/\mathbf{G_m}$ on $\mathbf{R}$.

It is clear from Equations (4.2) that the action of $\mathbf{G}^V_{\mathbf{m}}/\mathbf{G_m}$ on $\mathbf{R}$ leaves each $Y^{a,b}_{\ell,\mathfrak{m}}$ invariant. We may thus describe each $Y^{a,b}_{\ell,\mathfrak{m}}$ in terms of its orbits. This was done in [AE20b], Thm. 5.3, and we reproduce it and its Cor. 5.4 here for later use.

Theorem 4.3. *Let $\ell\colon E \to \mathbb{N}$ be an edge length function, $\mathfrak{m} \in C^1(G,\mathbb{Z})$, and let*

$$a\colon C^1(G,\mathbb{Z}) \to \mathbf{G_m}(\kappa), \quad b\colon C^1(G,\mathbb{Z}) \to \mathbf{G_m}(\kappa)$$

be characters. For each $n \in \mathbb{N}$ and $f \in C^0(G,\mathbb{Z})$, let p_f^n be the point on $\mathbf{P}_{\mathfrak{d}_f^{\mathfrak{m},n}}$ given by the coordinates

$$(b_e a_e^{\mathfrak{d}_f^{\mathfrak{m},n}(e)} : 1) \quad \text{for each } e \in E^{\mathfrak{o}} \text{ with } \mathfrak{d}_f^{\mathfrak{m},n}(e) \in \mathbb{Z}.$$

Then $Y_{\ell,\mathfrak{m}}^{a,b}$ is the union of the orbits of the p_f^n under the action of $\mathbf{G}_{\mathbf{m}}^V/\mathbf{G}_{\mathbf{m}}$. Furthermore, given $\alpha \in C^1(G, \frac{1}{2}\mathbb{Z})$, we have that $\mathbf{P}_\alpha^* \cap Y_{\ell,\mathfrak{m}}^{a,b} \neq \emptyset$ if and only if $\alpha = \mathfrak{d}_f^{\mathfrak{m},n}$ for certain $n \in \mathbb{N}$ and $f \in C^0(G,\mathbb{Z})$, and in this case $\mathbf{P}_\alpha^* \cap Y_{\ell,\mathfrak{m}}^{a,b}$ is the orbit of p_f^n and is dense in $\mathbf{P}_\alpha \cap Y_{\ell,\mathfrak{m}}^{a,b}$.

Recall from Section 2 the definition of the schemes $\mathbf{J}^{\mathbf{b}}$ and $\mathbf{R}_{\mathbf{J}^{\mathbf{b}}}$ and the stacks $\mathbf{S}^d$ and $\mathbf{J}^d$. Recall that given a point $s \in \mathbf{J}^{\mathbf{b}}$ corresponding to a collection of torsion-free, rank-one sheaves $(L_v\,;\, v \in V)$ with L_v of degree $\mathbf{b}_v$, fixing trivializations $L_v|_{N_e} \cong \kappa$ and $\mathcal{O}_{X_v}(N_e)|_{N_e} \cong \kappa$ for each $e \in E$ and each $v \in e$, we obtain an isomorphism between the fiber of $\mathbf{R}_{\mathbf{J}^{\mathbf{b}}}^E/\mathbf{J}^{\mathbf{b}}$ over s and $\mathbf{R}$. The isomorphism is uniquely defined once we specify that $\mathbf{P}_{\mathbf{J}^{\mathbf{b}}}^{\mathbf{c}+c}$ is taken to $\mathbf{P}_c$ for each $c \in C^1(G,\mathbb{Z})$. For simplification, we assume from now on that $\mathbf{c} = 0$.

The importance of the $Y_{\ell,\mathfrak{m}}^{a,b}$ stems from the following proposition.

Proposition 4.4. *Let $m \in \mathbb{N}$ and L_η be an invertible sheaf of degree d on the generic fiber of $\mathcal{X}^m/B$. Let $\mathcal{L}$ be a σ^m-admissible almost invertible extension of L_η to $\widetilde{\mathcal{X}}^m$. For each $n \in \mathbb{N}$ divisible by m, let $\mathcal{L}^n$ be the pullback of $\mathcal{L}$ to $\widetilde{\mathcal{X}}^n$, and for each $f \in C^0(G,\mathbb{Z})$ let*

$$L^n(f) := \mathcal{L}^n(f)|_{X^{n\ell}} \quad \text{and} \quad I^n(f) := \sigma_*^{n\ell} L^n(f).$$

Let $D^m \in \mathrm{Div}(H^m)$ be the divisor associated to $L^m(0)$. For each $e \in E^{\mathfrak{o}}$, let

$$\mathfrak{m}_e := \frac{1}{m} \sum_{i=1}^{m\ell_e - 1} i D^m(z_i^{\overline{e},m}),$$

and let $\mathfrak{m}$ be the unique element of $C^1(G,\mathbb{Q})$ satisfying this. Let $E_0 \subseteq E$ be the support of $\mathfrak{m}$ and put

$$K_v := I^m(0)_v \Big(\sum_{\substack{e \in E_0^{\mathfrak{o}} \\ \mathrm{h}_e = v}} N_e \Big) \quad \text{for each } v \in V(G),$$

where $I^m(0)_v$ is the torsion-free, rank-one sheaf generated by $I^m(0)$ on X_v. Let $\mathbf{b} \in C^0(G,\mathbb{Z})$ satisfying $\mathbf{b}_v = \deg(K_v)$ for each $v \in V(G)$. Fix trivializations $K_v|_{N_e} \cong \kappa$ and $\mathcal{O}_{X_v}(N_e)|_{N_e} \cong \kappa$ for each $e \in E$ and $v \in e$. Then the following three statements hold:

1. *The degree of $\mathbf{b}$ is d and the K_v are represented by a unique point $s \in \mathbf{J}^{\mathbf{b}}$.*
2. *For each $n \in \mathbb{N}$ with $m|n$ and $f \in C^0(G,\mathbb{Z})$, there is a unique (modulo action of $\mathbf{G}_{\mathbf{m}}^V/\mathbf{G}_{\mathbf{m}}$) point t_f^n on the fiber of $\mathbf{R}_{\mathbf{J}^{\mathbf{b}}}$ over s representing $I^n(f)$ with $c(t_f^n) = \mathfrak{d}_f^{\mathfrak{m},n}$.*
3. *There are characters*

$$a\colon C^1(G,\mathbb{Z}) \to \mathbf{G}_{\mathbf{m}}(\kappa) \quad \text{and} \quad b\colon C^1(G,\mathbb{Z}) \to \mathbf{G}_{\mathbf{m}}(\kappa)$$

such that, under the chosen trivializations, the equivariant isomorphism of the fiber of $\mathbf{R}_{\mathbf{J}^{\mathbf{b}}}$ over s with $\mathbf{R}$, taking $\mathbf{P}_{\mathbf{J}^{\mathbf{b}}}^c$ to $\mathbf{P}_c$ for each $c \in C^1(G,\mathbb{Z})$, takes a point on the *orbit of* t^n_f *for each* $n \in \mathbb{N}$ *divisible by* m *and* $f \in C^0(G,\mathbb{Z})$ *to the point on* $\mathbf{P}_{\mathfrak{d}_f^{\mathfrak{m},n}}$ *given by the coordinates*

$$(b_e a_e^{\mathfrak{d}_f^{\mathfrak{m},n}(e)} : 1) \quad \textit{for each } e \in E^{\mathfrak{o}} \textit{ with } \mathfrak{d}_f^{\mathfrak{m},n}(e) \in \mathbb{Z}.$$

(4) *In particular, the union of the orbits of all the* t^n_f *is a closed subset of the fiber of* $\mathbf{R}_{\mathbf{J}^{\mathfrak{b}}}$ *over* s *isomorphic to* $Y^{a,b}_{m\ell,m\mathfrak{m}}$.

