VORONOI TILINGS, TORIC ARRANGEMENTS AND DEGENERATIONS OF LINE BUNDLES II

OMID AMINI AND EDUARDO ESTEVES

Abstract. We describe limits of line bundles on nodal curves in terms of toric arrangements associated to Voronoi tilings of Euclidean spaces. These tilings encode information on the relationship between the possibly infinitely many limits, and ultimately give rise to a new definition of *limit linear series*. This article and its first and third part companion parts are the first in a series aimed to explore this new approach.

In the first part, we set up the combinatorial framework and showed how graphs weighted with integer lengths associated to the edges provide tilings of Euclidean spaces by polytopes associated to the graph itself and to its subgraphs.

In this part, we describe the *arrangements of toric varieties* associated to these tilings. Roughly speaking, the normal fan to each polytope in the tiling corresponds to a toric variety, and these toric varieties are glued together in an arrangement according to how the polytopes meet. We provide a thorough description of these toric arrangements from different perspectives: by using normal fans, as unions of torus orbits, by describing the (infinitely many) polynomial equations defining them in products of doubly infinite chains of projective lines, and as degenerations of algebraic tori.

These results will be of use in the third part to achieve our goal of describing all *stable limits* of a family of line bundles along a degenerating family of curves. Our main result there asserts that the collection of all these limits is parametrized by a connected 0-dimensional closed substack of the Artin stack of all torsion-free rank-one sheaves on the limit curve. The substack is the quotient of an arrangement of toric varieties as described in the present article by the torus of the same dimension acting on it.

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1. Introduction

This is a sequel to our work [AE20a]. Our aim here is to give the geometric significance of the combinatorial results we proved there. This will be done by using toric geometry and will be of use in [AE20b] to achieve our goal of describing stable limits of line bundles on nodal curves. In this introduction, after briefly recalling the results we proved in [AE20a], we explain the main contributions of the present work. A reader interested in algebraic geometry shall find a more detailed exposition and a clearer picture of the link to the algebraic geometry of curves in the introduction to the third part.

1.1. Stable limits of line bundles.

Let X be a stable curve of arithmetic genus g over an algebraically closed field κ. Recall that this means X is a reduced, projective and connected scheme of dimension one that has at most ordinary nodes as singularities, a nodal curve, and in addition has an ample canonical bundle. Stable curves appear on the boundary of the Deligne–Mumford compactification $\overline{\mathscr{M}}_g$ of the moduli space $\mathscr{M}_g$ of smooth projective curves of genus g.

The question which motivates the study undertaken in our series of papers starting with [AE20a], the current paper and [AE20b] is the following:

Question 1.1. *Let X be a stable curve of genus g and denote by x the corresponding point in the Deligne–Mumford compactification $\overline{\mathscr{M}}_g$.*

Describe all the possible limits of linear series $\mathfrak{g}^r_d$ over any sequence $X_1, X_2, \dots$ of smooth projective curves of genus g whose corresponding points $x_1, x_2, \dots$ in $\mathscr{M}_g$ converge to x.

Recall that a linear series $\mathfrak{g}^r_d$ on a projective curve Y is by definition a vector subspace of dimension $r+1$ of the space of global sections $H^0(Y, L)$ of a line bundle L of degree d on Y.

The above simple looking question turns out to be a difficult and multifaceted problem for which only partial results are known. In the case where X is a curve of compact type, a satisfactory answer was given in the pioneering work by Eisenbud and Harris [EH86], who derived several interesting consequences in the study of the geometry of the moduli space of curves [EH87a, EH87b]. Apart from the case of compact type curves, Medeiros and the second named author [EM02] gave an answer to the problem for curves with two components, and, recently, Osserman extended the above mentioned results to the case of curves of pseudo-compact type [Oss19b, Oss16], which cover both the cases considered previously in [EH86, EM02]. As far as the stable curves lying on the deeper strata of the moduli space are concerned, only little is known. The most interesting results in this direction are probably the ones obtained in the work by Bainbridge, Chen, Gendron, Grushevsky and Möller, [BCG$^+$18], which describes limits of holomorphic one forms, i.e., $\mathfrak{g}^0_{2g-2}$ obtained as global sections of the canonical sheaf, and its sequel [BCG$^+$19], which studies sections of powers of the canonical sheaf. The above question is also intimately linked to tropical and hybrid geometry; we refer to [AB15, MUW17, LM18, BJ16, Car15, Oss19a, He19, FJP20] for other results which go in the direction of partially answering the question we posed.

The main combinatorial object associated to a stable curve X is its dual graph and it is now well-understood that dual graphs and their geometry play a central role in understanding questions of the type described above. The dual graph $G = (V, E)$ of X consists of a vertex set V in one-to-one correspondence with the set of irreducible components of X, and an edge set E in one-to-one correspondence with the set of nodes. An edge lies between two vertices if the corresponding node lies on the two corresponding components.

In our series of work we explore a new approach to Question 1.1 based on the geometry of dual graphs and the use of Artin stacks and their "Hilbert schemes." In [AE20b], we give an answer to the following question:

Question 1.2. *Let X be a stable curve of genus g and denote by x the corresponding point in the Deligne–Mumford compactification $\overline{\mathscr{M}}_g$.*

Describe all the stable limits of line bundles over any sequence $X_1, X_2, \dots$ of smooth projective curves of genus g whose corresponding points $x_1, x_2, \dots$ in $\mathscr{M}_g$ converge to x.

More precisely, let $\pi\colon \mathcal{X} \to B$ be a one-parameter smoothing of X. Here, B is the spectrum of $\kappa[[t]]$, and π is a projective flat morphism whose generic fiber $\mathcal{X}_\eta$ is smooth and whose special fiber is isomorphic to X. We fix such an isomorphism. The total space $\mathcal{X}$ is regular except possibly at the nodes of X. For each edge $e \in E$, one can associate to the corresponding node N_e of X a positive integer ℓ_e, called the the singularity degree (or the thickness) of π at N_e. This is obtained by looking at the completion of the local ring of $\mathcal{X}$ at N_e, which is isomorphic to $\kappa[[u, v, t]]/(uv - t^{\ell_e})$. If $\ell_e = 1$ for an edge $e \in E$, then $\mathcal{X}$ is regular at N_e.

Let L_η be an invertible sheaf on the generic fiber. If $\mathcal{X}$ is regular, it extends to an invertible sheaf $\mathcal{L}$ on $\mathcal{X}$. It is not unique, as $\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(\sum f(v)X_v)$ is another extension, for every integer valued function $f\colon V \to \mathbb{Z}$. (Here X_v is the component corresponding to $v \in V$.) For a general family π, without the regularity assumption, the sheaf L_η extends to a relatively torsion-free, rank-one sheaf $\mathcal{I}$ on $\mathcal{X}/B$, that is, a B-flat coherent sheaf on $\mathcal{X}$ whose fibers over B are torsion-free, rank-one. Again, there is no uniqueness for such an extension. Furthermore, one could perform a finite base change to the family π, take the pullback of L_η to the new generic fiber and consider its torsion-free extensions. These will be extensions on a different total space, but the special fibers of all the families are the same, and thus the restrictions of all these extensions to X are torsion-free, rank-one sheaves that we call the *stable limits* of L_η.

We describe these stable limits of line bundles on stable curves in terms of Voronoi decompositions of Euclidean spaces associated to graphs and their associated arrangements of toric varieties.

1.2. **Voronoi tilings.** Let X be a nodal curve and let $G = (V, E)$ be its dual graph, with self loops removed. We denote by $\mathbb{E}$ the set of all the possible orientations of the edges in

G. Thus, for each edge there are two possible arrows, pointing to the two different vertices joined by that edge. For $e \in \mathbb{E}$ with end-points u and v, we write $e = uv$ if e has tail equal to u and head equal to v, and let $\overline{e}$ denote the reverse orientation of the same edge.

For a coefficient ring A, we consider the following two complexes:

$$d_A \colon C^0(G,A) \to C^1(G,A) \quad \text{and} \quad \partial_A \colon C_1(G,A) \to C_0(G,A).$$

Here, $C^0(G,A)$ denotes the A-module of functions $V \to A$ and $C_0(G,A)$ the free A-module generated by the vertices in V. Similarly, $C^1(G,A)$ is the A-module of functions $f \colon \mathbb{E} \to A$ which verify the condition $f(\overline{e}) = -f(e)$ for each $e \in \mathbb{E}$, and $C_1(G,A)$ is the quotient of the free A-module generated by $\mathbb{E}$ modulo the submodule generated by $e + \overline{e}$ for all $e \in \mathbb{E}$. Finally, the differentials are given by $d(f)(e) = f(v) - f(u)$ and $\partial(e) = v - u$ for each oriented edge $e = uv \in \mathbb{E}$.

The above A-modules come with natural isomorphisms $C_0(G,A) \to C^0(G,A)$, which takes v to the characteristic function χ_v, and $C_1(G,A) \to C^1(G,A)$, which takes e to $\chi_e - \chi_{\overline{e}}$. Also, we have bilinear forms $\langle\, , \rangle$ on $C_0(G,A)$ and $C_1(G,A)$ such that $\langle v, w\rangle = \delta_{v,w}$ for $v, w \in V$ and $\langle e, f\rangle = \delta_{e,f} - \delta_{e,\overline{f}}$ for $e, f \in \mathbb{E}$. The above isomorphisms induce bilinear forms on $C^0(G,A)$ and $C^1(G,A)$ as well. Denote by $d_A^* \colon C^1(G,A) \to C^0(G,A)$ the homomorphism corresponding to ∂_A under the isomorphisms. Then, d_A^* is the adjoint to d_A, that is, $\langle f, d_A^*(h)\rangle = \langle d_A(f), h\rangle$ for each $f \in C^0(G,A)$ and $h \in C^1(G,A)$.

Let $H_{0,A} := \{f \in C^0(G,A) \mid \sum f(v) = 0\}$ and $F_A := \mathrm{Im}(d_A)$. The Laplacian of G is $\Delta_A := d_A^* d_A$. The homomorphism $d_{\mathbb{R}}^*$ induces an isomorphism $F_{\mathbb{R}} \to H_{0,\mathbb{R}}$. Also, the bilinear form $\langle\, , \rangle$ induces a norm on $F_{\mathbb{R}}$ corresponding via $d_{\mathbb{R}}^*$ to the quadratic form q on $H_{0,\mathbb{R}}$ satisfying $q(f) = \langle f, \Delta_{\mathbb{R}}(f)\rangle$ for each $f \in C^0(G,\mathbb{R})$.

In [AE20a], we described certain tilings of $H_{0,\mathbb{R}}$ by polytopes. For instance, let $\Lambda_A := \mathrm{Im}(d_A^*)$. Then $\Lambda_{\mathbb{R}} = H_{0,\mathbb{R}}$ and $\Lambda_{\mathbb{Z}}$ is a sublattice of $H_{0,\mathbb{Z}}$ of finite index equal to the number of spanning trees of G, by the Kirchhoff Matrix-tree Theorem. The *standard Voronoi tiling* of G, denoted Vor_G, is the Voronoi decomposition of $H_{0,\mathbb{R}}$ with respect to $\Lambda_{\mathbb{Z}}$ and q: the tiles are

$$\mathrm{Vor}_q(\beta) := \{\eta \in H_{0,\mathbb{R}} \mid q(\eta - \beta) \leq q(\eta - \alpha) \text{ for every } \alpha \in \Lambda_{\mathbb{Z}} - \{\beta\}\}$$

for $\beta \in \Lambda_{\mathbb{Z}}$. This is one of the infinitely many different tilings we consider. The other ones are variations of this one that we call *twisted mixed Voronoi tilings* and denote $\mathrm{Vor}_{G,\ell}^{\mathfrak{m}}$. Though the standard Voronoi tiling is homogeneous, meaning all tiles are translates of the tile centered at the origin, $\mathrm{Vor}_G(O)$, a twisted mixed Voronoi tiling is obtained by putting together translations of the tiles $\mathrm{Vor}_H(O)$ associated to a collection of connected spanning subgraphs H of G. More precisely, the twisted mixed Voronoi tiling $\mathrm{Vor}_{G,\ell}^{\mathfrak{m}}$ depends on $\mathfrak{m} \in C^1(G,\mathbb{Z})$ (the "twisting") and an edge length function $\ell \colon E \to \mathbb{N}$; the tiles are the polytopes $d^*(\mathfrak{d}_f^{\mathfrak{m}}) + \mathrm{Vor}_{G_f^{\mathfrak{m}}}(O)$ for $f \in C^0(G,\mathbb{Z})$ with $G_f^{\mathfrak{m}}$ connected, where $\mathfrak{d}_f^{\mathfrak{m}} \in C^1(G,\mathbb{R})$ is a modification of $d_{\mathbb{Z}}(f)$ defined by

$$\mathfrak{d}_f^{\mathfrak{m}}(e) := \frac{1}{2}\big(\delta_e^{\mathfrak{m},\ell}(f) - \delta_{\overline{e}}^{\mathfrak{m},\ell}(f)\big), \text{ where } \delta_e^{\mathfrak{m},\ell}(f) := \Big\lfloor \frac{f(v) - f(u) + \mathfrak{m}_e}{\ell_e} \Big\rfloor \text{ for each } e = uv \in \mathbb{E},$$

and $G_f^{\mathfrak{m}}$ is the spanning subgraph of G obtaining by removing the edges $e \in \mathbb{E}$ for which $\mathfrak{d}_f^{\mathfrak{m}}(e) \notin \mathbb{Z}$. The construction of these tilings is briefly recalled in Section 2. We refer to [AE20a] for a thorough presentation.

1.3. **Toric arrangements.** Our aim in this paper is to associate arrangements of toric varieties to the twisted mixed Voronoi tilings we constructed in [AE20a].

To an arrangement of rational polytopes in a real vector space, rational with respect to a fixed full rank lattice, we describe how to associate an arrangement of toric varieties, a scheme whose irreducible components are the toric varieties associated to the polytopes, which intersect in toric subvarieties corresponding to the faces of those polytopes. The arrangement is called a *toric tiling* if it arises from a tiling of the whole vector space; see Subsection 3.1.

Let $G = (V, E)$ be a finite connected graph without loops. The lattice $C^1(G,\mathbb{Z})$ in $C^1(G,\mathbb{R})$ gives rise to an arrangement $\square_G$ of hypercubes hypercubes $\square_\alpha$ in $C^1(G,\mathbb{R})$ for $\alpha \in C^1(G,\mathbb{Z})$, where for each $\alpha \in C^1(G,\mathbb{Z})$, the points of $\square_\alpha$ are of the form $\alpha + \epsilon$ with $|\epsilon(e)| \leq \frac{1}{2}$ for all $e \in \mathbb{E}$. The arrangement of hypercubes $\square_G$ leads to an arrangement $\mathbf{R}$ of toric varieties $\mathbf{P}_\alpha$ for $\alpha \in C^1(G,\mathbb{Z})$, where each $\mathbf{P}_\alpha$ is isomorphic to the product $\prod_{e\in E} \mathbf{P}^1_e$; see Subsection 3.2. (Here $\mathbf{P}^1_e$ denotes a copy of the projective line $\mathbf{P}^1$.)

Consider a length function $\ell\colon E \to \mathbb{N}$ and a twisting $\mathfrak{m} \in C^1(G,\mathbb{Z})$. Let H be the subdivision of G given by the length function, and consider the Voronoi tiling $\mathrm{Vor}^{\mathfrak{m}}_{G,\ell}$ of $H_{0,\mathbb{R}}$. The tiling gives rise to an arrangement of toric varieties that we denote by $Y^{\mathrm{bt}}_{\ell,\mathfrak{m}}$ and call it the *basic toric tiling* associated to the triple consisting of the graph G, the edge length function ℓ and the twisting $\mathfrak{m}$; see Subsection 4.1.

We describe a natural embedding of $Y^{\mathrm{bt}}_{\ell,\mathfrak{m}}$ in $\mathbf{R}$. We do this by describing explicitly a natural embedding in $\mathbf{P}_0$ of the toric variety associated to $\mathrm{Vor}_G(O)$ in Theorem 3.10, the first main result of our paper. We use this to associate to each tile of $\mathrm{Vor}^{\mathfrak{m}}_{G,\ell}$ a toric subvariety of a certain $\mathbf{P}_\alpha$. We show that the union in $\mathbf{R}$ of these subvarieties has the structure of $Y^{\mathrm{bt}}_{\ell,\mathfrak{m}}$, our second main result; see Theorem 4.2 and Corollary 4.3.

We introduce moduli to $Y^{\mathrm{bt}}_{\ell,\mathfrak{m}}$, by deforming the equations of its irreducible components: To the extra data of characters

$$a\colon C^1(G,\mathbb{Z}) \to \mathbf{G_m}(\kappa) \quad \text{and} \quad b\colon H^1(G,\mathbb{Z}) \to \mathbf{G_m}(\kappa),$$

we associate a toric tiling $Y^{a,b}_{\ell,\mathfrak{m}}$ in $\mathbf{R}$, that lies in the rational equivalent class of $Y^{\mathrm{bt}}_{\ell,\mathfrak{m}}$, and coincides with it when the characters are trivial; see Subsection 4.2. (Here $\mathbf{G_m}$ denotes the multiplicative group.)

The extra data is necessary to describe the stable limits on the stable curve X of an invertible sheaf L_η in the situation of Subsection 1.1. We show in [AE20b] that the collection of stable limits is parameterized by a connected closed substack of dimension 0 of the Artin stack of all torsion-free, rank-one sheaves on X, and describe the substack as the quotient of $Y^{a,b}_{\ell,\mathfrak{m}}$ for certain ℓ, $\mathfrak{m}$, a and b by the natural action of a certain torus. The function ℓ is the one introduced in Subsection 4.2. The character a encodes partially the infinitesimal data of the arc drawn by π on the moduli of curves. The character b is an amalgamation of glueing data arising from certain limits of L_η, whereas the twisting $\mathfrak{m}$ keeps track of where those limits fail to be invertible. Furthermore, we show that all $Y^{a,b}_{\ell,\mathfrak{m}}$ arise in this way.