Proof. The first statement is immediate from the fact that $\deg \mathbf{b} = \deg I^m(0) = \deg L_\eta = d$.

As for the second statement, let $n \in \mathbb{N}$ divisible by m and $f \in C^0(G,\mathbb{Z})$. Let D^n denote the pullback of D^m to H^n: we have $D^n(v) = D^m(v)$ for each $v \in V(G)$ and $D^n(z_j^{\bar{e},n}) = 0$ for each $e \in \mathbb{E}$ and $j = 1,\dots,n\ell_e - 1$, unless $n|jm$ and $D^m(z^{\bar{e},n}_{jm/n}) = 1$, in which case $D^n(z_j^{\bar{e},n}) = 1$. Also, D^n is the associated divisor to $L^n(0)$. In particular, D^n is G-admissible. Notice that

$$\mathfrak{m}_e = \frac{1}{n}\sum_{i=1}^{m\ell_e-1}\frac{n}{m} iD^n(z^{\bar{e},n}_{ni/m}) = \frac{1}{n}\sum_{i=1}^{n\ell_e-1} iD^n(z_i^{\bar{e},n})$$

for each $e \in E^{\mathfrak{o}}$. Then, by Propositions 3.2 and 4.2,

$$I^n(f)_v \cong L^n(f)|_{X_v} \cong L^m(0)|_{X_v} \otimes \mathcal{O}_{X_v}\Big(\sum_{\substack{e\in\mathbb{E}\\ \mathfrak{t}_e=v}} \lfloor \mathfrak{d}_f^{\mathfrak{m},n}(e)\rfloor N_e + \sum_{\substack{e\in\mathbb{E}\\ \mathfrak{t}_e=v\\ \mathfrak{m}_e<0}} N_e\Big)$$

for each vertex $v \in V(G)$, where $I^n(f)_v$ is the restriction modulo torsion of $I^n(f)$ to X_v. Furthermore, since $\mathfrak{m}_e > 0$ if and only if $e \in E_0^{\mathfrak{o}}$, we have

$$I^n(f)_v \cong I^m(0)_v\Big(\sum_{\substack{e\in E_0^{\mathfrak{o}}\\ \mathfrak{h}_e=v}} N_e\Big) \otimes \mathcal{O}_{X_v}\Big(\sum_{\substack{e\in\mathbb{E}\\ \mathfrak{h}_e=v}} \lfloor \mathfrak{d}_f^{\mathfrak{m},n}(\bar{e})\rfloor N_e\Big) \cong K_v\Big(\sum_{\substack{e\in\mathbb{E}\\ \mathfrak{h}_e=v}} \lfloor -\mathfrak{d}_f^{\mathfrak{m},n}(e)\rfloor N_e\Big) \cong \mathcal{L}_v^{\mathfrak{d}_f^{\mathfrak{m},n}}(s),$$

the latter isomorphism following from the definition in Subsection 2.2. In addition, since $I^n(f) = \sigma^{n\ell}_* L^n(f)$, it follows that I^n_f fails to be invertible at a node N_e if and only if $\mathfrak{d}_f^{\mathfrak{m},n}(e) \notin \mathbb{Z}$. The statement follows now from Proposition 2.6.

We prove now Statement (3). First, given the trivializations, the $\mathcal{O}_{\mathcal{X}^n}(f)|_X$ depend on $a\colon C^1(G,\mathbb{Z}) \to \mathbf{G}_{\mathbf{m}}(\kappa)$ arising from the deformation $\mathcal{X}/B$, as explained in Subsection 3.2. More explicitly, $\mathcal{O}_{\mathcal{X}^n}(f)|_X$ is isomorphic to the subsheaf of

$$\bigoplus_{v\in V(G)} \mathcal{O}_{X_v}\Big(\sum_{\substack{e\in\mathbb{E}\\ \mathfrak{h}_e=v}} \mathfrak{d}_f^{0,n}(\bar{e})N_e\Big)$$

whose cokernel is supported at the nodes N_e for $e = uv \in E^{\mathfrak{o}}$ such that $\mathfrak{d}_f^{0,n}(e) \in \mathbb{Z}$ and is equal in a neighborhood of such a N_e to the quotient of the vector space

$$\mathcal{O}_{X_u}\big(\mathfrak{d}_f^{0,n}(e)N_e\big)|_{N_e} \oplus \mathcal{O}_{X_v}\big(-\mathfrak{d}_f^{0,n}(e)N_e\big)|_{N_e}$$

by a certain one-dimensional vector subspace. The space is identified with $\kappa \oplus \kappa$, under our choices of trivializations, and the subspace is that generated by $(a^{\mathfrak{d}_f^{0,n}(e)}, 1)$.

Second, for each $e = uv \in E^{\mathfrak{o}}$, consider the sheaf $I^m(m\mathfrak{m}_e\chi_u)$. Then $\mathfrak{d}^{\mathfrak{m},m}_{m\mathfrak{m}_e\chi_u}(e) \in \mathbb{Z}$, whence $I^m(m\mathfrak{m}_e\chi_u)$ is invertible at N_e. Furthermore, it is naturally in a neighborhood of N_e a subsheaf of $K_u \oplus K_v$ whose quotient is the same as the quotient of a one-dimensional vector subspace of $K_u|_{N_e} \oplus K_v|_{N_e}$. The latter is identified with $\kappa \oplus \kappa$, under our choices of trivializations. Define $b_e \in \kappa$ such that the one-dimensional vector subspace is generated by $(b_e, 1)$. Thus, we have an element $b\colon C^1(G, \mathbb{Z}) \to \mathbf{G_m}(\kappa)$. Its restriction to $H^1(G, \mathbb{Z})$ is denoted by b by abusing the notation.

It follows that, under the above trivializations, given $n \in \mathbb{N}$, $f \in C^0(G, \mathbb{Z})$ and $c \in C^1(G, \mathbb{Z})$ such that $|c_e - \mathfrak{d}_f^{\mathfrak{m},n}(e)| \leq 1/2$ for all $e \in \mathbb{E}$, the fiber of $\mathbf{P}^c_{\mathbf{J^b}}$ over s is identified with $\mathbf{P}_c$, and the point t_f^n on the fiber representing $I^n(f)$ corresponds to the point on $\mathbf{P}_{\mathfrak{d}_f^{\mathfrak{m},n}} \subseteq \mathbf{P}_c$ with coordinates

$$(b_e a_e^{c_e} : 1) \quad \text{for each } e \in E^{\mathfrak{o}} \text{ with } \mathfrak{d}_f^{\mathfrak{m},n}(e) \in \mathbb{Z}.$$

Indeed, for each $e = uv \in E^{\mathfrak{o}}$ with $\mathfrak{d}_f^{\mathfrak{m},n}(e) \in \mathbb{Z}$ we have that $d_f^{\mathfrak{m},n}(e) = \mathfrak{d}_g^{0,n}(e)$, where $g := f - n\mathfrak{m}_e\chi_u$. Furthermore, $I^n(f)$ is equal to the tensor product of $I^n(n\mathfrak{m}_e\chi_u)$ and $\mathcal{O}_{\mathcal{X}^n}(g)|_X$ in a neighborhood of N_e, thus equal in that neighborhood to the subsheaf of

$$K_u\Big(\mathfrak{d}_f^{\mathfrak{m},n}(e)N_e\Big) \oplus K_v\Big(-\mathfrak{d}_f^{\mathfrak{m},n}(e)N_e\Big)$$

whose quotient is, under the identification with $\kappa \oplus \kappa$ of the restriction of the latter sheaf to N_e, that of $\kappa \oplus \kappa$ by the subspace generated by $(b_e a_e^{\mathfrak{d}_f^{\mathfrak{m},n}(e)}, 1)$.

The final statement is now an application of Theorem 4.3, once we observe that

$$\mathfrak{d}^{\mathfrak{m},n}_{\ell,f} = \mathfrak{d}^{m\mathfrak{m},n/m}_{m\ell,f}$$

for each $f \in C^0(G, \mathbb{Z})$ and each $n \in \mathbb{N}$ divisible by m. □

Corollary 4.5. *Notations as in Proposition 4.4, the sheaf $I^n(f)$ is a flat degeneration of the sheaf $I^q(h)$ if $\mathbf{P}_{\mathfrak{d}_f^{\mathfrak{m},n}} \subseteq \mathbf{P}_{\mathfrak{d}_h^{\mathfrak{m},q}}$, or equivalently, for each $e \in \mathbb{E}$ the following two conditions are satisfied:*

(1) $|\mathfrak{d}_f^{\mathfrak{m},n}(e) - \mathfrak{d}_h^{\mathfrak{m},q}(e)| \leq 1/2$.
(2) If $\mathfrak{d}_f^{\mathfrak{m},n}(e) \in \mathbb{Z}$ then $\mathfrak{d}_h^{\mathfrak{m},q}(e) \in \mathbb{Z}$.

Proof. By Proposition 4.4, there is a closed subscheme Y of the fiber of $\mathbf{R_{J^b}}$ over s, with the induced reduced structure, which is the union of all the orbits of the t_f^n. Consider the quotient stack $\mathfrak{I} := [Y/(\mathbf{G}_\mathbf{m}^V/\mathbf{G_m})]$. Let s_f^n denote the image of t_f^n in $\mathfrak{I}$ for each n and f. As observed in Section 2, there is a relative torsion-free rank-one sheaf $\mathcal{I}$ on $X \times \mathfrak{I}/\mathfrak{I}$ such that $\mathcal{I}|_{X \times \{s_f^n\}} \cong I^n(f)$ for each n and f.

By Proposition 4.4, there is an equivariant isomorphism from Y to $Y^{a,b}_{m\ell,m\mathfrak{m}}$ for certain $\mathfrak{m}$, a and b, taking each t_f^n to the point p_f^n on $\mathbf{P}^*_{\mathfrak{d}_f^{\mathfrak{m},n}}$ given in Theorem 4.3. The isomorphism is equivariant, whence we may view $\mathfrak{I}$ as a quotient of $Y^{a,b}_{m\ell,m\mathfrak{m}}$ where each s^n_f is the image of the point p^n_f. Pulling back $\mathcal{I}$ we obtain a relative torsion-free rank-one sheaf $\mathcal{L}$ on $X \times Y^{a,b}_{m\ell,m\mathfrak{m}}/Y^{a,b}_{m\ell,m\mathfrak{m}}$ such that $\mathcal{L}|_{X\times\{y\}} \cong I^n(f)$ for each point y on the orbit of p^n_f.

Let $n, q \in \mathbb{N}$ divisible by m and $f, h \in C^0(G,\mathbb{Z})$. Suppose $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m},q}_h} \supseteq \mathbf{P}_{\mathfrak{d}^{\mathfrak{m},n}_f}$. It is now enough to observe that $\mathbf{P}^*_{\mathfrak{d}^{\mathfrak{m},q}_h} \cap Y^{a,b}_{m\ell,m\mathfrak{m}}$ is the orbit of p^q_h and is dense in $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m},q}_h} \cap Y^{a,b}_{n\ell,m\mathfrak{m}}$ by Theorem 4.3. ☐

Now we need only one technical statement before stating the main result of the section.

Proposition 4.6. *Let $f_1, f_2 \in C^0(G,\mathbb{Z})$ and $n_1, n_2 \in \mathbb{N}$. If $\mathfrak{d}^{\mathfrak{m},n_2}_{f_2} - \mathfrak{d}^{\mathfrak{m},n_1}_{f_1} \in H^1(G,\mathbb{Z})$, then $\mathfrak{d}^{\mathfrak{m},n_2}_{f_2} = \mathfrak{d}^{\mathfrak{m},n_1}_{f_1}$.*

Proof. Since $\mathfrak{d}^{\mathfrak{m},n_i}_{f_i} = \mathfrak{d}^{\mathfrak{m},pn_i}_{pf_i}$ for each $p \in \mathbb{N}$, we may assume $n_1 = n_2$. Then the proposition follows from [AE20b], Prop. 5.1. ☐

Theorem 4.7 (Degeneration)**.** *Let $\mathfrak{I}$ be the collection of stable limits of an invertible sheaf L_η of degree d on the generic fiber of $\mathcal{X}^m/B$ for given integer $m \in \mathbb{N}$ and smoothing $\mathcal{X}/B$ with singularity degrees $\ell\colon E \to \mathbb{N}$ of the nodal curve X. Then $\mathfrak{I}$ is the support of a reduced closed substack of $\mathbf{J}^d$, also denoted $\mathfrak{I}$. The inverse image of $\mathfrak{I}$ in $\mathbf{R}_{\mathbf{J}^\mathbf{b}}$ is an infinite disjoint union of connected closed subschemes of certain fibers of $\mathbf{R}_{\mathbf{J}^\mathbf{b}}$ over $\mathbf{J}^\mathbf{b}$. Each connected component is the image of each other, under the action of a unique element of $H^1(G,\mathbb{Z})$, and each maps isomorphically to the same closed subscheme $\mathfrak{Y} \subseteq \mathbf{S}^d$. Furthermore, each connected component, under an identification of the fiber containing it with $\mathbf{R}$, is equal to the subscheme $Y^{a,b}_{\ell,\mathfrak{m}}$ for certain choices of $\mathfrak{m} \in C^1(G,\mathbb{Z})$ and characters $a\colon C^1(G,\mathbb{Z}) \to \mathbf{G_m}(\kappa)$ and $b\colon H^1(G,\mathbb{Z}) \to \mathbf{G_m}(\kappa)$. In particular, $\mathfrak{Y}$ has pure dimension $|V| - 1$ and $\mathfrak{I}$ has pure dimension 0.*