The rest of the current paper is then devoted to a thorough study of the arrangements $Y^{a,b}_{\ell,\mathfrak{m}}$ and the way they are embedded in $\mathbf{R}$. Here is what we do:

- We give a description of the toric stratification of $Y^{a,b}_{\ell,\mathfrak{m}}$. More precisely, we define a natural action of the torus $\mathbf{G}^V_\mathfrak{m}/\mathbf{G}_\mathbf{m} \simeq \mathbf{G}^{|V|-1}_\mathfrak{m}$ on each $Y^{a,b}_{\ell,\mathfrak{m}}$ and characterize its orbits; see Subsection 5.1 and Theorem 5.3. This is the crucial result used in [AE20b].
- We describe a complete set of (infinitely many) equations for the embedding of $Y^{a,b}_{\ell,\mathfrak{m}}$ inside $\mathbf{R}$; see Theorem 6.3. Of course, this will be crucial in describing the moduli of the various $Y^{a,b}_{\ell,\mathfrak{m}}$ in the "Hilbert scheme" of $\mathbf{R}$.
- We show that each $Y^{a,b}_{\ell,\mathfrak{m}}$ can be obtained as an equivariant flat degeneration of the algebraic torus $\mathbf{G}^{|V|-1}_\mathfrak{m}$, and thus the quotient stack is the degeneration of a point; see Theorem 7.2. This is naturally a consequence of what we do in [AE20b], but it is shown independently here.

1.4. **Organization.** The layout of the paper is as follows. In Section 2, we recall the Voronoi tilings introduced in [AE20a]. In Section 3, we describe the toric variety associated to a given graph, and use this in Section 4 to define the arrangements of toric varieties associated to the Voronoi tilings of Section 2, and to describe the way the toric varieties associated to the graph and certain of its connected subgraphs are glued together. In Section 5, we describe the action of the algebraic torus $\mathbf{G}^{|V|-1}_\mathbf{m}$ on the arrangements of toric varieties and give a complete description of its orbits. In Section 6, we work out the equations giving our arrangements for the natural embedding in $\mathbf{R}$. Finally, in Section 7, we show that our toric arrangements are all obtained as equivariant degenerations of the algebraic torus $\mathbf{G}^{|V|-1}_\mathbf{m}$.

2. Voronoi tilings associated to graphs

The aim of this preliminary section is to recall the construction of the different types of Voronoi tilings associated to graphs with edge lengths and twisting carried out in detail in the first part of this work [AE20a]. In this paper we will freely use the definitions, notations and results of [AE20a]. For the convenience of the reader, we reproduce a few of them here and later refer to appropriate parts of [AE20a] for the others.

Let $G = (V, E)$ be a finite connected graph without loops. We denote by $\mathbb{E}$ the set of all the orientations of the edges in G. We use $\{u, v\}$ to denote a non-oriented edge connecting vertices u and v, even if not unique, and uv or vu for the two possible orientations of that edge. For an oriented edge $e = uv \in \mathbb{E}$, we denote by $\overline{e} = vu$ the edge $\{u, v\}$ with the reverse orientation. We call u and v the tail and head of the edge $e = uv$, respectively, and sometimes denote them by t_e and h_e. In particular, we have $\mathrm{t}_{\overline{e}} = \mathrm{h}_e$ and $\mathrm{h}_{\overline{e}} = \mathrm{t}_e$. For a subset $X \subseteq V$, we denote by $\mathbb{E}(X, V - X)$ the set of oriented edges of the form uv with $u \in X$ and $v \in V - X$. These sets are called (oriented) cuts in the graph.

Let A be one of the rings $\mathbb{R}$, $\mathbb{Q}$ or $\mathbb{Z}$. We use the standard notations $C^0(G, A)$ and $C^1(G, A)$ for the A-modules of 0- and 1-cochains with values in A, respectively: the first consists of all the functions $f : V \to A$, whereas the second consists of all the functions $g : \mathbb{E} \to A$ which verify $g(e) = -g(\overline{e})$ for each oriented edge e in $\mathbb{E}$. Similarly, we use $C_0(G, A)$ and $C_1(G, A)$ for the A-modules of 0- and 1-chains on G: the first is the free module generated by the vertices of G, the generator associated to $v \in V$ denoted by (v), whereas the second is the quotient of the free module generated by the oriented edges of G by the submodule generated by the relations $(e) = -(\overline{e})$ for each $e \in \mathbb{E}$, where (e) denotes the element associated to e in the free module and the quotient.

The cochain complex is the complex $C^\bullet(G, A) : C^0(G, A) \xrightarrow{d} C^1(G, A)$, where the differential d is defined by

$$d(f)(e) := d_{uv} f := f(v) - f(u)$$

for each $f \in C^0(G, A)$ and $e = uv \in \mathbb{E}$. On the other hand, the chain complex is the complex $C_\bullet(G, A) : C_1(G, A) \xrightarrow{\partial} C_0(G, A)$, where the boundary map ∂ is defined by

$$\partial(e) := (v) - (u)$$

for each $e = uv \in \mathbb{E}$.

As defined above, it is clear that the spaces $C_i(G, A)$ and $C^i(G, A)$ are canonically dual for $i = 0, 1$. Moreover, there are natural scalar products $\langle \, , \rangle$ on $C_0(G, A)$ and $C_1(G, A)$ that naturally identify the space $C_i(G, A)$ with $C^i(G, A)$ for $i = 0, 1$. Because of this, we simply use α_e for the value of $\alpha \in C^1(G, A)$ at an oriented edge $e \in \mathbb{E}$. Also, ∂ becomes identified with the adjoint d^* of d, and we get $\partial d = d^* d = \Delta$, the *Laplacian* of the finite graph G, which is defined by

$$\Delta(f)(v) = \sum_{\substack{e \in \mathbb{E} \\ \mathrm{h}_e = v}} f(\mathrm{h}_e) - f(\mathrm{t}_e)$$

for each $f \in C^0(G, A)$ and $v \in V$.

At some point in the text it will be more convenient to fix an orientation of the edges. By an orientation we mean a map $\mathfrak{o} : E \to \mathbb{E}$ which for each edge $\{u, v\}$ of G takes one of the oriented edges uv or vu as value. In other words, $\mathfrak{o}$ is a right inverse to the natural forgetful map $\mathbb{E} \to E$. We denote by $E^{\mathfrak{o}}$ the image of $\mathfrak{o}$. Given $e \in \mathbb{E}$, by an abuse of the notation, we will also denote by e its image in E. And given $e \in E$, we let $e^{\mathfrak{o}}$ denote $\mathfrak{o}(e)$.

Using an orientation $\mathfrak{o}$, we can identify the inclusion of the lattice $C^1(G, \mathbb{Z})$ in $C^1(G, \mathbb{R})$ with the standard lattice $\mathbb{Z}^E$ in $\mathbb{R}^E$; in this way, the scalar product on $C^1(G, \mathbb{R})$ becomes the standard Euclidean product on $\mathbb{R}^E$. We denote the corresponding norm by $\| \, \|$. Furthermore, under this identification, the Voronoi decomposition of $C^1(G, \mathbb{Z}) \subset C^1(G, \mathbb{R})$

with respect to $\| \ \|$ is the standard tiling of $\mathbb{R}^E$ by hypercubes $\square_\alpha$, one for each element $\alpha \in \mathbb{Z}^E$, defined by

$$\square_\alpha := \Big\{ x \in \mathbb{R}^E \,\Big|\, |x_e - \alpha_e| \leq \frac{1}{2} \text{ for all } e \in E \Big\}.$$

The submodule F_A of $C^1(G, A)$ defined by the image of d is called *the module of A-valued cuts*. By definition, it is generated by elements of the form $d(\chi_X)$ for subsets $X \subseteq V$, where χ_X is the characteristic function of X. One easily verifies that

$$d(\chi_X)(e) := \begin{cases} -1 & \text{if } e \in \mathbb{E}(X, V - X), \\ 1 & \text{if } \overline{e} \in \mathbb{E}(X, V - X), \\ 0 & \text{otherwise.} \end{cases}$$

We call $F_{\mathbb{Z}}$ *the cut lattice* and $F_{\mathbb{R}}$ *the cut space* of G, respectively. Elements of the form $d(\chi_X)$ are called *cut elements*.

Define Λ_A as the image by d^* of F_A, and let $H_{0,A}$ be defined by

$$H_{0,A} := \Big\{ f \in C^0(G, A) \,\Big|\, \sum_{v \in V} f(v) = 0 \Big\}.$$

It is easy to see that $\Lambda_A \subseteq H_{0,A}$, with equality if $A = \mathbb{R}$. The sublattice $\Lambda_{\mathbb{Z}} \subseteq H_{0,\mathbb{Z}}$ is called *the Laplacian lattice* of the graph.

We denote by $\mathrm{Vor}_{\|.\|}(F_{\mathbb{Z}})$ the Voronoi decomposition of $(F_{\mathbb{R}}, \|.\|)$ with respect to the cut lattice $F_{\mathbb{Z}}$. The map d^* induces an isomorphic Voronoi decomposition of $\Lambda_{\mathbb{R}}$ with respect to the sublattice $\Lambda_{\mathbb{Z}}$, that we denote by $\mathrm{Vor}_G(\Lambda_{\mathbb{Z}})$, or simply Vor_G. In fact, the cells of Vor_G are the projections under d^* of the hypercubes $\square_\alpha$ for $\alpha \in F_{\mathbb{Z}}$; see [AE20a, Thm. 3.14].

In [AE20a] we gave a detailed description of the Voronoi decomposition of $F_{\mathbb{R}}$ with respect to $F_{\mathbb{Z}}$. Let us briefly recall it here.

Denote by $\mathrm{Vor}_F(\beta)$ the Voronoi cell of a lattice element $\beta \in F_{\mathbb{Z}}$. Let $\mathcal{FP}$ be the face poset of the polytope $\mathrm{Vor}_F(O)$ of the origin in $F_{\mathbb{Z}}$; as a set, $\mathcal{FP}$ consists of the faces in $\mathrm{Vor}_F(O)$, and the inclusion between faces defines the partial order. We explain now the combinatorics of $\mathcal{FP}$.

First, by a spanning subgraph G' of G we mean a subgraph with $V(G') = V(G)$. A *cut subgraph* of G is by definition a spanning subgraph G' of G for which we can find a partition $V_1, \dots, V_s$ of V such that the edges of G' are all those edges in G that connect a vertex of V_i to a vertex of V_j for $j \neq i$. Recall as well that an acyclic orientation of a graph G is an orientation which does not contain any oriented cycle. In [AE20a] we called an orientation D of a subgraph G' of G *coherent acyclic* if G' is a cut subgraph of G given by a partition $V_1, \dots, V_s$ of $V(G)$ that is ordered consistently with D, that is, all the edges between V_i and V_j for $i < j$ get orientation in D from V_i to V_j.

A coherent acyclic orientation D can be viewed as the subset $\mathbb{E}(D) \subseteq \mathbb{E}$ of its oriented edges. The corresponding set of non-oriented edges is denoted $E(D)$. We define $\mathcal{CAC}$ as the set consisting of all the coherent acyclic orientations of cut subgraphs of G. The set

$\mathcal{CAC}$ has a natural poset structure: for D_1 and D_2 in $\mathcal{CAC}$, set $D_1 \preceq D_2$ if $\mathbb{E}(D_2) \subseteq \mathbb{E}(D_1)$. We proved the following theorem in [AE20a].

Theorem 2.1 ([AE20a, Thm. 3.19])**.** *The two posets $\mathcal{FP}$ and $\mathcal{CAC}$ are isomorphic.*

We briefly recall how the isomorphism is defined. First, for an element $x \in F_{\mathbb{R}}$, recall that the positive support of x, denoted $\mathrm{supp}^+(x)$, is defined as the set of all oriented edges $e \in \mathbb{E}$ with $x_e > 0$. A *bond element* in $F_{\mathbb{Z}}$ is any element of the form $d(\chi_X)$ for a nonempty $X \subsetneq V$ such that both the graphs $G[X]$ and $G[V-X]$ induced on X and $V-X$, respectively, are connected. (The graph $G[X]$ induced by G on X has vertex set X and edge set all those edges of G with both endpoints in X.)

We showed in [AE20a] that bond elements form a system of generators for $F_{\mathbb{Z}}$. Moreover, they define a hyperplane arrangement in $F_{\mathbb{R}}$ as follows. Let β be a bond element of $F_{\mathbb{Z}}$. The affine hyperplane F_β of $F_{\mathbb{R}}$ is defined by

$$F_\beta := \left\{ x \in F_{\mathbb{R}} \,|\, 2\langle x, \beta\rangle = \|\beta\|^2 \right\}.$$

Moreover, for each lattice point $\mu \in F_{\mathbb{Z}}$ and bond element β, we consider the affine hyperplane $F_{\mu,\beta} := \mu + F_\beta$. The cell $\mathrm{Vor}_F(\mu)$ is the open cell containing μ of the hyperplane arrangement given by the $F_{\mu,\beta}$ for all bond elements β.

Consider now the Voronoi cell $\mathrm{Vor}_F(O)$ and let $\mathfrak{f}$ be a face of this polytope. We define $\mathcal{U}_{\mathfrak{f}}$ as the set of all the bond elements β in $F_{\mathbb{Z}}$ such that $\mathfrak{f} \subset F_\beta$.

With these definitions, we proved that the isomorphism $\phi : \mathcal{FP} \to \mathcal{CAC}$ is simply given by

$$\phi(\mathfrak{f}) := \bigcup_{\beta \in \mathcal{U}_{\mathfrak{f}}} \mathrm{supp}^+(\beta).$$

The tiling given by $\mathrm{Vor}_{\|.\|}(F_{\mathbb{Z}})$ or $\mathrm{Vor}_G(\Lambda_{\mathbb{Z}})$ is regular in the sense that all the Voronoi cells are translations of each other. We recall now the generalization of the above picture to the case of graphs equipped with an integer length function $\ell : E \to \mathbb{N}$ and a twisting $\mathfrak{m} \in C^1(G, \mathbb{Z})$.

Let $G = (V, E)$ be a connected loopless finite graph with integer edge lengths $\ell : E \to \mathbb{N}$, and let H be the subdivision of G where each edge e is subdivided $\ell_e - 1$ times. Let $\mathfrak{m} \in C^1(G, \mathbb{Z})$. We call $\mathfrak{m}$ a *twisting*.

There is a natural subcomplex $\square_H^{\mathfrak{m}}$ of the standard Voronoi tiling of $C^1(G, \mathbb{R})$ by hypercubes, which is defined as follows. For an element $f \in C^0(G, \mathbb{Z})$, and for an edge $e = uv$ in $\mathbb{E}$, set $\delta_e^{\mathfrak{m}}(f) := \lfloor \frac{f(v)-f(u)+\mathfrak{m}_e}{\ell_e} \rfloor$, where e is viewed in E via the natural forgetful map $\mathbb{E} \to E$.

We reproduce the following definitions from [AE20a, Section 5]. For each $f \in C^0(G, \mathbb{Z})$, let:

- $\mathfrak{d}_f^{\mathfrak{m}} \in C^1(G, \mathbb{R})$ given by

$$\mathfrak{d}_f^{\mathfrak{m}}(e) := \begin{cases} \delta_e^{\mathfrak{m}}(f) & \text{if } \ell_e \,\big|\, f(v) - f(u) + \mathfrak{m}_e, \\ \delta_e^{\mathfrak{m}}(f) + \frac{1}{2} & \text{otherwise} \end{cases}$$

for each oriented edge $e = uv \in \mathbb{E}$;

- $G^{\mathfrak{m}}_f$ be the spanning subgraph of G containing the support of all oriented edges $e \in \mathbb{E}$ for which $\mathfrak{d}^{\mathfrak{m}}_f(e) \in \mathbb{Z}$;
- $\square_{\mathfrak{d}^{\mathfrak{m}}_f}$ be the hypercube of dimension $|E(G^{\mathfrak{m}}_f)|$ defined by

$$\square_{\mathfrak{d}^{\mathfrak{m}}_f} := \mathfrak{d}^{\mathfrak{m}}_f + \Big\{x \in C^1(G,\mathbb{R}) \,\Big|\, |x_e| \leq \frac{1}{2} \text{ for all } e, \text{ and } x_e = 0 \text{ for all } e \in \mathbb{E}(G) - \mathbb{E}(G^{\mathfrak{m}}_f)\Big\};$$

- $\square^{\mathfrak{m}}_{G,\ell}$ be the arrangement of hypercubes $\square_{\mathfrak{d}^{\mathfrak{m}}_f}$ in $C^1(G,\mathbb{R})$, so

$$\square^{\mathfrak{m}}_{G,\ell} := \bigcup_{f \in C^0(G,\mathbb{Z})} \square_{\mathfrak{d}^{\mathfrak{m}}_f}.$$

We use $\square^{\mathfrak{m}}_f$ instead of $\square_{\mathfrak{d}^{\mathfrak{m}}_f}$ and $\square^{\mathfrak{m}}_H$ instead of $\square^{\mathfrak{m}}_{G,\ell}$, if there is no risk of confusion.