Proof. Using the notation in Proposition 4.4, we have that $\mathfrak{I}$ is the collection of the I^n_f for $n \in \mathbb{N}$ divisible by m and $f \in C^0(G,\mathbb{Z})$. Let the K_v and $\mathbf{b}$ be as in Proposition 4.4, as well as $s \in \mathbf{J}^\mathbf{b}$ and the $t^n_f \in \mathbf{R}_{\mathbf{J}^\mathbf{b}}$ in Statements (1) and (2) therein.

Let $Y \subseteq \mathbf{R}_{\mathbf{J}^\mathbf{b}}$ be the union of the orbits of the t^n_f for all $n \in \mathbb{N}$ and $f \in C^0(G,\mathbb{Z})$. By Proposition 4.4, it is a closed subset of the fiber of $\mathbf{R}_{\mathbf{J}^\mathbf{b}}$ over s. We give it the reduced structure, so that Y becomes isomorphic to $Y^{a,b}_{\ell,\mathfrak{m}}$ for certain choices of $\mathfrak{m} \in C^1(G,\mathbb{Z})$ and characters $a\colon C^1(G,\mathbb{Z}) \to \mathbf{G_m}(\kappa)$ and $b\colon H^1(G,\mathbb{Z}) \to \mathbf{G_m}(\kappa)$, according to Proposition 4.4. Then Y is a connected closed subscheme of $\mathbf{R}_{\mathbf{J}^\mathbf{b}}$ of pure dimension $|V| - 1$.

We claim the quotient map induces a closed embedding $Y \to \mathbf{S}^d$. Since $\mathbf{S}^d$ is the quotient of $\mathbf{R}_{\mathbf{J}^\mathbf{b}}$ by the free action of the discrete group $H^1(G,\mathbb{Z})$, to prove our claim it is enough to prove that the translate of Y by any nonzero element $\gamma \in H^1(G,\mathbb{Z})$ does not intersect Y.

So, let $\gamma \in H^1(G,\mathbb{Z})$. Suppose that there are two points t_1 and t_2 on Y that differ by the action of γ. Let $c_1, c_2 \in C^1(G,\mathbb{Z})$ such that t_1 lies on the fiber of $\mathbf{P}^{c_1}_{\mathbf{J}^\mathbf{b}}$ over s and t_2 on the fiber of $\mathbf{P}^{c_2}_{\mathbf{J}^\mathbf{b}}$ over s. We may assume that $c_1 = c_2 + \gamma$, and thus $c(t_1) = c(t_2) + \gamma$. But, by construction of Y, there are $n_1, n_2 \in \mathbb{N}$ and $f_1, f_2 \in C^0(G,\mathbb{Z})$ such that $c(t_i) = \mathfrak{d}_{f_i}^{\mathfrak{m},n_i}$ for $i = 1, 2$. Then Proposition 4.6 implies that $\gamma = 0$.

Any two points on $\mathbf{R}_{\mathbf{J}^{\mathbf{b}}}$ representing $I^n(f)$ for each $n \in \mathbb{N}$ and $f \in C^0(G,\mathbb{Z})$ differ by the action of $H^1(G,\mathbb{Z}) \times \mathbf{G}_{\mathbf{m}}^V/\mathbf{G}_{\mathbf{m}}$. Since the action of $H^1(G,\mathbb{Z})$ commutes with that of $\mathbf{G}_{\mathbf{m}}^V/\mathbf{G}_{\mathbf{m}}$, it follows that all the points on $\mathbf{R}_{\mathbf{J}^{\mathbf{b}}}$ representing $I^n(f)$ are in $\widetilde{\tau}^{\gamma}(Y)$ for $\gamma \in H^1(G,\mathbb{Z})$. So the inverse image of $\mathfrak{I}$ in $\mathbf{R}_{\mathbf{J}^{\mathbf{b}}}$ is the union $\bigcup \widetilde{\tau}^{\gamma}(Y)$ for γ running in $H^1(G,\mathbb{Z})$. Now, it follows from what we proved above that the intersection of $\widetilde{\tau}^{\gamma}(Y)$ with $\widetilde{\tau}^{\gamma'}(Y)$ for distinct $\gamma, \gamma' \in H^1(G,\mathbb{Z})$ is empty. Thus the $\widetilde{\tau}^{\gamma}(Y)$ are the connected components of the inverse image.

It follows that each of the $\widetilde{\tau}^{\gamma}(Y)$ is mapped isomorphically to the same closed subscheme $\mathfrak{Y} \subseteq \mathbf{S}^d$, and that $\mathfrak{Y}$ has pure dimension $|V| - 1$. Then $\mathfrak{I}$ has the structure of a closed substack of $\mathbf{J}^d$ of pure dimension 0. All the statements have been proved. □

Remark 4.8. Each $\mathfrak{I}$ should be parameterized by a point on $\mathrm{Hilb}_{\mathbf{J}^d}$, the Hilbert stack of $\mathbf{J}^d$, and each $\mathfrak{Y}$ should be parameterized by a point on $\mathrm{Hilb}_{\mathbf{S}^d}^{\mathbf{G}_{\mathbf{m}}^V/\mathbf{G}_{\mathbf{m}}}$, the Hilbert stack parameterizing $\mathbf{G}_{\mathbf{m}}^V/\mathbf{G}_{\mathbf{m}}$-invariant closed substacks of $\mathbf{S}^d$, if this Hilbert stack exists! Moreover, the analysis of a few examples leads us to ask whether these points, as π and L_η vary, form a closed substack of $\mathrm{Hilb}_{\mathbf{J}^d}$ or $\mathrm{Hilb}_{\mathbf{S}^d}^{\mathbf{G}_{\mathbf{m}}^V/\mathbf{G}_{\mathbf{m}}}$. This substack should be regarded as a *new compactified Jacobian* for X.

The situation is simpler when X is of compact type. Then there are no cycles, so $\mathbf{R}_{\mathbf{J}^{\mathbf{b}}} = \mathbf{S}^d$. Furthermore, $\mathfrak{Y}$ is a fiber of $\mathbf{R}_{\mathbf{J}^{\mathbf{b}}}$ over $\mathbf{J}^{\mathbf{b}}$. Thus the "new compactified Jacobian" is $\mathbf{J}^{\mathbf{b}}$, which is of course nothing new in this case. The general situation, where the "new compactified Jacobian" is indeed a new construction, will be addressed in a future work.

Remark 4.9. There is a natural $\mathbf{G}_{\mathbf{m}}^V/\mathbf{G}_{\mathbf{m}}$-invariant relative torsion-free, rank-one, degree-d sheaf on $X \times \mathfrak{Y}/\mathfrak{Y}$ which is given by the restriction of the sheaf on $X \times \mathbf{S}^d$ we called $\mathcal{I}$ in Section 2. Being invariant, it is the pullback of a relative torsion-free, rank-one sheaf on $X \times \mathfrak{I}/\mathfrak{I}$. It is this sheaf that gives the structure of $\mathfrak{I}$ as a closed substack of $\mathbf{J}^d$.

5. Regeneration

Recall from Section 4 the definition of $\mathbf{R}$ and of the $Y_{\ell,\mathfrak{m}}^{a,b}$. Recall from Section 2 the definition of the schemes $\mathbf{J}^{\mathbf{b}}$ and $\mathbf{R}_{\mathbf{J}^{\mathbf{b}}}$ and the stacks $\mathbf{S}^d$ and $\mathbf{J}^d$. Recall that given a point $s \in \mathbf{J}^{\mathbf{b}}$ corresponding to a collection of torsion-free, rank-one sheaves $(L_v\,;\, v \in V)$ with L_v of degree $\mathbf{b}_v$, fixing trivializations $L_v|_{N_e} \cong \kappa$ and $\mathcal{O}_{X_v}(N_e)|_{N_e} \cong \kappa$ for each $e \in E$ and each $v \in e$, we obtain an isomorphism between the fiber of $\mathbf{R}_{\mathbf{J}^{\mathbf{b}}}^E/\mathbf{J}^{\mathbf{b}}$ over s and $\mathbf{R}$. The isomorphism is uniquely defined once we specify that $\mathbf{P}_{\mathbf{J}^{\mathbf{b}}}^{\mathbf{c}+c}$ is taken to $\mathbf{P}_c$ for each $c \in C^1(G,\mathbb{Z})$. For simplification, we assume from now on that $\mathbf{c} = 0$.

In this section, we give a converse to Theorem 4.7.

Theorem 5.1 (Regeneration)**.** *Let $\ell\colon E \to \mathbb{N}$ be a length function, $\mathfrak{m} \in C^1(G,\mathbb{Z})$ and let*

$$a\colon C^1(G,\mathbb{Z}) \to \mathbf{G}_{\mathbf{m}}(\kappa) \quad \textit{and} \quad b\colon H^1(G,\mathbb{Z}) \to \mathbf{G}_{\mathbf{m}}(\kappa)$$

be characters. Let $Y^{a,b}_{\ell,\mathfrak{m}}$ be the corresponding subscheme of $\mathbf{R}$. Let $\mathbf{b} \in C^0(G,\mathbb{Z})$. For each $v \in V$, let K_v be a torsion-free, rank-one sheaf of degree $\mathbf{b}_v$ on X_v. Let $s \in \mathbf{J}^{\mathbf{b}}$ be the corresponding point. Fix trivializations $K_v|_{N_e} \cong \kappa$ and $\mathcal{O}_{X_v}(N_e)|_{N_e} \cong \kappa$ for each $e \in E$ and $v \in e$. Let $Y \subseteq \mathbf{R}_{\mathbf{J}^{\mathbf{b}}}$ be the closed subscheme of the fiber over s corresponding to $Y^{a,b}_{\ell,\mathfrak{m}}$ under the isomorphism of the fiber with $\mathbf{R}$ induced by the trivializations. Let $\mathfrak{I} \subseteq \mathbf{J}^d$ be the image of Y. Then there exist a smoothing $\pi\colon \mathcal{X} \to B$ of X and an invertible sheaf L_η on the generic fiber of π such that the following two statements hold:

1. *The total space $\mathcal{X}$ is regular everywhere but possibly at the nodes N_e of X, where it has singularity degree ℓ_e for each $e \in \mathbb{E}(G)$, and at the remaining nodes of X where the sheaf $\oplus K_v$ is not invertible, where it has singularity degree 2.*
2. *$\mathfrak{I}$ is the collection of stable limits of L_η.*

The whole section is devoted to the proof of the above theorem.

Let G' denote the full dual graph of X, containing self loops. Consider a versal deformation V/M of X, as explained in Subsection 3.2. More precisely, M is the spectrum of the power series ring R over κ in the variables t_e for $e \in E(G')$, and variables $s_1, \dots, s_p$ for a certain integer p, and we have an isomorphism of R-algebras

$$\psi_e \colon \widehat{\mathcal{O}}_{V,N_e} \to R[[z_e, w_e]]/(z_e w_e - t_e)$$

for each $e \in E(G')$, with $z_e = 0$ corresponding to the component X_v of X and $w_e = 0$ to X_u if $e \in E(G)$ and $e^{\mathfrak{o}} = uv$. The pullbacks of z_e and w_e under ψ_e yield under restriction analytic local parameters of X_u and X_v at N_e, respectively. We may assume that the isomorphisms $\mathcal{O}_{X_v}(N_e)|_{N_e} \cong \kappa$ are given by them, for all $e \in E(G)$ and $v \in V(G)$ with $v \in e$.

We may then consider the smoothing $\pi\colon \mathcal{X} \to B$ of X induced by the map $B \to M$ pulling back the s_i to t, and pulling back t_e to $a_{e^{\mathfrak{o}}} t^{\ell_e}$ for each $e \in E(G)$, to t for each $e \in E(G') - E(G)$ if K_v is invertible at the corresponding node for $v \in e$, and to t^2 otherwise. Then π is as in the statement of the theorem. Also, as explained in Subsection 3.2, the sheaves $\mathcal{O}_{\mathcal{X}^n}(f)|_X$ are determined by the homomorphism a.

Let $\sigma\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ be the Cartier reduction of π. The composition $\widetilde{\pi} := \pi\sigma\colon \widetilde{\mathcal{X}} \to B$ is a smoothing of X^ℓ. As in Section 4, we will let $\widetilde{\pi}^n\colon \widetilde{\mathcal{X}}^n \to B$ denote the Cartier reduction of the extension of π by the base change map $\mu_n\colon B \to B$ for each $n \in \mathbb{N}$; it is a smoothing of $X^{n\ell}$.