We consider now the projection map $d^* : \square^{\mathfrak{m}}_H \to H_{0,\mathbb{R}}$. For each $f \in C^0(G,\mathbb{R})$ with connected subgraph $G^{\mathfrak{m}}_f$, define $\mathrm{Vor}^{\mathfrak{m}}_H(f)$ as the image $d^*(\square^{\mathfrak{m}}_f)$. We showed in [AE20a, Prop. 5.7] that

$$\mathrm{Vor}^{\mathfrak{m}}_H(f) = d^*(\mathfrak{d}^{\mathfrak{m}}_f) + \mathrm{Vor}_{G^{\mathfrak{m}}_f}(O),$$

where $\mathrm{Vor}_{G^{\mathfrak{m}}_f}(O)$ denotes the Voronoi cell of the origin in $H_{0,\mathbb{R}}$ for the graph $G^{\mathfrak{m}}_f$. Furthermore, we proved the following theorem:

Theorem 2.2 ([AE20a, Thm. 5.9]). *The set of polytopes $\mathrm{Vor}^{\mathfrak{m}}_H(f)$ for $f \in C^0(G,\mathbb{Z})$ with connected spanning subgraph $G^{\mathfrak{m}}_f$ is a tiling of $H_{0,\mathbb{R}}$. Each $\mathrm{Vor}^{\mathfrak{m}}_H(f)$ in this tiling is congruent to the Voronoi cell of $G^{\mathfrak{m}}_f$.*

We denote this tiling by $\mathrm{Vor}^{\mathfrak{m}}_{G,\ell}$ or simply $\mathrm{Vor}^{\mathfrak{m}}_H$ if there is no risk of confusion. If $\mathfrak{m} = 0$ we drop the upper index $\mathfrak{m}$, and if $\ell = 1$ we drop the lower index ℓ.

3. Toric tilings associated to graphs

The aim of this section is to introduce tilings by toric varieties associated to the tilings by polytopes of the previous section.

3.1. **Toric tilings.** Let N be a lattice and $M := N^\vee := \mathrm{Hom}(N,\mathbb{Z})$ be its dual. Let $\langle\, , \rangle$ denote the natural pairing of $M_\mathbb{R}$ and $N_\mathbb{R}$. A rational polyhedral complex $\mathscr{P}$ in $M_\mathbb{R}$ is by definition a (possibly infinite) collection of rational polyhedra in $M_\mathbb{R}$ verifying the following two properties:

- A face $\mathfrak{f}$ of a polyhedron $\mathrm{P} \in \mathscr{P}$ belongs to $\mathscr{P}$.
- For any pair of polyhedra P_1 and P_2 in $\mathscr{P}$, their intersection is either empty or a common face to both P_1 and P_2.

We consider only polyhedral complexes which are locally finite, that is, for which each point x on $M_\mathbb{R}$ lies in a neighborhood which intersects only finitely many members of $\mathscr{P}$.

Example 3.1. Our two natural examples are the tiling of $C^1(G,\mathbb{R})$ by hypercubes with lattice $C^1(G,\mathbb{Z})$, and the polyhedral complex given by the arrangement of polytopes $\mathrm{Vor}_H^{\mathfrak{m}}$ in $H_{0,\mathbb{R}}$ with lattice $M=\Lambda_{\mathbb{Z}}$, or the isomorphic arrangement in $F_{\mathbb{R}}$ with lattice $M=F_{\mathbb{Z}}$ induced by the isomorphism $d^*\colon F_{\mathbb{Z}}\to\Lambda_{\mathbb{Z}}$.

Let $\mathscr{P}$ be a rational polyhedral complex and let P be a polyhedron in $\mathscr{P}$. Denote by $M_{\mathrm{P},\mathbb{R}}$ the linear subspace of $M_{\mathbb{R}}$ parallel to P, thus $M_{\mathrm{P},\mathbb{R}}$ is the tangent space to any point in the relative interior of P. Define the lattice $M_{\mathrm{P}}:=M\cap M_{\mathrm{P},\mathbb{R}}$, and note that by the rationality of $\mathscr{P}$, the lattice M_{P} is of full rank in $M_{\mathrm{P},\mathbb{R}}$.

For each subset $\sigma\subseteq M_{\mathbb{R}}$ let

$$\sigma^\vee := \{v\in N_{\mathbb{R}}\,|\,\langle v,w\rangle\geq 0 \text{ for every } w\in\sigma\}.$$

Analogously, define $\sigma^\vee\subseteq M_{\mathbb{R}}$ for each $\sigma\subseteq N_{\mathbb{R}}$. Notice that $\sigma^\vee$ is a convex cone. If σ is a linear subspace, so is $\sigma^\vee$. Set $N_{\mathrm{P},\mathbb{R}}:=M_{\mathrm{P},\mathbb{R}}^\vee$ for each polyhedron P in $\mathscr{P}$, and put $N_{\mathrm{P}}:=N\cap N_{\mathrm{P},\mathbb{R}}$. Again by the rationality of $\mathscr{P}$, the lattice N_{P} is of full rank in $N_{\mathrm{P},\mathbb{R}}$. Also, if $\mathfrak{f}$ is a face of P then $N_{\mathrm{P},\mathbb{R}}\subset N_{\mathfrak{f},\mathbb{R}}$.

To the polyhedron P is associated its normal fan Σ_{P}, which is a rational fan in the vector space $N_{\mathbb{R}}/N_{\mathrm{P},\mathbb{R}}$ with respect to the lattice N/N_{P}. Precisely, to each face $\mathfrak{f}$ of P we associate the normal cone $\sigma_{\mathfrak{f},\mathrm{P}}$ in $N_{\mathbb{R}}/N_{\mathrm{P},\mathbb{R}}$ given by

$$\sigma_{\mathfrak{f},\mathrm{P}} := \{v\in N_{\mathbb{R}}\,|\,\langle v,w_1-w_2\rangle\geq 0 \text{ for every } w_1\in\mathfrak{f} \text{ and } w_2\in\mathrm{P}\}/N_{\mathrm{P},\mathbb{R}}.$$

(Notice that $\sigma_{\mathfrak{f},\mathrm{P}}\subseteq N_{\mathfrak{f},\mathbb{R}}/N_{\mathrm{P},\mathbb{R}}$.) Then Σ_{P} is the collection of the $\sigma_{\mathfrak{f},\mathrm{P}}$ as $\mathfrak{f}$ runs through the faces of P.

Let $\mathbf{P}_{\mathrm{P}}$ be the toric variety corresponding to the normal fan Σ_{P}. Note that $\mathbf{P}_{\mathrm{P}}$ is complete if and only if Σ_{P} has full support in $N_{\mathbb{R}}/N_{\mathrm{P},\mathbb{R}}$, if and only if P is a polytope, that is, a bounded polyhedron.

Let P be a polyhedron in $\mathscr{P}$, and let $\mathfrak{f}$ be a face of P. Consider the normal cone $\sigma_{\mathfrak{f},\mathrm{P}}$ in $N_{\mathbb{R}}/N_{\mathrm{P},\mathbb{R}}$. The orbit closure associated to $\mathfrak{f}$ in the toric variety $\mathbf{P}_{\mathrm{P}}$ is isomorphic to the toric variety associated to the fan

$$\Big\{\,\big(\sigma_{\mathfrak{f}',\mathrm{P}}+N_{\mathfrak{f},\mathbb{R}}\big)/N_{\mathfrak{f},\mathbb{R}}\,\Big|\,\mathfrak{f}' \text{ a face of } \mathrm{P} \text{ contained in } \mathfrak{f}\Big\}.$$

The natural pairing induces an isomorphism $(N_{\mathbb{R}}/N_{\mathfrak{f},\mathbb{R}})^\vee\simeq M_{\mathfrak{f},\mathbb{R}}$, and the dual of the cone $\big(\sigma_{\mathfrak{f}',\mathrm{P}}+N_{\mathfrak{f},\mathbb{R}}\big)/N_{\mathfrak{f},\mathbb{R}}$ in $M_{\mathfrak{f},\mathbb{R}}$ is the dual of $\sigma_{\mathfrak{f}',\mathfrak{f}}$. It follows that the orbit closure associated to $\mathfrak{f}$ in $\mathbf{P}_{\mathrm{P}}$ is isomorphic to $\mathbf{P}_{\mathfrak{f}}$, and we thus get a natural embedding $\mathbf{P}_{\mathfrak{f}}\hookrightarrow\mathbf{P}_{\mathrm{P}}$.

Definition 3.2. The *toric arrangement* $\mathbf{P}_{\mathscr{P}}$ associated to $\mathscr{P}$ is defined as follows: Consider the disjoint union of the toric varieties $\mathbf{P}_{\mathrm{P}}$ for all P in $\mathscr{P}$, and identify $\mathbf{P}_{\mathfrak{f}}$ with the orbit closure associated to $\mathfrak{f}$ in $\mathbf{P}_{\mathrm{P}}$, as described above, for each face $\mathfrak{f}$ of each polyhedron P.

When $\mathscr{P}$ is a tiling of $M_{\mathbb{R}}$ by polytopes, that is, when the polyhedra are polytopes covering $M_{\mathbb{R}}$, we say that $\mathbf{P}_{\mathscr{P}}$ is a *toric tiling*.

The following proposition is straightforward.

Proposition 3.3. *Assume that $\mathscr{P}$ is a locally finite rational polyhedral complex in $M_{\mathbb{R}}$. Then the toric arrangement $\mathbf{P}_{\mathscr{P}}$ is naturally a scheme locally of finite type.*

We are not careful enough in Definition 3.2 to define completely the scheme structure of $\mathbf{P}_{\mathscr{P}}$, as we do not specify the scheme structure in neighborhoods of the intersections of the various $\mathbf{P}_{\mathrm{P}}$. However, whatever structure is given under the conditions imposed by Definition 3.2, Proposition 3.3 holds. Furthermore, we will see definite scheme structures being given in the cases mentioned in Example 3.1, in Subsection 3.2 below and in Subsection 4.1.

3.2. **Toric tiling $\mathbf{R}$ associated to the tiling of $C^1(G,\mathbb{R})$ by hypercubes.** Consider a finite connected loopless graph $G=(V,E)$. Let $M := C^1(G,\mathbb{Z})$ and apply the construction above to the associated Voronoi tiling of $M_{\mathbb{R}}$, the tiling by hypercubes $\square_\alpha$ for $\alpha \in C^1(G,\mathbb{Z})$. We get a toric tiling, denoted $\mathbf{R}$, that we describe below.

Since we have a natural pairing $\langle . , . \rangle$ on $M_{\mathbb{R}}$, we may identify M with the dual lattice $N := M^\vee$. We may thus view the normal fan Σ_α of each hypercube $\square_\alpha \subseteq M_{\mathbb{R}}$ in the same space $M_{\mathbb{R}}$. It is the standard fan given by the basis axes. More precisely, consider the family $\mathcal{A}$ of all subsets $A \subset \mathbb{E}(G)$ which have the property that for each edge $\{u,v\} \in E(G)$ at most one of the two possible orientations uv or vu are in A, in other words, A is an orientation of a subset, denoted $E(A)$, of edges in E. The *positive cone* σ_A associated to A is defined as $\sigma_A := \sum_{e\in A} \mathbb{R}_{\geq 0}(\chi_e - \chi_{\bar{e}})$. Then Σ_α consists of all the cones σ_A.

It follows that each maximal dimensional toric variety $\mathbf{P}_\alpha$ in the tiling $\mathbf{R}$ is isomorphic to the product $\prod_{e\in E} \mathbf{P}^1$. And the tiling has the following description: Fix an orientation $\mathfrak{o}$ for the set of edges of the graph. Consider an infinite number of copies $\mathbf{P}^1_{e,i}$ of $\mathbf{P}^1$, indexed by the $e \in E^{\mathfrak{o}}$ and integers $i \in \mathbb{Z}$. We distinguish two points $0_{e,i}$ and $\infty_{e,i}$ on each $\mathbf{P}^1_{e,i}$. In fact, for later use, we give coordinates $(\mathrm{x}_{e,i} : \mathrm{x}_{\bar{e},i})$ to each $\mathbf{P}^1_{e,i}$, with $\mathrm{x}_{e,i} = 0$ corresponding to $0_{e,i}$ and $\mathrm{x}_{\bar{e},i} = 0$ to $\infty_{e,i}$. For each edge $e \in E^{\mathfrak{o}}$, we define $\mathbf{R}_e$ as the locally finite scheme obtained by gluing $\mathbf{P}^1_{e,i}$ and $\mathbf{P}^1_{e,i+1}$ over the points $0_{e,i}$ and $\infty_{e,i+1}$ for each $i \in \mathbb{Z}$. The toric tiling $\mathbf{R}$ is then the product of the $\mathbf{R}_e$ over the edges $e \in E^{\mathfrak{o}}$. In this description,

$$\mathbf{P}_\alpha = \prod_{e\in E^{\mathfrak{o}}} \mathbf{P}^1_{e,\alpha_e} \quad \text{for each } \alpha \in C^1(G,\mathbb{Z}).$$

We may likewise describe the other toric varieties in the tiling $\mathbf{R}$, the ones associated to the proper faces of the hypercubes. First, for each edge $e \in E^{\mathfrak{o}}$ and half integer $a \in \frac{1}{2}\mathbb{Z} - \mathbb{Z}$, we set $\mathbf{P}_{e,a} := \{0_{e,\lfloor a \rfloor}\} = \{\infty_{e,\lceil a \rceil}\} \subset \mathbf{R}_e$. Then we extend the definition of $\mathbf{P}_\alpha$ by setting

$$\mathbf{P}_\alpha := \prod_{e\in E^{\mathfrak{o}}} \mathbf{P}_{e,\alpha_e} \quad \text{for each } \alpha \in C^1(G,\frac{1}{2}\mathbb{Z}).$$

The whole tiling $\mathbf{R}$ is completely described by the toric varieties $\mathbf{P}_\alpha$ and the natural existing inclusions between them.

Removing from each $\mathbf{P}_\alpha$ all those $\mathbf{P}_\beta$ contained in it, we obtain the interior of $\mathbf{P}_\alpha$, an open subscheme denoted $\mathbf{P}^*_\alpha$. The $\mathbf{P}^*_\alpha$ for $\alpha \in C^1(G, \frac{1}{2}\mathbb{Z})$ form a stratification of $\mathbf{R}$.

Finally, we may describe $\mathbf{P}_0$ as a gluing of affine schemes. For each oriented edge $e \in \mathbb{E}$, we denote by $\mathbf{A}^1_e$ the open subscheme of $\mathbf{P}^1_{e^{\mathfrak{o}},0}$ corresponding to $\mathrm{x}_{\bar{e},0} \neq 0$ and by $\mathbf{G}_{\mathbf{m},e}$ the open subscheme corresponding to $\mathrm{x}_{e,0}\mathrm{x}_{\bar{e},0} \neq 0$. Notice that $\mathbf{G}_{\mathbf{m},e} = \mathbf{G}_{\mathbf{m},\bar{e}}$, and we may thus abuse the notation by forgetting the orientation of e. For each $A \in \mathcal{A}$, let $S_A := \sigma_A^\vee \cap M$, the semigroup of integral points of the dual cone $\sigma_A^\vee$. The toric variety $\mathbf{P}_0$ is covered by the affine open subsets $\prod_{e\in A} \mathbf{A}^1_e \times \prod_{e \in E-E(A)} \mathbf{G}_{\mathbf{m},e}$, each isomorphic to $\mathrm{Spec}\big(\kappa[S_A]\big)$.

3.3. **Toric variety associated to the Voronoi cell of a graph.** Consider a finite connected loopless graph G with uniform edge lengths equal to one with fixed orientation $\mathfrak{o}$, and denote by $\mathrm{Vor}_F(O)$ the Voronoi cell of the origin in $F_\mathbb{R}$ with respect to $F_\mathbb{Z}$. In [AE20a, Section 3], we described the face poset of $\mathrm{Vor}_F(O)$ in terms of the coherent acyclic orientations of subgraphs of G, a description we reviewed in Section 2. In this section, we use those results to describe the normal fan of $\mathrm{Vor}_F(O)$, and to provide a complete description of the corresponding toric variety, which we denote by $\mathbf{P}_G$.

Recall the description of the face poset $\mathcal{FP}$ of the Voronoi cell $\mathrm{Vor}_F(O)$: The faces are in bijection with coherent acyclic orientations D of cut subgraphs of G, and under this bijection, the face $\mathfrak{f}_D$ associated to D has support in the affine plane F_D given by $F_D = \bigcap_{\beta\in\mathcal{U}_D} F_\beta$, where $F_\beta = \big\{x \in F_\mathbb{R}, \big| \, 2\langle x,\beta\rangle = \|\beta\|^2\big\}$ and $\mathcal{U}_D$ is the set of all bond elements β of $F_\mathbb{Z}$ with $\mathrm{supp}^+(\beta) \subseteq \mathbb{E}(D)$; see [AE20a, Subsection 3.4].

The pairing $\langle . , . \rangle$ on $M_\mathbb{R} = C^1(G,\mathbb{R})$ gives the orthogonal decomposition $M_\mathbb{R} = F_\mathbb{R} \oplus \ker(d^*)$, which allows us to identify further $F_\mathbb{R}$ with its dual $F_\mathbb{R}^\vee$. Under this identification, the normal cone σ_D to the face $\mathfrak{f}_D$ of $\mathrm{Vor}_F(O)$ is the cone in $F_\mathbb{R}$ generated by the $\beta \in \mathcal{U}_D$, that is,

$$\sigma_D = \sum_{\beta\in\mathcal{U}_D} \mathbb{R}_{\geq 0}\beta.$$

Therefore, we get the following result.

Proposition 3.4. *The normal fan Σ_G of the Voronoi polytope $\mathrm{Vor}_F(O)$ consists of the cones σ_D for $D \in \mathcal{CAC}$, where σ_D is the cone generated by all the $\beta \in \mathcal{U}_D$. In particular, the rays of Σ_G are in bijection with the bond elements of $F_\mathbb{Z}$.*

For each $D \in \mathcal{CAC}$, consider the corresponding cone $\sigma_D \in \Sigma_G$. The dual cone $\sigma_D^\vee \subseteq F_\mathbb{R}$ consists of all the elements of $F_\mathbb{R}$ which are non-negative on σ_D. Denote by $S_D := \sigma_D^\vee \cap F_\mathbb{Z}$ the semigroup of integral points in $\sigma_D^\vee$, and let $U_D := \mathrm{Spec}(\kappa[S_D])$. The toric variety $\mathbf{P}_G$ is obtained by gluing the affine varieties U_D.