Recall from Theorem 4.3 that $Y^{a,b}_{\ell,\mathfrak{m}}$ is a union of orbits in $\mathbf{R}$ under the action of $\mathbf{G}^V_{\mathbf{m}}/\mathbf{G}_{\mathbf{m}}$, the orbits of the p^n_f for $f \in C^0(G,\mathbb{Z})$ and $n \in \mathbb{N}$, where p^n_f is a point on $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m},n}_f}$ whose coordinates are given in terms of a and b, as made precise in that theorem. Now, for each $f \in C^0(G,\mathbb{Z})$ and $n \in \mathbb{N}$, let I^n_f be the torsion-free, rank-one sheaf on X parameterized by the point $t^n_f \in Y$ corresponding to p^n_f. For each $e = uv \in \mathbb{E}$, write

$$f(v) - f(u) + n\mathfrak{m}_e = n\ell_e \delta^{\mathfrak{m},n}_e(f) + n\ell_e - i^n_e(f), \tag{5.1}$$

where $i_e^n(f)$ is an integer satisfying $0 < i_e^n(f) \leq n\ell_e$. We can write $I_f^n = \sigma_*^{n\ell} L_f^n$ for a certain almost invertible sheaf L_f^n on $X^{n\ell}$, admissible for $\sigma^{n\ell}$, such that L_f^n has degree zero on each rational smooth curve $Z_i^{e,n}$ contracted by $\sigma^{n\ell}$ onto N_e for each $e = uv \in \mathbb{E}$, unless $\mathfrak{d}_f^{\mathfrak{m},n}(e) \notin \mathbb{Z}$ and $i = i_e^n(f)$.

The sheaf L_f^n is not unique. Its associated divisor in H^n, denoted henceforth by D_f^n, is. At any rate, with the choices we have already made, we have the following claim.

Claim 5.2. *Notation as above, for each $f, h \in C^0(G, \mathbb{Z})$ and $n \in \mathbb{N}$, the sheaves*

$$L_h^n \quad \textit{and} \quad L_f^n \otimes \mathcal{O}_{\widetilde{\mathcal{X}}^n}(\widetilde{g})|_{X^{n\ell}} \tag{5.2}$$

have the same degrees on the exceptional components of $X^{n\ell}$ and the same restrictions to X_v for each $v \in V(G)$, where $\widetilde{g}$ is the canonical extension of $g := h - f$ with respect to D_f^n. In particular,

$$D_h^n = D_f^n + \mathrm{div}(\widetilde{g}).$$

Proof. Indeed, both sheaves in (5.2) are $\sigma^{n\ell}$-admissible. Furthermore, for each $e = uv \in \mathbb{E}$, the sheaf on the right in (5.2) has degree 0 on $Z_i^{e,n}$ for each $i = 1, \dots, n\ell_e - 1$, unless

$$g(u) - g(v) + i_e^n(f) = n\ell_e \Big\lfloor \frac{g(u) - g(v) + i_e^n(f)}{n\ell_e} \Big\rfloor + i.$$

Substituting for $i_e^n(f)$ from Equation (5.1) we get

$$g(u) - g(v) + f(u) - f(v) + n\ell_e \delta_e^{\mathfrak{m},n}(f) + n\ell_e - n\mathfrak{m}_e = n\ell_e \Big\lfloor \frac{g(u) - g(v) + i_e^n(f)}{n\ell_e} \Big\rfloor + i,$$

whence

$$h(v) - h(u) + n\mathfrak{m}_e = n\ell_e \Big(\delta_e^{\mathfrak{m},n}(f) - \Big\lfloor \frac{g(u) - g(v) + i_e^n(f)}{n\ell_e} \Big\rfloor \Big) + n\ell_e - i.$$

Comparing with Equation (5.1), with f replaced by h, it follows that both sheaves in (5.2) have the same degree on each exceptional component $Z_i^{e,n}$.

Now, given $v \in V(G)$ and $q \in C^0(G, \mathbb{Z})$, it follows from Propositions 2.2 and 3.2 that

$$L_q^n|_{X_v} \cong (I_q^n)_v \cong K_v \otimes \mathcal{O}_{X_v}\Big(\sum_{\substack{e \in \mathbb{E} \\ \mathrm{t}_e = v}} \lfloor \mathfrak{d}_q^{\mathfrak{m},n}(e) \rfloor N_e \Big), \tag{5.3}$$

where $(I_q^n)_v$ is the restriction of I_q^n to X_v modulo torsion. In addition, by Proposition 4.2,

$$\mathcal{O}_{\widetilde{\mathcal{X}}^n}(\widetilde{g})|_{X_v} \cong \mathcal{O}_{X_v}\Big(\sum_{\substack{e \in \mathbb{E} \\ \mathrm{t}_e = v}} \lfloor \mathfrak{d}_g^{\mathfrak{p},n}(e) \rfloor N_e + \sum_{\substack{e \in \mathbb{E} \\ \mathrm{t}_e = v \\ \mathfrak{p}_e < 0}} N_e \Big), \tag{5.4}$$

where $\mathfrak{p} \in C^1(G, \mathbb{Z})$ satisfies

$$\mathfrak{p}_e = \frac{1}{n} \sum_{i=1}^{n\ell_e - 1} i D_f^n(z_i^{\bar{e},n}) \tag{5.5}$$

for each $e \in E^{\mathfrak{o}}$.

Since D_f^n is the divisor associated to L_f^n, we have that $n\mathfrak{p}_e = i_{\bar{e}}^n(f)$ for each $e \in E^{\mathfrak{o}}$ with $i_{\bar{e}}^n(f) \neq n\ell_e$, whence $n\mathfrak{p}_e = n\ell_e - i_e^n(f)$. The latter holds even if $i_{\bar{e}}^n(f) = n\ell_e$, because then $\mathfrak{p}_e = 0$ and $i_e^n(f) = n\ell_e$. Now, if $e \in \mathbb{E} - E^{\mathfrak{o}}$, then $n\mathfrak{p}_{\bar{e}} = i_e^n(f)$, whence $n\mathfrak{p}_e = -i_e^n(f)$, unless $i_e^n(f) = n\ell_e$, in which case $n\mathfrak{p}_e = 0$. To summarize,

$$n\mathfrak{p}_e = \begin{cases} n\ell_e - i_e^n(f) & \text{if } \mathfrak{p}_e \geq 0, \\ -i_e^n(f) & \text{if } \mathfrak{p}_e < 0. \end{cases} \tag{5.6}$$

Finally, for each $e \in \mathbb{E}$,

$$\begin{aligned} n\ell_e \lfloor \mathfrak{d}_h^{\mathfrak{m},n}(e) \rfloor =& h(\mathrm{h}_e) - h(\mathrm{t}_e) + n\mathfrak{m}_e - n\ell_e + i_e^n(h) \\ =& h(\mathrm{h}_e) - h(\mathrm{t}_e) - f(\mathrm{h}_e) + f(\mathrm{t}_e) - i_e^n(f) + n\ell_e \lfloor \mathfrak{d}_f^{\mathfrak{m},n}(e) \rfloor + i_e^n(h) \\ =& g(\mathrm{h}_e) - g(\mathrm{t}_e) + n\mathfrak{p}_e - n\mathfrak{p}_e - i_e^n(f) + n\ell_e \lfloor \mathfrak{d}_f^{\mathfrak{m},n}(e) \rfloor + i_e^n(h) \\ =& n\ell_e \lfloor \mathfrak{d}_g^{\mathfrak{p},n}(e) \rfloor + \rho - n\mathfrak{p}_e - i_e^n(f) + n\ell_e \lfloor \mathfrak{d}_f^{\mathfrak{m},n}(e) \rfloor + i_e^n(h) \end{aligned}$$