Consider the inclusion $F_\mathbb{R} \hookrightarrow M_\mathbb{R}$. We have:

Proposition 3.5. *The lattice $F_\mathbb{Z}$ coincides with the lattice $M \cap F_\mathbb{R}$. In particular, the dual map $N \to F_\mathbb{Z}^\vee$ is surjective.*

Proof. The elements of $M\cap F_\mathbb{R}$ are of the form $d(f)$ for $f \in C^0(G,\mathbb{R})$ such that $f(v)-f(u) \in \mathbb{Z}$ for each $e = uv \in \mathbb{E}(G)$. Since G is connected, f has to be of the form $h + c$ for

$h \in C^0(G,\mathbb{Z})$ and a constant $c \in \mathbb{R}$, and so $d(f) = d(h) \in F_{\mathbb{Z}}$. The second statement follows from the first, as the first implies that $M/F_{\mathbb{Z}}$ is torsion-free. $\square$

Remark 3.6. The pairing $\langle .\,,.\rangle$ is integral on $M = C^1(G,\mathbb{Z})$ and moreover identifies M with its dual N. Under this identification, the surjection $N \to F_{\mathbb{Z}}^\vee$ gets identified with the restriction $M \to F_{\mathbb{Z}}$ of the orthogonal projection map.

For each $D \in \mathcal{CAC}$, recall that the set of oriented edges of D is denoted by $\mathbb{E}(D)$, and gives the cone $\sigma_{\mathbb{E}(D)}$ in the normal fan Σ_0 to $\square_0$. Since $\mathrm{supp}^+(\beta) \subseteq \mathbb{E}(D)$ for each $\beta \in \mathcal{U}_D$, the inclusion $F_{\mathbb{R}} \hookrightarrow M_{\mathbb{R}}$ restricts to an inclusion of cones $\sigma_D \hookrightarrow \sigma_{\mathbb{E}(D)}$.

Proposition 3.7. *We have $\sigma_D = F_{\mathbb{R}} \cap \sigma_{\mathbb{E}(D)}$. Also, $\mathbb{R}\sigma_D = F_{\mathbb{R}} \cap \mathbb{R}\sigma_{\mathbb{E}(D)}$.*

Proof. Let f be an element of $C^0(G,\mathbb{R})$ such that $d(f)$ belongs to $\mathbb{R}\sigma_{\mathbb{E}(D)}$. Adding a constant to f if necessary, we may assume that the minimum of f over the vertices is zero. Let $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_l$ be all the values taken by f, and denote by X_i the set of all vertices v with $f(v_i) \geq \alpha_i$ for each $i = 1,\dots,l$. Notice that $X_1 \supset X_2 \supset \cdots \supset X_l$. We can write

$$f = \alpha_1\chi_{X_1} + (\alpha_2-\alpha_1)\chi_{X_2} + \cdots + (\alpha_l - \alpha_{l-1})\chi_{X_l},$$

so that

$$d(f) = \alpha_1 d(\chi_{X_1}) + (\alpha_2-\alpha_1)d(\chi_{X_2}) + \cdots + (\alpha_l - \alpha_{l-1})d(\chi_{X_l}).$$

Since the X_i form a decreasing chain of subsets, it follows that the supports $\mathrm{supp}^+(d(\chi_{X_i}))$ are consistent in their orientations. Moreover, since $\alpha_1 > 0$ and the α_i form an increasing sequence, $\mathrm{supp}^+(d(\chi_{X_i}))$ is a subset of $\mathrm{supp}^+(d(f))$. Each $d(\chi_{X_i})$ is an element of $F_{\mathbb{Z}}$ and is thus a sum of bond elements $\beta_{i,j}$ with $\mathrm{supp}^+(\beta_{i,j}) \subseteq \mathrm{supp}^+(d(\chi_{X_i}))$; see [AE20a, Prop. 3.34]. If $d(f) \in \sigma_{\mathbb{E}(D)}$, that is, $\mathrm{supp}^+(d(f)) \subset \mathbb{E}(D)$ then all the $\beta_{i,j}$ belong to $\mathcal{U}_D$. This implies that $d(f)$ is a sum of bond elements belonging to $\mathcal{U}_D$, that is, $d(f) \in \sigma_D$, and the first statement follows.

As for the second statement, notice that at any rate all the $\beta_{i,j}$ belong to $F_{\mathbb{R}} \cap \mathbb{R}\sigma_{\mathbb{E}(D)}$. Thus we may suppose that $f = \chi_X$ for some cut $X \subset V$. Since $D \in \mathcal{CAC}$, there is a partition $V_1,\dots,V_s$ of V such that $e \in \mathbb{E}(D)$ if and only if e is an edge connecting V_i to V_j with $i < j$. We may suppose $G[V_i]$ is connected for each i. Let $X_i := X \cap V_i$ for each i. Any edge connecting $V_i - X_i$ to X_i would be in $\mathrm{supp}^+(d(f))$ but would not be supported over $E(D)$. Thus $\mathbb{E}(V_i - X_i, X_i) = \emptyset$ for each i. Furthermore, since $G[V_i]$ is connected, either $X_i = \emptyset$ or $X_i = V_i$. It follows that X is a union of some of the V_i. We need only prove now that $d(\chi_{V_i}) \in \mathbb{R}\sigma_D$ for each i. But this follows because $d(\chi_{V_i} + \cdots + \chi_{V_s}) \in F_{\mathbb{R}} \cap \sigma_{\mathbb{E}(D)}$, and thus $d(\chi_{V_i} + \cdots + \chi_{V_s}) \in \sigma_D$ for each i by the first statement. $\square$

We prove now the surjectivity of the dual map.

Proposition 3.8. *Notations as above, the map of dual cones $\sigma_{\mathbb{E}(D)}^\vee \to \sigma_D^\vee$ is surjective. Furthermore, the induced map on integral points $\sigma_{\mathbb{E}(D)}^\vee \cap M \to \sigma_D^\vee \cap F_{\mathbb{Z}}$ is surjective.*

Proof. Let ϕ be an element of the dual cone $\sigma_D^\vee$ in $F_\mathbb{R}$ (resp. in $F_\mathbb{Z}$). We want to show the existence of an element $\eta \in \sigma_{\mathbb{E}(D)}^\vee$ (resp. $\eta \in \sigma_{\mathbb{E}(D)}^\vee \cap M$), thus an element $\eta \in C^1(G,\mathbb{R})$ (resp. $\eta \in C^1(G,\mathbb{Z})$) non-negative on every edge of $\mathbb{E}(D)$, whose orthogonal projection to $F_\mathbb{R}$ is ϕ.

Since ϕ is in $\sigma_D^\vee$, for each bond element $\beta \in \mathcal{U}_D$ we have $\langle \phi, \beta\rangle \geq 0$. Now, it follows from [AE20a, Prop. 3.34] that any element $\alpha \in F_\mathbb{R}$ with $\mathrm{supp}^+(\alpha) \subset \mathbb{E}(D)$ is a linear combination with positive coefficients of bond elements in $\mathcal{U}_D$. Then

$$\langle \phi, \alpha\rangle \geq 0 \qquad \text{for all } \alpha \in F_\mathbb{R} \text{ with } \mathrm{supp}^+(\alpha) \subset \mathbb{E}(D).$$

This shows that for each $f \in C^0(G,\mathbb{R})$ with $\mathrm{supp}^+(d(f)) \subset \mathbb{E}(D)$, we have

$$\langle \phi, d(f)\rangle = \langle d^*(\phi), f\rangle \geq 0.$$

In particular, for each cut $X \subseteq V$ with $\mathbb{E}(V-X,X) \subset \mathbb{E}(D)$, we have $d^*(\phi)(X) \geq 0$. Furthermore, if $\phi \in F_\mathbb{Z}$ then $d^*(\phi) \in C^0(G,\mathbb{Z})$.

It follows thus from the next proposition that there exists $\eta \in C^1(G,\mathbb{R})$ (resp. $\eta \in C^1(G,\mathbb{Z})$) with $\eta(e) \geq 0$ for all $e \in \mathbb{E}(D)$ such that

$$d^*(\eta) = d^*(\phi).$$

Since $\eta - \phi \in \ker(d^*)$, it follows that η projects to ϕ. ☐

Recall that $H_{0,\mathbb{R}}$ is the set of all elements $h \in C^0(G,\mathbb{R})$ with $\sum_{v\in V} h(v) = 0$. Moreover, the map $d^* : C^1(G,\mathbb{R}) \to C^0(G,\mathbb{R})$ surjects onto $H_{0,\mathbb{R}}$. We are interested in characterizing those elements of $H_{0,\mathbb{R}}$ which are in the image of elements $\eta \in C^1(G,\mathbb{R})$ that are positive on all oriented edges in $\mathbb{E}(D)$. We have the following theorem.

Proposition 3.9. *Let $G = (V,E)$ be a finite loopless graph and D an orientation of a subset of E. Let $h \in C^0(G,\mathbb{R})$ be an element in $H_{0,\mathbb{R}}$. The following two statements are equivalent:*

(1) For all subsets $X \subseteq V$ with $\mathbb{E}(V-X,X) \subseteq \mathbb{E}(D)$, we have $h(X) \geq 0$.

(2) There exists $\eta \in C^1(G,\mathbb{R})$ with $d^*(\eta) = h$ such that $\eta(e) \geq 0$ for all $e \in \mathbb{E}(D)$.

Moreover, in the case $h \in C^0(G,\mathbb{Z})$, we can modify (2) *by requiring $\eta \in C^1(G,\mathbb{Z})$.*

Proof. The implication (2) ⇒ (1) follows from observing that for an element η as in (2), we have for each subset $X \subseteq V$:

$$h(X) = d^*(\eta)(X) = \eta\big(\mathbb{E}(V-X,X)\big) \geq 0.$$

We prove the other implication, (1) ⇒ (2). Denote by V^+ the set of all vertices v with $h(v) > 0$, by V^- the set of all vertices v with $h(v) < 0$, and by V^0 the set of all vertices v with $h(v) = 0$. We proceed by induction on $|V^+| + |V^-|$. If this quantity is zero, then $h \equiv 0$, and $\eta = 0$ does the job.

Suppose now we have $|V^+| + |V^-| = n > 0$. Since $h \in H_{0,\mathbb{R}}$, both V^+ and V^- are nonempty. We claim there exists an oriented path P compatible with D which connects a vertex v of V^- to a vertex u of V^+. Indeed, let X be the set of vertices that can be connected to a vertex of V^- by an oriented path compatible with D, in the sense that no oriented edge e in P satisfies $\overline{e} \in \mathbb{E}(D)$. Then $\mathbb{E}(V-X, X) \subset \mathbb{E}(D)$. If $X \cap V^+ = \emptyset$ then $h(X) = h(V^-) < 0$, which would contradict the assumption on h.

Let P be such an oriented path from a vertex v in V^- to a vertex u in V^+. We view P as an element of $C^1(G, \mathbb{Z})$ taking value one at each oriented edge $e \in \mathbb{E}(P)$. For each $\epsilon \geq 0$, define $h_\epsilon := h - \epsilon d^*(P) = h - \epsilon\chi_u + \epsilon\chi_v$. So $h_\epsilon(u) = h(u) - \epsilon$, $h_\epsilon(v) = h(v) + \epsilon$, and $h_\epsilon(w) = h(w)$ for any w different from u and v. Let ϵ be the greatest nonnegative real number such that $h_\epsilon(u) \geq 0$, $h_\epsilon(v) \leq 0$ and $h_\epsilon(X) \geq 0$ for every cut $X \subseteq V$ with $\mathbb{E}(V-X, X) \subset \mathbb{E}(D)$. If $h \in C^0(G, \mathbb{Z})$ then ϵ is an integer, and so $h_\epsilon \in C^0(G, \mathbb{Z})$.

If $h_\epsilon(u) = 0$ or $h_\epsilon(v) = 0$, then the value of $|V^+| + |V^-|$ for h_ϵ has decreased, hence by the induction hypothesis there exists $\eta_\epsilon \in C^1(G, \mathbb{R})$ with η_ϵ nonnegative on $\mathbb{E}(D)$ such that $d^*(\eta_\epsilon) = h_\epsilon$. Also, η_ϵ is integral if so is h. Let $\eta := \eta_\epsilon + \epsilon P$. Then η is nonnegative on $\mathbb{E}(D)$ and $d^*(\eta) = h$. Furthermore, if h is integral, so is η. This finishes the proof in this case.

In the remaining case, we get a nonempty proper subset $X \subsetneq V$ of vertices with $\mathbb{E}(V - X, X) \subset \mathbb{E}(D)$ such that $h_\epsilon(X) = 0$. In this case, one verifies that both the restrictions of h_ϵ to X and to $V - X$ satisfy the conditions of the proposition. A second induction on the size of V then shows the existence of $\eta_1 \in C^1(G[X], \mathbb{R})$ and $\eta_2 \in C^1(G[V-X], \mathbb{R})$ such that η_1 is nonnegative on all edges of $\mathbb{E}(D)$ with both endpoints in X, and η_2 is nonnegative on all edges of $\mathbb{E}(D)$ with both endpoints in $V - X$, and such that $d^*(\eta_1) = h_{\epsilon|X}$ in $G[X]$ and $d^*(\eta_2) = h_{\epsilon|V-X}$ in $G[V - X]$. Furthermore, we may take η_1 and η_2 integral in the case h, and so h_ϵ is integral. Define $\eta_\epsilon \in C^1(G, \mathbb{R})$ to be equal to η_1 on $G[X]$, equal to η_2 on $G[V - X]$, and equal to zero on all edges of $\mathbb{E}(V - X, X)$. We have $d^*(\eta_\epsilon) = h_\epsilon$, and η_ϵ is obviously nonnegative on all edges of $\mathbb{E}(D)$. Moreover, it is integral if h is integral. Taking now $\eta := \eta_\epsilon + \epsilon P$, we conclude as before. □

Consider now the map of cones $\sigma^\vee_{\mathbb{E}(D)} \to \sigma^\vee_D$. Using Proposition 3.8, we get a surjective map of semigroups

$$S_{\mathbb{E}(D)} = \sigma^\vee_{\mathbb{E}(D)} \cap M \longrightarrow S_D = \sigma^\vee_D \cap F_{\mathbb{Z}},$$

which induces a closed embedding of $U_D \to U_{\mathbb{E}(D)}$ in the affine variety

$$U_{\mathbb{E}(D)} := \operatorname{Spec}[\kappa[S_{\mathbb{E}(D)}]] \simeq \prod_{e \in \mathbb{E}(D)} \mathbf{A}^1_e \times \prod_{e \in E - E(D)} \mathbf{G}_{\mathbf{m},e},$$

which is the affine open subvariety of $\mathbf{P}_0$ corresponding to the cone $\sigma_{\mathbb{E}(D)}$.

The maps $U_D \to U_{\mathbb{E}(D)}$ glue to a map $\alpha_0 \colon \mathbf{P}_G \to \mathbf{P}_0$ satisfying $\alpha_0^{-1}(U_{\mathbb{E}(D)}) = U_D$ for each coherent acyclic orientation D of a cut subgraph of G. Thus α_0 is a closed embedding. Its image is described by the following theorem:

Theorem 3.10. *Notations as above, the map α_0 induces an isomorphism from $\mathbf{P}_G$ to the subvariety of $\mathbf{P}_0 = \prod_{e\in E^{\mathfrak{o}}}\mathbf{P}_{e,0}$ given by the equations*

$$\forall \text{ oriented cycle } \gamma \text{ in } G, \qquad \prod_{e\in\gamma\cap E^{\mathfrak{o}}} \mathrm{x}_{e,0} \prod_{e\in\bar\gamma\cap E^{\mathfrak{o}}} \mathrm{x}_{\bar e,0} = \prod_{e\in\bar\gamma\cap E^{\mathfrak{o}}} \mathrm{x}_{e,0} \prod_{e\in\gamma\cap E^{\mathfrak{o}}} \mathrm{x}_{\bar e,0}.$$

Before proving the theorem we need to introduce notation and prove a proposition.

For each subset $A\subset\mathbb{E}(G)$ which is an orientation of a subset of edges of G, let $\kappa[\mathrm{x}_A]$ be the polynomial ring generated by the variables x_e for $e\in A$, and define

$$\mathbf{A}^A := \mathrm{Spec}[\kappa[\mathrm{x}_A]] = \prod_{e\in A}\mathbf{A}^1_e.$$

If D is an acyclic orientation of the whole graph G, then $U_{\mathbb{E}(D)} = \mathbf{A}^{\mathbb{E}(D)}$, and we get that α_0 restricts to an embedding $U_D\hookrightarrow\mathbf{A}^{\mathbb{E}(D)}$. In this case, we have the following proposition.

Proposition 3.11. *Let D be an acyclic orientation of the whole graph G. Then the embedding $U_D\hookrightarrow\mathbf{A}^{\mathbb{E}(D)}$ identifies U_D with the subvariety of the affine space $\mathbf{A}^{\mathbb{E}(D)}$ given by the equations*

$$(*)\quad \forall \text{ oriented cycle } \gamma \text{ in } G, \qquad \prod_{e\in\gamma\cap\mathbb{E}(D)}\mathrm{x}_e = \prod_{e\in\bar\gamma\cap\mathbb{E}(D)}\mathrm{x}_e.$$

Proof. Let α denote the surjection $\kappa[S_{\mathbb{E}(D)}]\to\kappa[S_D]$. We show first that for each oriented cycle γ in G, the difference $\prod_{e\in\gamma\cap\mathbb{E}(D)}\mathrm{x}_e - \prod_{e\in\bar\gamma\cap\mathbb{E}(D)}\mathrm{x}_e$ is in the kernel of α.

We view each oriented cycle γ as the element $\sum_{e\in\gamma}(\chi_e-\chi_{\bar e})\in M = C^1(G,\mathbb{Z})$, which we will also denote by γ. Also, we let

$$\gamma_+ := \sum_{e\in\gamma\cap\mathbb{E}(D)}(\chi_e-\chi_{\bar e}) \quad\text{ and }\quad \gamma_- := \sum_{e\in\bar\gamma\cap\mathbb{E}(D)}(\chi_e-\chi_{\bar e}).$$

Thus $\mathrm{supp}^+(\gamma_\pm)\subseteq\mathbb{E}(D)$ and $\gamma=\gamma_+-\gamma_-$.

Now, if γ is an oriented cycle, $\langle\gamma,\beta\rangle = 0$ for every $\beta\in F_\mathbb{R}$. Thus $\langle\gamma_+,\beta\rangle = \langle\gamma_-,\beta\rangle$ for every $\beta\in F_\mathbb{R}$, that is, the orthogonal projections of γ_+ and γ_- in $F_\mathbb{R}$ coincide.

It follows that $\alpha(\prod_{e\in\gamma\cap\mathbb{E}(D)}\mathrm{x}_e) = \alpha(\mathrm{x}^{\gamma_+}) = \alpha(\mathrm{x}^{\gamma_-}) = \alpha(\prod_{e\in\bar\gamma\cap\mathbb{E}(D)}\mathrm{x}_e)$, which shows that the kernel of α contains the difference $\prod_{e\in\gamma\cap\mathbb{E}(D)}\mathrm{x}_e - \prod_{e\in\bar\gamma\cap\mathbb{E}(D)}\mathrm{x}_e$.

We now prove that the differences $\prod_{e\in\gamma\cap\mathbb{E}(D)}\mathrm{x}_e - \prod_{e\in\bar\gamma\cap\mathbb{E}(D)}\mathrm{x}_e$ generate the kernel of α. Indeed, the kernel of α is generated by the elements of the form $\mathrm{x}^{\eta_1}-\mathrm{x}^{\eta_2}$ for η_1 and η_2 in M verifying

(1) $\mathrm{supp}^+(\eta_i)\subseteq\mathbb{E}(D)$ for $i=1,2$ and
(2) $\langle\eta_1,\beta\rangle = \langle\eta_2,\beta\rangle$ for every $\beta\in\mathcal{U}_D$.

Cancelling out the common factors, we may further assume that $\mathrm{supp}^+(\eta_1)\cap\mathrm{supp}^+(\eta_2)=\emptyset$. Note that, since D is an acyclic orientation of the whole graph, the $\beta\in\mathcal{U}_D$ generate $F_\mathbb{R}$, whence (2) implies that $\eta_1-\eta_2$ is orthogonal to the cut lattice.

Consider the oriented graph G' on the vertex set $V(G)$ which for each oriented edge $e \in \mathbb{E}(D)$ contains $\eta_1(e)$ oriented edges parallel to e and $\eta_2(e)$ oriented edges parallel to $\overline{e}$. Since $\eta_1 - \eta_2$ is orthogonal to the cut lattice, we get that $\langle \eta_1 - \eta_2, d(\chi_v)\rangle = 0$ for the cut defined by each vertex v in G, which implies that the graph G' is Eulerian, that is, at each vertex v the in-degree of v is equal to its out-degree. Then the (oriented) edges of the Eulerian oriented graph G' can be partitioned into an edge-disjoint union of oriented cycles $\gamma_1, \dots, \gamma_N$. Notice that for each $i = 1, \dots, N$ and each $e \in \mathbb{E}$ there is at most one oriented edge of γ_i among those parallel to e, and thus we can view γ_i as an oriented cycle of G as well. Thus, we have

$$\eta_1 = \sum_{i=1}^{N} \gamma_{i,+} \quad \text{and} \quad \eta_2 = \sum_{i=1}^{N} \gamma_{i,-}.$$

Let $\mathrm{A}_i := \mathrm{x}^{\gamma_{i,+}}$ and $\mathrm{B}_i := \mathrm{x}^{\gamma_{i,-}}$ for $i = 1, \dots, N$. Put $\mathrm{B}_0 := 1$ and $\mathrm{A}_{N+1} := 1$. Since

$$\mathrm{x}^{\eta_1} - \mathrm{x}^{\eta_2} = \mathrm{A}_1 \cdots \mathrm{A}_N - \mathrm{B}_1 \cdots \mathrm{B}_N = \sum_{i=0}^{N-1} \mathrm{B}_0 \cdots \mathrm{B}_i(\mathrm{A}_{i+1} - \mathrm{B}_{i+1})\mathrm{A}_{i+2} \cdots \mathrm{A}_{N+1},$$

it follows that $\mathrm{x}^{\eta_1} - \mathrm{x}^{\eta_2} \in (\mathrm{x}^{\gamma_{1,+}} - \mathrm{x}^{\gamma_{1,-}}, \dots, \mathrm{x}^{\gamma_{N,+}} - \mathrm{x}^{\gamma_{N,-}})$, finishing the proof. □

We can now prove Theorem 3.10.

Proof of Theorem 3.10. By Proposition 3.11, the equations giving U_D in $U_{\mathbb{E}(D)}$ are just the dehomogenizations of the equations given in the theorem, if D is an acyclic orientation of the whole graph. Now, every $D \in \mathcal{CAC}$ can be completed to an acyclic orientation of the whole graph G. Indeed, as the two posets $\mathcal{FP}$ and $\mathcal{CAC}$ are isomorphic by Theorem 2.1, and the vertices of $\mathrm{Vor}_F(O)$ correspond to acyclic orientations of the whole graph by [AE20a, Lem. 3.42], the statement is equivalent to saying that every face of $\mathrm{Vor}_F(O)$ contains a vertex of $\mathrm{Vor}_F(O)$. (Or we may use Lemma 3.12 below.) Thus, the U_D for acyclic orientations D of the whole graph form a covering of $\mathbf{P}_G$. It follows that the points on $\mathbf{P}_G$ satisfy the equations given in the theorem.

Moreover, to conclude the proof it is enough to show that every point p on $\mathbf{P}_0$ satisfying the equations given in the theorem lies on $U_{\mathbb{E}(D)}$ for a certain acyclic orientation D of the whole graph G. This is the case, as it follows from the equations that for each oriented cycle γ in G, the product $\prod_{e\in\gamma} \mathrm{x}_{e,0}$ vanishes on p if and only if so does $\prod_{e\in\gamma} \mathrm{x}_{\overline{e},0}$. Thus, if we let $A \subseteq \mathbb{E}$ be the subset of all oriented edges e such that $\mathrm{x}_{e,0}$ vanishes on p we have that A is an orientation of a subset of edges of G satisfying the following property: If γ is an oriented cycle in G, then $\gamma \cap A = \emptyset$ if and only if $\gamma \cap \overline{A} = \emptyset$. The following lemma finishes the proof. □

Lemma 3.12. *Let $A \subseteq \mathbb{E}$ be an orientation of a subset of edges of G such that for each oriented cycle γ of G, we have that $\gamma \cap A = \emptyset$ if and only if $\overline{\gamma} \cap A = \emptyset$. Then A can be completed to an acyclic orientation of all the edges of G.*

Proof. Let G' be the subgraph of G obtained by removing all the edges supporting A. Give G' any acyclic orientation, for instance, following an ordering of the vertices, and extend it to an orientation $\mathfrak{o}$ of G by adding A. We claim $\mathfrak{o}$ is acyclic. Indeed, if γ is an oriented cycle of G following this orientation, since G' has an induced acyclic orientation, then there must be $e \in \gamma \cap A$. But then, by hypothesis, there is $e' \in \gamma \cap \overline{A}$, hence $e' \in \gamma$ which does not follow the orientation $\mathfrak{o}$, a contradiction. ☐

3.4. **Toric orbits.** The toric variety $\mathbf{P}_G$ is naturally stratified by its toric orbits. We describe now this stratification of $\mathbf{P}_G$. More precisely, let D be a coherent acyclic orientation of a cut subgraph of G. Denote by T_D the torus orbit associated to the normal cone $\sigma_D \in \Sigma_G$ to $\mathfrak{f}_D$. Our aim is to describe the closure $\overline{T}_D$ of T_D in $\mathbf{P}_G$.

Let $Y_1, \dots, Y_d$ be the connected components of the graph obtained by removing all edges of $E(D)$. By ordering the Y_i appropriately, we may and will assume that an oriented edge $e \in \mathbb{E}(G)$ is in D if and only if e connects Y_i to Y_j for $i < j$. For each $i = 1, \dots, d$, let $M_i := C^1(Y_i, \mathbb{Z})$ and $N_i := M_i^\vee$. Also, let $F_{i,\mathbb{R}}$ and $F_{i,\mathbb{Z}}$ be the cut space and the cut lattice of Y_i. Denote by Σ_i the normal fan of $\mathrm{Vor}_{F_i}(O) \subset F_{i,\mathbb{R}}$. Also, let E_i be the set of edges of Y_i and $\mathfrak{o}_i$ the orientation induced from the orientation $\mathfrak{o}$ of G. As before, we will identify $M_{i,\mathbb{R}}$ with $N_{i,\mathbb{R}}$ for each i. We may also view each $M_{i,\mathbb{R}}$ in $M_\mathbb{R}$ under extension by zero.

Consider first the subvariety $\mathbf{P}_{D,0}$ of $\mathbf{P}_0$, which consists of all the points $x \in \mathbf{P}_0$ whose e-th coordinates $(x_{e,0} : x_{\bar{e},0})$ for $e \in \mathbb{E}(D)$ satisfy $x_{e,0} = 0$. This is the closure of the torus associated to the cone $\sigma_{\mathbb{E}(D)}$ in the normal fan of the hypercube $\square_0$. It is thus the toric variety associated to the fan $\Sigma_{D,0}$ in $C^1(G,\mathbb{R})/\mathbb{R}\sigma_{\mathbb{E}(D)}$ consisting of the cones

$$\frac{\sigma_{\mathbb{E}(D')} + \mathbb{R}\sigma_{\mathbb{E}(D)}}{\mathbb{R}\sigma_{\mathbb{E}(D)}}$$

for $D' \in \mathcal{CAC}$ with $D' \preceq D$, or equivalently, $\mathbb{E}(D') \supseteq \mathbb{E}(D)$.

The subvariety $\mathbf{P}_{D,0}$ is naturally isomorphic to the product $\prod_{i=1}^d \mathbf{P}_{i,0}$ where each $\mathbf{P}_{i,0}$ itself is isomorphic to the product $\prod_{e \in E_i^{\mathfrak{o}_i}} \mathbf{P}^1_e$. The isomorphism is induced by the isomorphism

$$\bigoplus_{i=1}^d C^1(Y_i,\mathbb{R}) \longrightarrow C^1(G,\mathbb{R}) \longrightarrow \frac{C^1(G,\mathbb{R})}{\mathbb{R}\sigma_{\mathbb{E}(D)}}, \tag{3.1}$$

as it takes the product of the standard fans $\Sigma_{i,0}$ associated to the Y_i onto $\Sigma_{D,0}$. In particular, denoting by $\mathbf{P}_{Y_i}$ the toric varieties associated to the fans Σ_i, Theorem 3.10 yields embeddings $\mathbf{P}_{Y_i} \hookrightarrow \mathbf{P}_{i,0}$. Taking the product of these embeddings, we thus get an embedding

$$\alpha_D : \prod_{i=1}^d \mathbf{P}_{Y_i} \longhookrightarrow \mathbf{P}_{D,0}.$$

Proposition 3.13. *Notations as above, the image of α_D is $\mathbf{P}_G \cap \mathbf{P}_{D,0}$.*

Proof. The equations defining $\mathbf{P}_G \cap \mathbf{P}_{D,0}$ in $\mathbf{P}_0$ are those given by Theorem 3.10 and the equations $\mathrm{x}_{e,0} = 0$ for all $e \in \mathbb{E}(D)$. Now, if γ is an oriented cycle in Y_i, for each i, the equation prescribed by Theorem 3.10 for $\mathbf{P}_G$ is the same as the equation prescribed for $\mathbf{P}_{Y_i}$ in $\mathbf{P}_{i,0}$. Thus we need only show that the equation given by an oriented cycle γ in G not supported entirely in any of the Y_i is a consequence of the equations $\mathrm{x}_{e,0} = 0$ for all $e \in \mathbb{E}(D)$.

Indeed, let γ be such a cycle. Since γ is not entirely supported in any Y_i, there is $e \in \gamma$ connecting Y_i to Y_j for $i \neq j$. Assume, without loss of generality as it will be clear, that $i < j$. Since γ is a cycle there will be $f \in \gamma$ connecting Y_l to Y_m for $l > m$. Thus $e \in \mathbb{E}(D)$ and $\overline{f} \in \mathbb{E}(D)$, whence $\mathrm{x}_{e,0} = 0$ and $\mathrm{x}_{\overline{f},0} = 0$ on $\mathbf{P}_G \cap \mathbf{P}_{D,0}$. But then the equation associated to γ given by Theorem 3.10 is satisfied on $\mathbf{P}_G \cap \mathbf{P}_{D,0}$ because both sides are equal to 0. □

Theorem 3.14. *Notations as above, the embedding α_D induces an isomorphism from the product of toric varieties associated to the connected components of $G - E(D)$ to the closure of the orbit T_D in $\mathbf{P}_G$.*

Proof. Composition (3.1) induces an isomorphism from $\oplus_i F_{i,\mathbb{R}}$ onto the image of $F_\mathbb{R}$ in the quotient. This image is isomorphic to the quotient of $F_\mathbb{R}$ by $F_\mathbb{R} \cap \mathbb{R}\sigma_{\mathbb{E}(D)}$. But the latter is $\mathbb{R}\sigma_D$, by Proposition 3.7. So we obtain a natural isomorphism

$$\bigoplus_{i=1}^{d} F_{i,\mathbb{R}} \longrightarrow \frac{F_\mathbb{R}}{\mathbb{R}\sigma_D}. \tag{3.2}$$

The orbit closure of T_D is isomorphic to the toric variety $\mathbf{P}(\Sigma_D)$ defined by the fan Σ_D in $F_\mathbb{R}/\mathbb{R}\sigma_D$ consisting of the cones

$$\tau_{D'} := \Big(\sigma_{D'} + \mathbb{R}\sigma_D\Big)/\mathbb{R}\sigma_D$$

for $D' \in \mathcal{CAC}$ with $D' \preceq D$, or equivalently, $\mathbb{E}(D') \supseteq \mathbb{E}(D)$. To finish the proof, we will show that the fan Σ_D is the sum of the fans Σ_i under the isomorphism (3.2).

For each $D' \in \mathcal{CAC}$ with $\mathbb{E}(D') \supseteq \mathbb{E}(D)$, if we let D'_i denote the orientation of the edges in the graph Y_i given by D' for each $i = 1, \dots, d$, it follows that D'_i is a coherent acyclic orientation of Y_i. Let σ_i denote the corresponding cone in $F_{i,\mathbb{R}}$. We claim that, under the isomorphism (3.2), the sum of cones $\oplus_i \sigma_i$ is taken to $\tau_{D'}$.

Indeed, each σ_i is generated by elements of the form $\beta|_{\mathbb{E}(Y_i)}$ where $\beta := d(\chi_Z)$ for cuts $Z \subseteq V$ contained in $V(Y_i)$ such that $\mathrm{supp}^+(\beta|_{\mathbb{E}(Y_i)}) \subseteq \mathbb{E}(D'_i)$. Let W be the union of Z with the $V(Y_l)$ for $l > i$ and put $\alpha := d(\chi_W)$. Clearly, $\alpha|_{\mathbb{E}(Y_i)} = \beta|_{\mathbb{E}(Y_i)}$. Also, $\alpha \in \sigma_{\mathbb{E}(D')}$ and thus $\alpha \in \sigma_{D'}$ by Proposition 3.7. Furthermore, if β^* denotes the extension by zero of $\beta|_{\mathbb{E}(Y_i)}$ to $C^1(G, \mathbb{R})$, then $\alpha - \beta^* \in \mathbb{R}\sigma_{\mathbb{E}(D)}$. Hence the image of $\beta|_{\mathbb{E}(Y_i)}$ in $F_\mathbb{R}/\mathbb{R}\sigma_D$ is in $\tau_{D'}$.

Conversely, $\sigma_{D'}$ is generated by elements of the form $\beta := d(\chi_Z)$ for cuts $Z \subseteq V$ such that $\mathrm{supp}^+(\beta) \subseteq \mathbb{E}(D')$. Put $Z_i := Z \cap V(Y_i)$ for each i and set $\beta_i := d(\chi_{Z_i})$. Then $\mathrm{supp}^+(\beta_i|_{\mathbb{E}(Y_i)}) \subseteq \mathbb{E}(D'_i)$ and thus $\beta_i|_{\mathbb{E}(Y_i)} \in \sigma_i$ by Proposition 3.7. Let $\beta_i^* \in C^1(G, \mathbb{R})$ be the extension by zero of $\beta_i|_{\mathbb{E}(Y_i)}$ for each i. Then $\beta - \sum_i \beta_i^* \in \mathbb{R}\sigma_{\mathbb{E}(D)}$.

On the other hand, if D_i' is a coherent acyclic orientation of a cut subgraph of Y_i for each $i = 1, \dots, d$, let D' denote the orientation in the graph G given by the D_i' and the D_i. It is clear that D' is a coherent acyclic orientation of a cut subgraph of G that induces D_i' on each graph Y_i and satisfies $\mathbb{E}(D') \supseteq \mathbb{E}(D)$. It follows that the fan Σ_D is the sum of the fans Σ_i under the isomorphism (3.2).