for a certain integer ρ satisfying $0 \leq \rho < n\ell_e$. Since $0 < \rho + i_e^n(h) < 2n\ell_e$, and Equation (5.6) yields that $n\mathfrak{p}_e + i_e^n(f)$ is a multiple of $n\ell_e$, it follows that $\rho + i_e^n(h) = n\ell_e$ and

$$\lfloor \mathfrak{d}_h^{\mathfrak{m},n}(e) \rfloor = \begin{cases} \lfloor \mathfrak{d}_g^{\mathfrak{p},n}(e) \rfloor + \lfloor \mathfrak{d}_f^{\mathfrak{m},n}(e) \rfloor & \text{if } \mathfrak{p}_e \geq 0, \\ \lfloor \mathfrak{d}_g^{\mathfrak{p},n}(e) \rfloor + \lfloor \mathfrak{d}_f^{\mathfrak{m},n}(e) \rfloor + 1 & \text{if } \mathfrak{p}_e < 0. \end{cases} \tag{5.7}$$

It follows now from Isomorphisms (5.3) for $q = h$ and $q = f$, and from Equation (5.4), that the restrictions to X_v of both sheaves in (5.2) are equal, for each $v \in V(G)$, finishing the proof of our claim. □

Notice as well that, since $\rho + i_e^n(h) = n\ell_e$, if $\mathfrak{d}_h^{\mathfrak{m},n}(e) \in \mathbb{Z}$, then $i_e^n(h) = n\ell_e$ and thus $\rho = 0$, implying that also $\mathfrak{d}_g^{\mathfrak{p},n}(e) \in \mathbb{Z}$. And if $\mathfrak{d}_f^{\mathfrak{m},n}(e) \in \mathbb{Z}$ then $\mathfrak{p}_e = 0$. Now, if both I_f^n and I_h^n are invertible at N_e, then all of $\mathfrak{d}_f^{\mathfrak{m},n}(e), \mathfrak{d}_h^{\mathfrak{m},n}(e), \mathfrak{d}_g^{\mathfrak{p},n}(e)$ are integers, $\mathfrak{p}_e = 0$ and it follows from (5.7) that

$$\mathfrak{d}_h^{\mathfrak{m},n}(e) = \mathfrak{d}_g^{\mathfrak{p},n}(e) + \mathfrak{d}_h^{\mathfrak{m},n}(e). \tag{5.8}$$

Furthermore, since we have already shown that the sheaves in (5.2) have the same degrees on the exceptional components of $X^{n\ell}$, it follows that also the pushforward $\sigma_*^{n\ell}\mathcal{O}_{\widetilde{\mathcal{X}}^n}(\widetilde{g})|_{X^{n\ell}}$ is invertible at N_e. Finally, the gluing data of I_f^n and I_h^n at N_e are $b_e a_e^{\mathfrak{d}_f^{\mathfrak{m},n}(e)}$ and $b_e a_e^{\mathfrak{d}_h^{\mathfrak{m},n}(e)}$, respectively, as given by the coordinates of p_f^n and p_h^n. As for the gluing data of $\sigma_*^{n\ell}\mathcal{O}_{\widetilde{\mathcal{X}}^n}(\widetilde{g})|_{X^{n\ell}}$ at N_e, since $\mathfrak{p}_e = 0$, it is the same as if we assumed that $\widetilde{g}$ is the canonical extension of g with respect to the divisor 0; it follows from Subsection 3.2 that the gluing data is $a_e^{\mathfrak{d}_g^{0,n}(e)}$. Clearly, $\mathfrak{d}_g^{0,n}(e) = \mathfrak{d}_g^{\mathfrak{p},n}(e)$, and hence it follows from (5.8) that the sheaves in (5.2) have the same gluing data in a neighborhood of $Z^{e,n}$.

Since the chains $Z^{e,n}$ are curves of compact type, an invertible sheaf on each of them is determined by its degrees on the components. It follows that the sheaves in (5.2) coincide up to the chosen gluing along the intersections of the chains $Z^{e,n}$ with the rest of $X^{n\ell}$ for the $e \in \mathbb{E}$ such that either $\mathfrak{d}_h^{\mathfrak{m},n}(e)$ or $\mathfrak{d}_f^{\mathfrak{m},n}(e)$ is not an integer.

In fact, we have not yet specified the gluing giving L_f^n along $Z^{e,n}$ if I_f^n is not invertible at N_e. We do it now: For each oriented edge $e = uv \in E^{\mathfrak{o}}$ such that $\mathfrak{d}_f^{\mathfrak{m},n}(e) \notin \mathbb{Z}$, give L_f^n the gluing along $Z^{e,n}$ such that the pushforwards under $\sigma^{n\ell}$ of

$$L_{f_e}^n \quad \text{and} \quad L_f^n \otimes \mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{r}\big)|_{X^{n\ell}} \tag{5.9}$$

have the same gluing data at N_e, where $\widetilde{r}$ is the canonical extension of $r := (n\ell_e - i_e^n(f))\chi_u$ with respect to the divisor D_f^n, and $f_e := f + r$. Notice that $I_{f_e}^n$ is invertible at N_e, and thus the gluing data for $L_{f_e}^n$ along $Z^{e,n}$ is determined from that of $I_{f_e}^n$ at N_e. From the above prescription, the gluing data for L_f^n is now fixed everywhere for each $f \in C^0(G,\mathbb{Z})$ and $n \in \mathbb{N}$.

Finally, with the above choices, we can claim more than Claim 5.2:

Claim 5.3. *Notation as above, for each $f, h \in C^0(G,\mathbb{Z})$ and $n \in \mathbb{N}$, we have*

$$L_h^n \cong L_f^n \otimes \mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{g}\big)|_{X^{n\ell}},$$

where $\widetilde{g}$ is the canonical extension of $g := h - f$ with respect to the divisor D_f^n in H^n associated to L_f^n.

Proof. Indeed, given $e \in E^{\mathfrak{o}}$, notice first that, if $q \in C^0(G,\mathbb{Z})$ is such that I_q^n is invertible at N_e, we have proved that the gluing data along $Z^{e,n}$ of

$$L_q^n \quad \text{and} \quad L_{f_e}^n \otimes \mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{g}\big)|_{X^{n\ell}}$$

coincide, where $g := q - f_e$ and $\widetilde{g}$ is the canonical extension of g with respect to the divisor $D_{f_e}^n$ in H^n associated to $L_{f_e}^n$. It follows that the gluing data along $Z^{e,n}$ of

$$L_q^n \quad \text{and} \quad L_f^n \otimes \mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{g} + \widetilde{r}\big)|_{X^{n\ell}}$$

coincide, where $\widetilde{r}$ is the canonical extension of $r := (n\ell_e - i_e^n(f))\chi_u$ with respect to the divisor D_f^n. But $D_{f_e}^n = D_f^n + \mathrm{div}(\widetilde{r})$ by Claim 5.2. It thus follows from [AE20a], Prop. 2.11, that $\widetilde{r} + \widetilde{g}$ is the canonical extension of $r + g$ with respect to the divisor D_f^n. Notice that $r + g = q - f$.