Therefore, the toric variety $\mathbf{P}(\Sigma_D)$ associated to Σ_D is isomorphic to the product of the toric varieties associated to the Σ_i, namely $\prod \mathbf{P}_{Y_i}$. □

4. Mixed toric tilings I: Scheme structure

4.1. **Basic toric tiling associated to ℓ and $\mathfrak{m}$.** In this section, we describe the toric tiling associated to the mixed tiling defined by a graph G, an edge length function ℓ and a twisting $\mathfrak{m}$.

Consider thus the general situation described in [AE20a, Section 5] and reviewed in Section 2. Let G be a loopless connected graph. Let $\ell : E \to \mathbb{N}$ be an edge length function, and $\mathfrak{m} \in C^1(G, \mathbb{Z})$ a twisting. Let $\mathrm{Vor}_H^{\mathfrak{m}}$ be the Voronoi decomposition of $H_{0,\mathbb{R}}$. The maximal dimensional cells of $\mathrm{Vor}_H^{\mathfrak{m}}$ are of the form $\mathrm{Vor}_H^{\mathfrak{m}}(f)$ for $f \in C^0(G, \mathbb{Z})$ with connected $G_f^{\mathfrak{m}}$. We have $\mathrm{Vor}_H^{\mathfrak{m}}(f) = d^*(\mathfrak{d}_f^{\mathfrak{m}}) + \mathrm{Vor}_{G_f^{\mathfrak{m}}}(O)$. By Theorem 3.10 applied to $G_f^{\mathfrak{m}}$, the toric variety $\mathbf{P}_{G_f^{\mathfrak{m}}}$ embeds naturally in $\mathbf{P}_{\mathfrak{d}_f^{\mathfrak{m}}}$, and is given by the equations associated to the oriented cycles in $G_f^{\mathfrak{m}}$. We denote this subvariety of $\mathbf{P}_{\mathfrak{d}_f^{\mathfrak{m}}}$ by $\mathbf{P}_{\ell,\mathfrak{m},f}$. Denote by $Y_{\ell,\mathfrak{m}}^{\mathrm{bt}}$ the union of the $\mathbf{P}_{\ell,\mathfrak{m},f}$ in $\mathbf{R}$. We call this union the *basic toric tiling* or the *basic toric arrangement* associated to G, ℓ and $\mathfrak{m}$. The names are justified by the following theorem and its corollary.

First, we introduce notation and a lemma. For each $\mathfrak{m} \in C^1(G, \mathbb{Z})$, each $f \in C^0(G, \mathbb{Z})$ and each $n \in \mathbb{N}$, set

$$\delta_e^{\mathfrak{m},n}(f) := \Big\lfloor \frac{f(\mathrm{h}_e) - f(\mathrm{t}_e) + n\mathfrak{m}_e}{n\ell_e} \Big\rfloor \quad \text{for each } e \in \mathbb{E},$$

and let $\mathfrak{d}_f^{\mathfrak{m},n} \in C^1(G, \frac{1}{2}\mathbb{Z})$ be defined by putting

$$\mathfrak{d}_f^{\mathfrak{m},n}(e) = \frac{1}{2}(\delta_e^{\mathfrak{m},n}(f) - \delta_{\bar{e}}^{\mathfrak{m},n}(f))$$

for each $e \in \mathbb{E}$. It is the same definition given in Section 2 for $n = 1$, in which case the superscript is dropped.

Lemma 4.1. *Let $f, h \in C^0(G, \mathbb{Z})$ and $n, p \in \mathbb{N}$. Let γ be an oriented cycle in G such that $\mathfrak{d}_h^{\mathfrak{m},p}(e) \in \mathbb{Z}$ for each $e \in \gamma$. Then there is $e_1 \in \gamma$ such that $\mathfrak{d}_f^{\mathfrak{m},n}(e_1) < \mathfrak{d}_h^{\mathfrak{m},p}(e_1)$ if and only if there is $e_2 \in \gamma$ such that $\mathfrak{d}_f^{\mathfrak{m},n}(e_2) > \mathfrak{d}_h^{\mathfrak{m},p}(e_2)$.*

Proof. Since $\mathfrak{d}_h^{\mathfrak{m},p}(e) \in \mathbb{Z}$ for each $e \in \gamma$, it follows that

$$\sum_{e \in \gamma} \mathfrak{d}_h^{\mathfrak{m},p}(e) = \sum_{e \in \gamma} \frac{\mathfrak{m}_e}{\ell_e}. \tag{4.1}$$

Now, also

$$\sum_{e\in\gamma}\frac{f(\mathrm{h}_e)-f(\mathrm{t}_e)+n\mathfrak{m}_e}{n\ell_e}=\sum_{e\in\gamma}\frac{\mathfrak{m}_e}{\ell_e}. \tag{4.2}$$

Suppose there is $e_1\in\gamma$ such that $\mathfrak{d}_f^{\mathfrak{m},n}(e_1)<\mathfrak{d}_h^{\mathfrak{m},p}(e_1)$. Since $\mathfrak{d}_h^{\mathfrak{m},p}(e_1)$ is an integer, it follows that the summand on the left-hand side of Equation (4.2) corresponding to e_1 is smaller than $\mathfrak{d}_h^{\mathfrak{m},p}(e_1)$. But then it follows from Equations (4.1) and (4.2) that there is $e_2\in\gamma$ such that the summand on the left-hand side of Equation (4.2) corresponding to e_2 is bigger than $\mathfrak{d}_h^{\mathfrak{m},p}(e_2)$. Since the latter is an integer, it follows that $\mathfrak{d}_f^{\mathfrak{m},n}(e_2)>\mathfrak{d}_h^{\mathfrak{m},p}(e_2)$.

If there is $e_1\in\gamma$ such that $\mathfrak{d}_f^{\mathfrak{m},n}(e_1)>\mathfrak{d}_h^{\mathfrak{m},p}(e_1)$, apply the above argument to $\overline{\gamma}$ to conclude. □

Theorem 4.2. *Let $f_1,f_2\in C^0(G,\mathbb{Z})$ with both $G^{\mathfrak{m}}_{f_1}$ and $G^{\mathfrak{m}}_{f_2}$ connected. Then $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_{f_1}}\cap\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_{f_2}}\neq\emptyset$ if and only if the Voronoi cells $\mathrm{Vor}^{\mathfrak{m}}_H(f_1)$ and $\mathrm{Vor}^{\mathfrak{m}}_H(f_2)$ intersect. Furthermore, in this case,*

$$\mathbf{P}_{\ell,\mathfrak{m},f_1}\cap\mathbf{P}_{\ell,\mathfrak{m},f_2}=\mathbf{P}_{\ell,\mathfrak{m},f_1}\cap\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_{f_2}}=\mathbf{P}_{\ell,\mathfrak{m},f_2}\cap\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_{f_1}} \tag{4.3}$$

and the intersection $\mathbf{P}_{\ell,\mathfrak{m},f_1}\cap\mathbf{P}_{\ell,\mathfrak{m},f_2}$ is the closure of the torus orbit of the corresponding face of $\mathrm{Vor}^{\mathfrak{m}}_H(f_i)$ for $i=1,2$. Also, letting $G^{\mathfrak{m}}_{f_1,f_2}$ be the graph obtained from the intersection of the spanning subgraphs $G^{\mathfrak{m}}_{f_1}$ and $G^{\mathfrak{m}}_{f_2}$ of G by keeping only the edges e with $\mathfrak{d}^{\mathfrak{m}}_{f_1}(e)=\mathfrak{d}^{\mathfrak{m}}_{f_2}(e)$, there is a natural isomorphism

$$\prod_{i=1}^{d}\mathbf{P}_{Z_i}\to\mathbf{P}_{\ell,\mathfrak{m},f_1}\cap\mathbf{P}_{\ell,\mathfrak{m},f_2},$$

where $Z_1,\dots,Z_d$ are the components of $G^{\mathfrak{m}}_{f_1,f_2}$.

Proof. Let $X_1,\dots,X_q\subseteq V$ be the level subsets of f_2-f_1, in increasing order. Let D_1 be the coherent acyclic orientation of the cut subgraph of $G^{\mathfrak{m}}_{f_1}$ induced by the ordered partition $X_1,\dots,X_q$ of V, and D_2 that of the cut subgraph of $G^{\mathfrak{m}}_{f_2}$ induced by the same partition in the reverse order, $X_q,\dots,X_1$.

If $e=uv\in\mathbb{E}(D_1)$ then

$$\frac{f_2(v)-f_2(u)+\mathfrak{m}_e}{\ell_e}>\frac{f_1(v)-f_1(u)+\mathfrak{m}_e}{\ell_e}. \tag{4.4}$$

As the right-hand side above is an integer, it follows that $\mathfrak{d}^{\mathfrak{m}}_{f_2}(e)>\mathfrak{d}^{\mathfrak{m}}_{f_1}(e)$. Conversely, if $e=uv\in\mathbb{E}(G^{\mathfrak{m}}_{f_1})$ satisfies $\mathfrak{d}^{\mathfrak{m}}_{f_2}(e)>\mathfrak{d}^{\mathfrak{m}}_{f_1}(e)$ then (4.4) holds, and thus $u\in X_i$ and $v\in X_j$ for $i<j$, that is, $e\in\mathbb{E}(D_1)$. To summarize,

$$\mathbb{E}(D_1):=\{e\in\mathbb{E}(G^{\mathfrak{m}}_{f_1})\,|\,\mathfrak{d}^{\mathfrak{m}}_{f_2}(e)>\mathfrak{d}^{\mathfrak{m}}_{f_1}(e)\}\quad\text{and}\quad\mathbb{E}(D_2):=\{e\in\mathbb{E}(G^{\mathfrak{m}}_{f_2})\,|\,\mathfrak{d}^{\mathfrak{m}}_{f_1}(e)>\mathfrak{d}^{\mathfrak{m}}_{f_2}(e)\},$$

the second equality by analogy with the first. It follows as well that $G^{\mathfrak{m}}_{f_1,f_2}$ is the subgraph of $G^{\mathfrak{m}}_{f_i}$ obtained by removing the edges e in $E(D_i)$ for each $i=1,2$. In addition, the collection of connected components of $G^{\mathfrak{m}}_{f_i}[X_1],\dots,G^{\mathfrak{m}}_{f_i}[X_q]$ is $Z_1,\dots,Z_d$ for $i=1,2$.

If $\mathrm{Vor}_H^{\mathfrak{m}}(f_1)$ and $\mathrm{Vor}_H^{\mathfrak{m}}(f_2)$ intersect then, by [AE20a, Prop. 5.12],

$$\mathfrak{d}_{f_1}^{\mathfrak{m}} + \frac{1}{2}\chi_{D_1} = \mathfrak{d}_{f_2}^{\mathfrak{m}} + \frac{1}{2}\chi_{D_2},$$

where χ_{D_i} is the characteristic function of D_i, taking value $+1$ at $e \in \mathbb{E}(D_i)$, value -1 at $e \in \mathbb{E}$ with $\bar{e} \in \mathbb{E}(D_i)$, and value 0 elsewhere, for $i = 1, 2$. In particular, $|\mathfrak{d}_{f_2}^{\mathfrak{m}}(e) - \mathfrak{d}_{f_1}^{\mathfrak{m}}(e)| \le 1$ for each $e \in \mathbb{E}$ with equality only if $e \in \mathbb{E}(G_{f_1}^{\mathfrak{m}}) \cap \mathbb{E}(G_{f_2}^{\mathfrak{m}})$, in fact, if and only if $e \in \mathbb{E}(D_1)$ and $\bar{e} \in \mathbb{E}(D_2)$ or $\bar{e} \in \mathbb{E}(D_1)$ and $e \in \mathbb{E}(D_2)$. At any rate, it follows that $\mathbf{P}_{\mathfrak{d}_{f_1}^{\mathfrak{m}}} \cap \mathbf{P}_{\mathfrak{d}_{f_2}^{\mathfrak{m}}} \neq \emptyset$.

Assume now that $\mathbf{P}_{\mathfrak{d}_{f_1}^{\mathfrak{m}}} \cap \mathbf{P}_{\mathfrak{d}_{f_2}^{\mathfrak{m}}} \neq \emptyset$. Then $|\mathfrak{d}_{f_2}^{\mathfrak{m}}(e) - \mathfrak{d}_{f_1}^{\mathfrak{m}}(e)| \le 1$, with equality only if $\mathfrak{d}_{f_1}^{\mathfrak{m}}(e)$ and $\mathfrak{d}_{f_2}^{\mathfrak{m}}(e)$ are integers. Also $\mathfrak{d}_{f_2}^{\mathfrak{m}}(e) = \mathfrak{d}_{f_1}^{\mathfrak{m}}(e)$ if neither $\mathfrak{d}_{f_1}^{\mathfrak{m}}(e)$ nor $\mathfrak{d}_{f_2}^{\mathfrak{m}}(e)$ is an integer. Thus $\mathbf{P}_{\mathfrak{d}_{f_1}^{\mathfrak{m}}} \cap \mathbf{P}_{\mathfrak{d}_{f_2}^{\mathfrak{m}}} = \mathbf{P}_\alpha$, where $\alpha_e := 1/2(\mathfrak{d}_{f_1}^{\mathfrak{m}}(e) + \mathfrak{d}_{f_2}^{\mathfrak{m}}(e))$ for each $e \in \mathbb{E}(G)$, unless $|\mathfrak{d}_{f_2}^{\mathfrak{m}}(e) - \mathfrak{d}_{f_1}^{\mathfrak{m}}(e)| = 1/2$, in which case α_e is the half integer between $\mathfrak{d}_{f_1}^{\mathfrak{m}}(e)$ and $\mathfrak{d}_{f_2}^{\mathfrak{m}}(e)$.

Observe that $G_{f_1,f_2}^{\mathfrak{m}}$ is the spanning subgraph of G obtained by keeping all oriented edges e such that $\alpha_e \in \mathbb{Z}$. Also,

$$\alpha = \mathfrak{d}_{f_1}^{\mathfrak{m}} + \frac{1}{2}\chi_{D_1} = \mathfrak{d}_{f_2}^{\mathfrak{m}} + \frac{1}{2}\chi_{D_2}. \tag{4.5}$$

In particular, by [AE20a, Prop. 5.12], the Voronoi cells $\mathrm{Vor}_H^{\mathfrak{m}}(f_1)$ and $\mathrm{Vor}_H^{\mathfrak{m}}(f_2)$ intersect.

Now, $\mathbf{P}_{\ell,\mathfrak{m},f_1} \cap \mathbf{P}_{\mathfrak{d}_{f_2}^{\mathfrak{m}}}$ is the subvariety of $\mathbf{P}_{\ell,\mathfrak{m},f_1}$ given by the equations $\mathrm{x}_{e,\mathfrak{d}_{f_1}^{\mathfrak{m}}(e^{\mathfrak{o}})} = 0$ for each $e \in \mathbb{E}(D_1)$. It follows from Proposition 3.13 and Theorem 3.10 that $\mathbf{P}_{\ell,\mathfrak{m},f_1} \cap \mathbf{P}_{\mathfrak{d}_{f_2}^{\mathfrak{m}}}$ is defined in $\mathbf{P}_\alpha$ by the following equations: For each $i = 1, \dots, d$ and each oriented cycle γ in Z_i,

$$\prod_{e \in \gamma \cap E^{\mathfrak{o}}} \mathrm{x}_{e,\alpha_e} \prod_{e \in \bar{\gamma} \cap E^{\mathfrak{o}}} \mathrm{x}_{\bar{e},\alpha_e} = \prod_{e \in \bar{\gamma} \cap E^{\mathfrak{o}}} \mathrm{x}_{e,\alpha_e} \prod_{e \in \gamma \cap E^{\mathfrak{o}}} \mathrm{x}_{\bar{e},\alpha_e}. \tag{4.6}$$

As the same description applies to $\mathbf{P}_{\ell,\mathfrak{m},f_2} \cap \mathbf{P}_{\mathfrak{d}_{f_1}^{\mathfrak{m}}}$, we obtain (4.3) and the last statement of the theorem.

Furthermore, by Theorem 3.14, the intersection $\mathbf{P}_{\ell,\mathfrak{m},f_1} \cap \mathbf{P}_{\ell,\mathfrak{m},f_2}$ is the closure of the torus orbit of the face $\mathfrak{f}_i$ of $\mathrm{Vor}_H^{\mathfrak{m}}(f_i)$ corresponding to D_i for each $i = 1, 2$. By [AE20a, Prop. 5.12], the two faces $\mathfrak{f}_1$ and $\mathfrak{f}_2$ coincide with $\mathrm{Vor}_H^{\mathfrak{m}}(f_1) \cap \mathrm{Vor}_H^{\mathfrak{m}}(f_2)$. The proof of the theorem is finished. ☐

From Theorem 4.2, we get directly the following corollary.

Corollary 4.3. *The structure on the standard toric tiling $Y_{\ell,\mathfrak{m}}^{\mathrm{bt}}$ is obtained as follows: Take the disjoint union*

$$\bigsqcup_{\substack{f \in C^0(G,\mathbb{Z}) \\ G_f^{\mathfrak{m}} \text{connected}}} \mathbf{P}_{\ell,\mathfrak{m},f}.$$

For f and h in $C^0(G,\mathbb{Z})$ with both $G_f^{\mathfrak{m}}$ and $G_h^{\mathfrak{m}}$ connected such that the corresponding Voronoi cells share a face $\mathfrak{f} := \mathrm{Vor}_H^{\mathfrak{m}}(f) \cap \mathrm{Vor}_H^{\mathfrak{m}}(h) \neq \emptyset$, identify the closure in $\mathbf{P}_{\ell,\mathfrak{m},f}$ of the torus orbit associated to the face $\mathfrak{f}$ of $\mathrm{Vor}_H^{\mathfrak{m}}(f)$ with the closure in $\mathbf{P}_{\ell,\mathfrak{m},h}$ of the torus orbit associated to the face $\mathfrak{f}$ of $\mathrm{Vor}_H^{\mathfrak{m}}(h)$.

4.2. **General toric tiling $Y^{a,b}_{\ell,\mathfrak{m}}$.** We are now finally in position to define the most general form of our toric tilings. Let $\ell : E \to \mathbb{N}$ be an edge length function, and $\mathfrak{m} \in C^1(G,\mathbb{Z})$ a twisting. Let $a : C^1(G,\mathbb{Z}) \to \mathbf{G_m}(\kappa) = \kappa^*$ and $b : H^1(G,\mathbb{Z}) \to \mathbf{G_m}(\kappa) = \kappa^*$ be two characters. To the quadruple $(\ell,\mathfrak{m},a,b)$ we are going to associate the toric tiling $Y^{a,b}_{\ell,\mathfrak{m}}$, which is a modification of the standard tiling $Y^{\mathrm{bt}}_{\ell,\mathfrak{m}}$.

To simplify the notation, for each $e \in \mathbb{E}$ we will let $a_e := a(\chi_e - \chi_{\overline{e}})$. Then $a_e a_{\overline{e}} = 1$ for each $e \in \mathbb{E}$. In addition, we will also denote by b any character of $C^1(G,\mathbb{Z})$ extending b, and let $b_e := b(\chi_e - \chi_{\overline{e}})$ for each $e \in \mathbb{E}$ as before.

To each $f \in C^0(G,\mathbb{Z})$ we associate the subvariety $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$ of $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$ given by the equations

$$\forall \text{ oriented cycle } \gamma \text{ in } G^{\mathfrak{m}}_f,$$

$$\prod_{e\in\overline{\gamma}\cap E^{\mathfrak{o}}} b_e a_e^{\mathfrak{d}^{\mathfrak{m}}_f(e)} \prod_{e\in\gamma\cap E^{\mathfrak{o}}} \mathrm{X}_{e,\mathfrak{d}^{\mathfrak{m}}_f(e)} \prod_{e\in\overline{\gamma}\cap E^{\mathfrak{o}}} \mathrm{X}_{\overline{e},\mathfrak{d}^{\mathfrak{m}}_f(e)} = \prod_{e\in\gamma\cap E^{\mathfrak{o}}} b_e a_e^{\mathfrak{d}^{\mathfrak{m}}_f(e)} \prod_{e\in\overline{\gamma}\cap E^{\mathfrak{o}}} \mathrm{X}_{e,\mathfrak{d}^{\mathfrak{m}}_f(e)} \prod_{e\in\gamma\cap E^{\mathfrak{o}}} \mathrm{X}_{\overline{e},\mathfrak{d}^{\mathfrak{m}}_f(e)},$$

where $G^{\mathfrak{m}}_f$ is the spanning subgraph of G consisting of all the edges of G for which $\mathfrak{d}^{\mathfrak{m}}_f$ is an integer. It is the same $\mathbf{P}_{\ell,\mathfrak{m},f}$ as before, if a and b are the trivial characters. The equation corresponding to γ may also be written in the format:

$$\prod_{e\in\gamma} \mathrm{X}_{e,\mathfrak{d}^{\mathfrak{m}}_f(e^{\mathfrak{o}})} = \prod_{e\in\gamma} b_e a_e^{\mathfrak{d}^{\mathfrak{m}}_f(e^{\mathfrak{o}})} \prod_{e\in\overline{\gamma}} \mathrm{X}_{e,\mathfrak{d}^{\mathfrak{m}}_f(e^{\mathfrak{o}})}, \tag{4.7}$$

where we recall that $e^{\mathfrak{o}} = e$ if $e \in E^{\mathfrak{o}}$ and $e^{\mathfrak{o}} = \overline{e}$ otherwise.

We denote by $Y^{a,b}_{\ell,\mathfrak{m}}$ the union of the $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$ for those $f \in C^0(G,\mathbb{Z})$ for which $G^{\mathfrak{m}}_f$ is connected.

The statements in Theorem 4.2 applies to the $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$, with the same proof given there, with the only obvious modification in Equation (4.6). As a consequence we obtain the analogous to Corollary 4.3.

Theorem 4.4. *The structure on $Y^{a,b}_{\ell,\mathfrak{m}}$ is given as follows: Take the disjoint union*

$$\bigsqcup_{\substack{f\in C^0(G,\mathbb{Z})\\ G^{\mathfrak{m}}_f \text{connected}}} \mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}.$$

For f and h in $C^0(G,\mathbb{Z})$ with both $G^{\mathfrak{m}}_f$ and $G^{\mathfrak{m}}_h$ connected such that the corresponding Voronoi cells share a face $\mathfrak{f} := \mathrm{Vor}^{\mathfrak{m}}_H(f) \cap \mathrm{Vor}^{\mathfrak{m}}_H(h) \neq \emptyset$, identify the closure in $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$ of the torus orbit associated to the face $\mathfrak{f}$ of $\mathrm{Vor}^{\mathfrak{m}}_H(f)$ with the closure in $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},h}$ of the torus orbit associated to the face $\mathfrak{f}$ of $\mathrm{Vor}^{\mathfrak{m}}_H(h)$.

The restriction that $G^{\mathfrak{m}}_f$ be connected for $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$ to be part of $Y^{a,b}_{\ell,\mathfrak{m}}$ is in fact not necessary, as the next proposition shows.

Proposition 4.5. *For each $f \in C^0(G,\mathbb{Z})$ there is $h \in C^0(G,\mathbb{Z})$ with $G^{\mathfrak{m}}_h$ connected such that $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f} = \mathbf{P}^{a,b}_{\ell,\mathfrak{m},h} \cap \mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$.*

Proof. First we claim that there is $h \in C^0(G,\mathbb{Z})$ such that $G^{\mathfrak{m}}_h$ is connected, contains $G^{\mathfrak{m}}_f$ as a subgraph, and $|\mathfrak{d}^{\mathfrak{m}}_h(e) - \mathfrak{d}^{\mathfrak{m}}_f(e)| \leq 1/2$ for each $e \in \mathbb{E}(G)$. And then we will see that this h works.

Let $G_1, \dots, G_n$ be the connected components of $G^{\mathfrak{m}}_f$. Let $V_i := V(G_i)$ for each $i = 1, \dots, n$. Up to reordering, assume by induction that there is $h \in C^0(G,\mathbb{Z})$ such that

$$|\mathfrak{d}^{\mathfrak{m}}_h(e) - \mathfrak{d}^{\mathfrak{m}}_f(e)| \leq 1/2 \quad \text{for each } e \in \mathbb{E}(G), \tag{4.8}$$

such that $G^{\mathfrak{m}}_f$ is a subgraph of $G^{\mathfrak{m}}_h$, and that $G_1, \dots, G_i$ are contained in the same connected component of $G^{\mathfrak{m}}_h$. (For $i = 1$, simply take $h := f$.) Set $S_i := V_1 \cup \cdots \cup V_i$ and $T_i := V_{i+1} \cup \cdots \cup V_n$. Let q be the smallest nonnegative integer such that $g := h + q\chi_{T_i}$ satisfies $\mathfrak{d}^{\mathfrak{m}}_g(e) \in \mathbb{Z}$ for some oriented edge $e \in \mathbb{E}(S_i, T_i)$. Let e' be such an edge. Its target belongs to V_j for some $j > i$. We may assume $j = i + 1$. Clearly, $\mathfrak{d}^{\mathfrak{m}}_g(e) = \mathfrak{d}^{\mathfrak{m}}_h(e)$ for every $e \in \mathbb{E}(G)$ connecting two vertices either both inside S_i or outside. Thus $G^{\mathfrak{m}}_f$ is a subgraph of $G^{\mathfrak{m}}_g$ and $G_1, \dots, G_i$ are contained in the same connected component of $G^{\mathfrak{m}}_g$. Furthermore, since $e' \in \mathbb{E}(G^{\mathfrak{m}}_g)$, also G_{i+1} is in the same component.

We will now prove that

$$|\mathfrak{d}^{\mathfrak{m}}_g(e) - \mathfrak{d}^{\mathfrak{m}}_f(e)| \leq 1/2 \tag{4.9}$$

for each $e \in \mathbb{E}(G)$. As observed before, $\mathfrak{d}^{\mathfrak{m}}_g(e) = \mathfrak{d}^{\mathfrak{m}}_h(e)$, and thus (4.9) holds for all $e \in \mathbb{E}(G)$ by (4.8), unless e or $\overline{e}$ belongs to $\mathbb{E}(S_i, T_i)$. Let $e \in \mathbb{E}(S_i, T_i)$. Then $\mathfrak{d}^{\mathfrak{m}}_f(e)$ is not an integer. If $\mathfrak{d}^{\mathfrak{m}}_h(e) \in \mathbb{Z}$ then $q = 0$, and thus $\mathfrak{d}^{\mathfrak{m}}_g(e) = \mathfrak{d}^{\mathfrak{m}}_h(e)$. If $\mathfrak{d}^{\mathfrak{m}}_h(e) \notin \mathbb{Z}$ then $\mathfrak{d}^{\mathfrak{m}}_h(e) = \mathfrak{d}^{\mathfrak{m}}_f(e)$ by (4.8) and $\mathfrak{d}^{\mathfrak{m}}_h(e) \leq \mathfrak{d}^{\mathfrak{m}}_g(e) \leq \mathfrak{d}^{\mathfrak{m}}_h(e) + 1/2$. In any case, we conclude that (4.9) now holds for every $e \in \mathbb{E}(G)$.

By induction, the claim is proved. It follows immediately from it that $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f} \subseteq \mathbf{P}_{\mathfrak{d}^{m}_h}$. Furthermore, since $G^{\mathfrak{m}}_h$ contains $G^{\mathfrak{m}}_f$ as a subgraph, which implies that every oriented cycle γ in $G^{\mathfrak{m}}_f$ is an oriented cycle of $G^{\mathfrak{m}}_h$ satisfying $\mathfrak{d}^{\mathfrak{m}}_h(e) = \mathfrak{d}^{\mathfrak{m}}_f(e)$ for every $e \in \gamma$, all the equations defining $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$ in $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$ are equations defining $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},h}$ in $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_h}$, whence $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f} \supseteq \mathbf{P}^{a,b}_{\ell,\mathfrak{m},h} \cap \mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$.

Let now γ be an oriented cycle in $G^{\mathfrak{m}}_h$, and consider the corresponding equation (4.7) among those defining $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},h}$. It remains to show that the equation is satisfied on $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$.

If $\mathfrak{d}^{\mathfrak{m}}_f(e) \in \mathbb{Z}$ for every $e \in \gamma$ then γ is in $G^{\mathfrak{m}}_f$. Since also $\mathfrak{d}^{\mathfrak{m}}_f(e) = \mathfrak{d}^{\mathfrak{m}}_h(e)$ for each $e \in \gamma$ by (4.8), it follows that the equations corresponding to γ for $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$ and for $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},h}$ are the same, and thus (4.7) is satisfied on $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$.

On the other hand, if there is $e \in \gamma$ such that $\mathfrak{d}^{\mathfrak{m}}_f(e) \notin \mathbb{Z}$, then Lemma 4.1 yields $e_1, e_2 \in \gamma$ such that

$$\mathfrak{d}^{\mathfrak{m}}_f(e_1) = \mathfrak{d}^{\mathfrak{m}}_h(e_1) - 1/2 \quad \text{and} \quad \mathfrak{d}^{\mathfrak{m}}_f(e_2) = \mathfrak{d}^{\mathfrak{m}}_h(e_2) + 1/2.$$

But then $\mathrm{x}_{\overline{e}_1,\mathfrak{d}^{\mathfrak{m}}_h(e_1^{\circ})} = 0$ (resp. $\mathrm{x}_{e_2,\mathfrak{d}^{\mathfrak{m}}_h(e_2^{\circ})} = 0$) on $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$, which implies that the right-hand side (resp. left-hand side) of (4.7) vanishes on $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$. Thus (4.7) is satisfied on $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$. □

Proposition 4.6. *The toric tilings $Y^{a,b}_{\ell,\mathfrak{m}}$ and $Y^{\mathrm{bt}}_{\ell,\mathfrak{m}}$ have the same combinatorial support in* $\mathbf{R}$*, in the sense that they have the same set of supporting* $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$*. Moreover, they are rationally equivalent.*

Proof. The claim about the set of supporting $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$ is obvious from the definition. The second statement is clear as in each $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$ there is a rational deformation of $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$ to $\mathbf{P}_{\ell,\mathfrak{m},f}$ obtained by deforming the characters a and b to the trivial characters. □

5. Mixed toric tilings II: Orbit structure

In this section we describe actions of $\mathbf{G}_{\mathbf{m}}^{|V|-1}$ on $\mathbf{R}$ and on the arrangements of toric varieties $Y^{a,b}_{\ell,\mathfrak{m}}$, and provide a complete description of the orbits.

5.1. **The action of $\mathbf{G}_{\mathbf{m}}^{|V|-1}$.** There is a natural action of the group of characters of $C^1(G,\mathbb{Z})$ on $\mathbf{R}$. Namely, each character $a\colon C^1(G,\mathbb{Z})\to\mathbf{G}_{\mathbf{m}}(\kappa)=\kappa^*$ acts on the point p of $\mathbf{P}_\alpha=\prod_{e\in E^{\mathfrak{o}}}\mathbf{P}^1_{e,\alpha_e}$ with coordinates $(x_{e,\alpha_e}:x_{\overline{e},\alpha_e})$, for each $\alpha\in C^1(G,\mathbb{Z})$, by taking it to the point on the same $\mathbf{P}_\alpha$ with coordinates $(a_e x_{e,\alpha_e}:x_{\overline{e},\alpha_e})$. Clearly, the action takes $\mathbf{P}_\alpha$ to itself for each $\alpha\in C^1(G,\frac{1}{2}\mathbb{Z})$.

The boundary map $d^*\colon C^1(G,\mathbb{Z})\to C^0(G,\mathbb{Z})$ induces thus a natural action of the group of characters of $C^0(G,\mathbb{Z})$ on $\mathbf{R}$. Given a character $c\colon C^0(G,\mathbb{Z})\to\mathbf{G}_{\mathbf{m}}(\kappa)$, since $d^*(\chi_e-\chi_{\overline{e}})=\chi_v-\chi_u$ for each $e=uv\in\mathbb{E}$, we have that c acts on the point p of $\mathbf{P}_\alpha=\prod_{e\in E^{\mathfrak{o}}}\mathbf{P}^1_{e,\alpha_e}$ with coordinates $(x_{e,\alpha_e}:x_{\overline{e},\alpha_e})$, for each $\alpha\in C^1(G,\mathbb{Z})$, by taking it to the point on the same $\mathbf{P}_\alpha$ with coordinates $(c_v x_{e,\alpha_e}:c_u x_{\overline{e},\alpha_e})$. Here, as before, $c_v:=c(\chi_v)$ for each $v\in V$.

Clearly, if $c_v=c_u$ for all $u,v\in V$, the action of c is trivial. Thus we need only consider characters induced from those of the quotient $C^0(G,\mathbb{Z})/\mathbb{Z}\chi_V$. The latter has character group isomorphic to $\mathbf{G}_{\mathbf{m}}^{|V|-1}$. We will loosely refer to this action as the action of $\mathbf{G}_{\mathbf{m}}^{|V|-1}$ on $\mathbf{R}$.

Finally, it follows from the equations defining the subvariety $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$ of $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$ for each $f\in C^0(G,\mathbb{Z})$ that the action of $\mathbf{G}_{\mathbf{m}}^{|V|-1}$ on $\mathbf{R}$ restricts to an action on $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$, and thus induces an action on $Y^{a,b}_{\ell,\mathfrak{m}}$ for any choices of edge length function $\ell:E\to\mathbb{N}$, twisting $\mathfrak{m}\in C^1(G,\mathbb{Z})$ and characters $a:C^1(G,\mathbb{Z})\to\kappa^*$ and $b:H^1(G,\mathbb{Z})\to\kappa^*$.

Moreover, the action on the open locus of $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$ where all the e-th coordinates for $e\in\mathbb{E}(G^{\mathfrak{m}}_f)\cap E^{\mathfrak{o}}$ are nonzero is transitive, if $G^{\mathfrak{m}}_f$ is connected. Indeed, setting $x_e:=\mathrm{x}_{e,\mathfrak{d}^{\mathfrak{m}}_f(e)}/\mathrm{x}_{\overline{e},\mathfrak{d}^{\mathfrak{m}}_f(e)}$ for each such e, the equations defining $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$ on the open locus become

$$\prod_{e\in\overline{\gamma}\cap E^{\mathfrak{o}}} b_e a_e^{\mathfrak{d}^{\mathfrak{m}}_f(e)}\prod_{e\in\gamma\cap E^{\mathfrak{o}}} x_e=\prod_{e\in\gamma\cap E^{\mathfrak{o}}} b_e a_e^{\mathfrak{d}^{\mathfrak{m}}_f(e)}\prod_{e\in\overline{\gamma}\cap E^{\mathfrak{o}}} x_e$$

for all oriented cycles γ in $G^{\mathfrak{m}}_f$. For each point p on the open locus, we may use the x_e to define the character $y\colon C^1(G^{\mathfrak{m}}_f,\mathbb{Z})\to\kappa^*$ for which, following notation, $y_e:=x_e(p)b_e^{-1}a_e^{-\mathfrak{d}^{\mathfrak{m}}_f(e)}$ for each $e\in\mathbb{E}(G^{\mathfrak{m}}_f)\cap E^{\mathfrak{o}}$. The equations for the x_e correspond thus to requiring that y restricts to the trivial character on $H^1(G^{\mathfrak{m}}_f,\mathbb{Z})$. Since $d^*\colon C^1(G^{\mathfrak{m}}_f,\mathbb{Z})\to C^0(G^{\mathfrak{m}}_f,\mathbb{Z})$ has kernel $H^1(G_f^{\mathfrak{m}},\mathbb{Z})$, it follows that y is induced from a character of the image of d^*, which, since $G_f^{\mathfrak{m}}$ is connected, is the subgroup of all $h\in C^0(G_f^{\mathfrak{m}},\mathbb{Z})$ with $\sum h(v)=0$. Thus the character of the image of d^* extends to a character of $C^0(G_f^{\mathfrak{m}},\mathbb{Z})=C^0(G,\mathbb{Z})$, which we denote $c\colon C^0(G,\mathbb{Z})\to\kappa^*$. It follows now that p is the result of the action of c on the point q on the open locus of $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},f}$ satisfying $x_e(q)=b_e a_e^{\mathfrak{d}_f^{\mathfrak{m}}(e)}$ for each $e\in\mathbb{E}(G_f^{\mathfrak{m}})\cap E^{\mathfrak{o}}$.