Thus, given $q \in C^0(G,\mathbb{Z})$ such that $\mathfrak{d}_q^{\mathfrak{m},n}(e) \in \mathbb{Z}$, the gluing data along $Z^{e,n}$ of these three sheaves,

$$L_q^n, \quad L_h^n \otimes \mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{p}\big)|_{X^{n\ell}} \quad \text{and} \quad L_f^n \otimes \mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{g}\big)|_{X^{n\ell}}, \tag{5.10}$$

coincide, where $\widetilde{p}$ (resp. $\widetilde{g}$) is the canonical extension of $p := q - h$ (resp. $g := q - f$) with respect to the divisor D_h^n (resp. D_f^n). But then so do the gluing data of

$$L_h^n \otimes \mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{p} + \widetilde{r}\big)|_{X^{n\ell}} \quad \text{and} \quad L_f^n \otimes \mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{g} + \widetilde{r}\big)|_{X^{n\ell}},$$

where $\widetilde{r}$ is the canonical extension of $r := h - q$ with respect to the divisor D_q^n. Since all three sheaves in (5.10) have the same associated divisor in H^n, it follows again from [AE20a], Prop 2.11, that $\widetilde{p} + \widetilde{r}$ is the canonical extension of $p + r = 0$ and that $\widetilde{g} + \widetilde{r}$ is the canonical extension of $g+r=h-f$ with respect to D^n_f. Since D^n_f is G-admissible, $\widetilde{p}+\widetilde{r}=0$. So the sheaves above are exactly those in (5.2), and we have concluded that they have the same gluing data along $Z^{e,n}$ for every $e\in E^{\mathfrak{o}}$, whence our claim. □

We need two more claims to finish the proof of the theorem.

Claim 5.4. *Notation as above, L^n_0 is the pullback of L^1_0 under the map $X^{n\ell}\to X^{\ell}$.*

Proof. Notice that $p^n_0=p^1_0$ for every $n\in\mathbb{Z}$, since $\mathfrak{d}^{\mathfrak{m},n}_0=\mathfrak{d}^{\mathfrak{m},1}_0$, and thus $I^n_0=I^1_0$. Thus the restrictions of L^n_0 and L^1_0 to the components X_v for $v\in V(G)$ coincide. Also, $i^n_e(0)=ni^1_e(0)$ for each $e\in\mathbb{E}$, and thus D^n_0 is the pullback of D^1_0. Thus the restriction of L^n_0 to $Z^{e,n}$ coincides with that of the pullback of L^1_0 for each $e\in\mathbb{E}$.

Also, the gluing data of L^n_0 along $Z^{e,n}$ for each $e\in E^{\mathfrak{o}}$ such that $\mathfrak{d}^{\mathfrak{m},1}_0(e)\in\mathbb{Z}$ is the gluing data of the pullback of L^1_0. On the other hand, by construction, for each $e=uv\in E^{\mathfrak{o}}$ such that $\mathfrak{d}^{\mathfrak{m},1}_0(e)\notin\mathbb{Z}$, the gluing data of L^n_0 along $Z^{e,n}$ is such that the gluing data at N_e of the pushforwards of

$$L^n_{g_n}\quad\text{and}\quad L^n_0\otimes\mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{g}_n\big)|_{X^{n\ell}}$$

coincide, where $g_n:=(n\ell_e-i^n_e(0))\chi_u$ and $\widetilde{g}_n$ is the canonical extension of g_n with respect to D^n_0. Same for L^1_0, for n replaced by 1. Clearly, since $i^n_e(0)=ni^1_e(0)$, we have that $g_n=ng_1$, whence $\mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{g}_n\big)$ is the pullback of $\mathcal{O}_{\widetilde{\mathcal{X}}}\big(\widetilde{g}_1\big)$ to $\widetilde{\mathcal{X}}^n$. Finally, the gluing data of the pushforward $I^n_{g_n}$ at N_e is given by $p^n_{g_n}$, whereas that of $I^1_{g_1}$ is given by $p^1_{g_1}$. Since $g_n=ng_1$, we have that $\mathfrak{d}^{\mathfrak{m},n}_{g_n}=\mathfrak{d}^{\mathfrak{m},1}_{g_1}$, and thus $p^n_{g_n}=p^1_{g_1}$. It follows that the gluing data of L^n_0 and of the pullback of L^1_0 coincide everywhere, proving the claim. □

Claim 5.5. *Notation as above, there is an almost invertible sheaf $\mathcal{L}$ on $\widetilde{\mathcal{X}}$ whose restriction to X^{ℓ} is L^1_0.*

Proof. The sheaf L^1_0 fails to be invertible only at the nodes of X^{ℓ} lying over nodes of X other than the N_e, at which the corresponding K_v is not invertible. Let $\widehat{\mathcal{X}}$ denote the blowup of $\widetilde{\mathcal{X}}$ at these nodes and by $\widehat{X}$ the special fiber of $\widehat{\mathcal{X}}/B$. The curve $\widehat{X}$ is obtained by separating the branches of every node in the center of the blowup and connecting them by a smooth rational curve. Then L^1_0 is the pushforward of an invertible sheaf on $\widehat{X}$, having degree one on the added rational curves, by [EP16], Prop. 5.5, p. 77. Since this sheaf is invertible, since the relative Picard scheme of $\widehat{\mathcal{X}}/B$ is smooth over B, and since B is the spectrum of $\kappa[[t]]$, it follows that it extends to an invertible sheaf on $\widehat{\mathcal{X}}$. The pushforward of this extension under the blowup map $\widehat{\mathcal{X}}\to\widetilde{X}$ is the required sheaf $\mathcal{L}$. □

Proof of Theorem 5.1. We claim that $L_\eta:=\mathcal{L}|_{X_\eta}$ is the required sheaf. Indeed, we need only prove that $L^n_h=\mathcal{L}^n(h)|_{X^{n\ell}}$ for each $h\in C^0(G,\mathbb{Z})$ and $n\in\mathbb{N}$, where $\mathcal{L}^n$ is the pullback of $\mathcal{L}$ to $\widetilde{\mathcal{X}}^n$. But notice that $\mathcal{L}^n|_{X^{n\ell}}=L^n_0$ by Claims 5.4 and 5.5. And

$$\mathcal{L}^n(h)=\mathcal{L}^n\otimes\mathcal{O}_{\widetilde{\mathcal{X}}^n}\big(\widetilde{h}\big)$$

for each $h\in C^0(G,\mathbb{Z})$, where $\widetilde{h}$ is the canonical extension of h with respect to the divisor D^n_0 on H^n associated to L^n_0. Applying Claim 5.3 with $f=0$ finishes the proof. □

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