5.2. $Y^{a,b}_{\ell,\mathfrak{m}}$ **as a union of orbits.** In this section we describe $Y^{a,b}_{\ell,\mathfrak{m}}$ as a union of orbits under the action of $\mathbf{G}_{\mathbf{m}}^{|V|-1}$. We need first two preliminary results.

Proposition 5.1. *Let $f_1,f_2\in C^0(G,\mathbb{Z})$ and $n\in\mathbb{N}$. If $\mathfrak{d}_{f_2}^{\mathfrak{m},n}-\mathfrak{d}_{f_1}^{\mathfrak{m},n}\in H^1(G,\mathbb{Z})$ then $\mathfrak{d}_{f_2}^{\mathfrak{m},n}=\mathfrak{d}_{f_1}^{\mathfrak{m},n}$.*

Proof. Since $\mathfrak{d}_{f_2}^{\mathfrak{m},n}-\mathfrak{d}_{f_1}^{\mathfrak{m},n}\in C^1(G,\mathbb{Z})$, it follows that $\mathfrak{d}_{f_2}^{\mathfrak{m},n}(e)\in\mathbb{Z}$ if and only if $\mathfrak{d}_{f_1}^{\mathfrak{m},n}(e)\in\mathbb{Z}$, for each $e\in\mathbb{E}$. Then $\mathfrak{d}_{f_2}^{\mathfrak{m},n}(e)-\mathfrak{d}_{f_1}^{\mathfrak{m},n}(e)=\delta_e^{\mathfrak{m},n}(f_2)-\delta_e^{\mathfrak{m},n}(f_1)$.

Let $h:=f_2-f_1$ and $\gamma:=\mathfrak{d}_{f_2}^{\mathfrak{m},n}-\mathfrak{d}_{f_1}^{\mathfrak{m},n}$. Suppose by contradiction that $\gamma\neq 0$. Let G' be the subgraph of G obtained by keeping only the edges in the support of γ. Then there is a connected component G'' of G' containing an edge. Let v be a vertex of G'' where $h(v)$ is maximum. Since $\gamma\in H^1(G,\mathbb{Z})$, we have

$$\sum_{\substack{e\in\mathbb{E}(G'')\\ \mathrm{h}_e=v}}\gamma_e=\sum_{\substack{e\in\mathbb{E}(G)\\ \mathrm{h}_e=v}}\gamma_e=0. \tag{5.1}$$

On the other hand, for each $e=uv\in\mathbb{E}(G'')$ write

$$f_i(v)-f_i(u)+n\mathfrak{m}_e=\delta_e^{\mathfrak{m},n}(f_i)n\ell_e+\rho_i(e),$$

where $0\leq\rho_i(e)<n\ell_e$ for $i=1,2$. Then

$$h(v)-h(u)+\rho_1(e)=(\delta_e^{\mathfrak{m},n}(f_2)-\delta_e^{\mathfrak{m},n}(f_1))n\ell_e+\rho_2(e).$$

Hence, since $h(v)\geq h(u)$, we have that

$$\gamma_e=\delta_e^{\mathfrak{m},n}(f_2)-\delta_e^{\mathfrak{m},n}(f_1)\geq 0\quad\text{for each } e=uv\in\mathbb{E}(G'').$$

But then Equation (5.1) yields that $\gamma_e=0$ for each $e\in\mathbb{E}(G'')$ with $\mathrm{h}_e=v$, an absurd. ☐

Lemma 5.2. *Let $n\in\mathbb{N}$ and $f\in C^0(G,\mathbb{Z})$. Let $g\in C^0(G,\mathbb{Z})$ be defined by $g(v)=\lfloor f(v)/n\rfloor$ for each $v\in V(G)$. Let $e=uv\in\mathbb{E}(G)$. Then the following statements hold:*

(1) If $\mathfrak{d}_f^{\mathfrak{m},n}(e)\in\mathbb{Z}$ then $\mathfrak{d}_g^{\mathfrak{m}}(e)=\mathfrak{d}_f^{\mathfrak{m},n}(e)$.

(2) If $\mathfrak{d}_f^{\mathfrak{m},n}(e)\notin\mathbb{Z}$ then

$$|\mathfrak{d}_g^{\mathfrak{m}}(e)-\mathfrak{d}_f^{\mathfrak{m},n}(e)|\leq\frac{1}{2}.$$

Proof. Observe first that

$$\frac{f(v)-f(u)+n\mathfrak{m}_e}{n\ell_e}-\frac{1}{\ell_e}<\frac{g(v)-g(u)+\mathfrak{m}_e}{\ell_e}<\frac{f(v)-f(u)+n\mathfrak{m}_e}{n\ell_e}+\frac{1}{\ell_e} \tag{5.2}$$

for each $e=uv\in\mathbb{E}$. Observe as well that the middle number above is in $(1/\ell_e)\mathbb{Z}$.

If $\mathfrak{d}_f^{\mathfrak{m},n}(e)\in\mathbb{Z}$ then

$$\frac{f(v)-f(u)+n\mathfrak{m}_e}{n\ell_e}\in\mathbb{Z}$$

and thus it follows from Inequalities (5.2) and the observation thereafter that

$$\frac{g(v)-g(u)+\mathfrak{m}_e}{\ell_e}=\frac{f(v)-f(u)+n\mathfrak{m}_e}{n\ell_e},$$

and thus $\mathfrak{d}_g^{\mathfrak{m}}(e)=\mathfrak{d}_f^{\mathfrak{m},n}(e)$.

On the other, suppose $\mathfrak{d}_f^{\mathfrak{m},n}(e)\notin\mathbb{Z}$. If $\mathfrak{d}_g^{\mathfrak{m}}(e)\notin\mathbb{Z}$ either then $\mathfrak{d}_f^{\mathfrak{m},n}(e)=\mathfrak{d}_g^{\mathfrak{m}}(e)$. Indeed, if this is not the case there is an integer c_e satisfying either the inequalities

$$\frac{f(v)-f(u)+n\mathfrak{m}_e}{n\ell_e}<c_e<\frac{g(v)-g(u)+\mathfrak{m}_e}{\ell_e}$$

or the reverse inequalities. If the displayed inequalities hold then

$$c_e<\frac{g(v)-g(u)+\mathfrak{m}_e}{\ell_e}<\frac{f(v)-f(u)+n\mathfrak{m}_e}{n\ell_e}+\frac{1}{\ell_e}<c_e+\frac{1}{\ell_e},$$

an absurd, as the second number above is in $(1/\ell_e)\mathbb{Z}$. The reverse inequalities yield a similar contradiction.

Finally, suppose $\mathfrak{d}_f^{\mathfrak{m},n}(e)\notin\mathbb{Z}$ but $\mathfrak{d}_g^{\mathfrak{m}}(e)\in\mathbb{Z}$. Then Inequalities (5.2) are equivalent to

$$\mathfrak{d}_g^{\mathfrak{m}}(e)-\frac{1}{\ell_e}<\frac{f(v)-f(u)+n\mathfrak{m}_e}{n\ell_e}<\mathfrak{d}_g^{\mathfrak{m}}(e)+\frac{1}{\ell_e},$$

from which follows that $\mathfrak{d}_f^{\mathfrak{m},n}(e)=\mathfrak{d}_g^{\mathfrak{m}}(e)-1/2$ if the middle term above is smaller than $\mathfrak{d}_g^{\mathfrak{m}}(e)$, or $\mathfrak{d}_f^{\mathfrak{m},n}(e)=\mathfrak{d}_g^{\mathfrak{m}}(e)+1/2$ otherwise. In any case,

$$\mathfrak{d}_f^{\mathfrak{m},n}(e)=\mathfrak{d}_g^{\mathfrak{m}}(e)\pm\frac{1}{2}.$$

□

Theorem 5.3. *Let $\mathfrak{o}$ be an orientation for the edges of G. Let $\ell\colon E\to\mathbb{N}$ be a length function, $\mathfrak{m}\in C^1(G,\mathbb{Z})$, and let*

$$a\colon C^1(G,\mathbb{Z})\to\kappa^*\quad\textit{and}\quad b\colon C^1(G,\mathbb{Z})\to\kappa^*$$

be characters. For each $n\in\mathbb{N}$ and $f\in C^0(G,\mathbb{Z})$, let p_f^n be the point on $\mathbf{P}_{\mathfrak{d}_f^{\mathfrak{m},n}}$ given by the coordinates

$$(b_e a_e^{\mathfrak{d}_f^{\mathfrak{m},n}(e)}:1)\quad\textit{for each } e\in E^{\mathfrak{o}}\textit{ with } \mathfrak{d}_f^{\mathfrak{m},n}(e)\in\mathbb{Z}.$$

Then $Y_{\ell,\mathfrak{m}}^{a,b}$ is the union of the orbits of the p_f^n under the action of $\mathbf{G}_{\mathbf{m}}^{|V|-1}$.

(We are also denoting by b its restriction to $H^1(G,\mathbb{Z})$.)

Proof. As defined in Section 4.2, the toric arrangement $Y^{a,b}_{\ell,\mathfrak{m}}$ is the union of the toric subvarieties $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},g}$ of $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_g}$ as g varies in $C^0(G,\mathbb{Z})$. (By Proposition 4.5, we do not need to restrict to those g for which $G^{\mathfrak{m}}_g$ is connected.) Each subvariety $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},g}$ is given by the equations:

$$\forall \text{ oriented cycle } \gamma \text{ in } G^{\mathfrak{m}}_g,$$

$$\prod_{e\in\gamma} \mathrm{x}_{e,\mathfrak{d}^{\mathfrak{m}}_g(e^{o})} = \prod_{e\in\gamma} b_e a_e^{\mathfrak{d}^{\mathfrak{m}}_g(e^{o})} \prod_{e\in\bar{\gamma}} \mathrm{x}_{e,\mathfrak{d}^{\mathfrak{m}}_g(e^{o})}. \tag{5.3}$$

Let $f\in C^0(G,\mathbb{Z})$ and $n\in\mathbb{N}$. Let $g\in C^0(G,\mathbb{Z})$ satisfying $g(v):=\lfloor f(v)/n\rfloor$ for each $v\in V$. Lemma 5.2 yields $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m},n}_f}\subseteq \mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_g}$. In particular, $p^n_f\in \mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_g}$. We will now show that $p^n_f\in \mathbf{P}^{a,b}_{\ell,\mathfrak{m},g}$.

Let γ be an oriented cycle in $G^{\mathfrak{m}}_g$. We show that Equation (5.3) is satisfied on p^n_f. First, if $\mathfrak{d}^{\mathfrak{m},n}_f(e)\in\mathbb{Z}$ for each $e\in\gamma$, then $\mathfrak{d}^{\mathfrak{m},n}_f(e)=\mathfrak{d}^{\mathfrak{m}}_g(e)$ for each $e\in\gamma\cup\bar{\gamma}$ by Lemma 5.2. It follows that $(\mathrm{x}_{e,\mathfrak{d}^{\mathfrak{m}}_g(e)}:\mathrm{x}_{\bar{e},\mathfrak{d}^{\mathfrak{m}}_g(e)})=(b_e a_e^{\mathfrak{d}^{\mathfrak{m}}_g(e)}:1)$ on p^n_f for each $e\in E^{o}$ with $e\in\gamma\cup\bar{\gamma}$, and then Equation (5.3) is satisfied on p^n_f.

Suppose now there is $e\in\gamma$ such that $\mathfrak{d}^{\mathfrak{m},n}_f(e)\notin\mathbb{Z}$. By Lemma 5.2 and Lemma 4.1, there are $e_1,e_2\in\gamma$ such that

$$\mathfrak{d}^{\mathfrak{m},n}_f(e_1)=\mathfrak{d}^{\mathfrak{m}}_g(e_1)-1/2 \quad\text{and}\quad \mathfrak{d}^{\mathfrak{m},n}_f(e_2)=\mathfrak{d}^{\mathfrak{m}}_g(e_2)+1/2.$$

But then $\mathrm{x}_{\bar{e}_1,\mathfrak{d}^{\mathfrak{m}}_g(e^{o}_1)}=0$ (resp. $\mathrm{x}_{e_2,\mathfrak{d}^{\mathfrak{m}}_g(e^{o}_2)}=0$) on p^n_f, which implies that the right-hand side (resp. left-hand side) of Equation (5.3) vanishes on p^n_f. Since both sides vanish, the equation is satisfied.

Let now p be a point on $\mathbf{P}^{a,b}_{\ell,\mathfrak{m},g}$ for some $g\in C^0(G,\mathbb{Z})$. We will show that p is on the orbit of p^n_f for certain $n\in\mathbb{N}$ and $f\in C^0(G,\mathbb{Z})$. This will finish the proof.

First, let $\mathbb{E}(p)\subseteq\mathbb{E}$ be the set of $e\in\mathbb{E}$ such that $\mathfrak{d}^{\mathfrak{m}}_g(e)\in\mathbb{Z}$ and $\mathrm{x}_{e,\mathfrak{d}^{\mathfrak{m}}_g(e^{o})}(p)=0$. Let $E(p)\subseteq E(G)$ be the support of $\mathbb{E}(p)$. Notice that if $e\in\mathbb{E}(p)$, then $\mathrm{x}_{\bar{e},\mathfrak{d}^{\mathfrak{m}}_g(e^{o})}(p)\neq 0$, and thus $\bar{e}\notin\mathbb{E}(p)$. Furthermore, since p satisfies Equation (5.3) for each oriented cycle γ in $G^{\mathfrak{m}}_g$, for each such cycle, if $e\in\gamma\cap\mathbb{E}(p)$, then there is $e'\in\bar{\gamma}\cap\mathbb{E}(p)$. It follows that there is no oriented cycle in $\mathbb{E}(p)$. In short, $\mathbb{E}(p)$ is an acyclic orientation of $E(p)$.

We claim there are $n\in\mathbb{N}$ and $f\in C^0(G,\mathbb{Z})$ such that $\mathfrak{d}^{\mathfrak{m},n}_f(e)=\mathfrak{d}^{\mathfrak{m}}_g(e)$ for each $e\in\mathbb{E}$ supported away from $E(p)$ and $\mathfrak{d}^{\mathfrak{m},n}_f(e)=\mathfrak{d}^{\mathfrak{m}}_g(e)+1/2$ for each $e\in\mathbb{E}(p)$. In particular, $G^{\mathfrak{m},n}_f$, the subgraph of G obtained by removing all edges $e\in\mathbb{E}$ for which $\mathfrak{d}^{\mathfrak{m},n}_f(e)\notin\mathbb{Z}$, is also obtained from $G^{\mathfrak{m}}_g$ by removing all edges of $E(p)$.

Indeed, we will let $\rho\in C^0(G,\mathbb{Z})$ be positive and small enough, and let $n\in\mathbb{N}$ be large enough such that $f\in C^0(G,\mathbb{Z})$ with $f(v)=ng(v)+\rho(v)$ for each $v\in V$ satisfies the desired conditions. First, if $(1/n)\rho$ is small enough we may assume that $\mathfrak{d}^{\mathfrak{m},n}_f(e)=\mathfrak{d}^{\mathfrak{m}}_g(e)$ whenever $\mathfrak{d}^{\mathfrak{m}}_g(e)\notin\mathbb{Z}$. Second, we set ρ to be constant on each connected component of the subgraph obtained from $G^{\mathfrak{m}}_g$ by removing all edges in $E(p)$. Furthermore, the complementary subgraph, obtained by keeping only the edges in $E(p)$, has $\mathbb{E}(p)$ as orientation, and since there is no oriented cycle in $\mathbb{E}(p)$, we may choose ρ such that $\rho(v)>\rho(u)$ for each

$e = uv \in \mathbb{E}(p)$. If $(1/n)\rho$ is small enough, it follows that $\mathfrak{d}_f^{\mathfrak{m},n}(e) = \mathfrak{d}_g^{\mathfrak{m}}(e) + 1/2$ for each $e \in \mathbb{E}(p)$. And if $e = uv \in \mathbb{E}$ is supported away from $E(p)$ and $\mathfrak{d}_g^{\mathfrak{m}}(e) \in \mathbb{Z}$, then e is an edge of $G_g^{\mathfrak{m}}$ supported away from $E(p)$, whence $\rho(v) = \rho(u)$ and so $\mathfrak{d}_f^{\mathfrak{m},n}(e) = \mathfrak{d}_g^{\mathfrak{m}}(e)$. The proof of the claim is finished.

Let n and f be as in the claim. Since $|\mathfrak{d}_f^{\mathfrak{m},n}(e) - \mathfrak{d}_g^{\mathfrak{m}}(e)| \leq 1/2$, with equality only if $\mathfrak{d}_g^{\mathfrak{m}}(e) \in \mathbb{Z}$, for each $e \in \mathbb{E}$, we have that $\mathbf{P}_{\mathfrak{d}_f^{\mathfrak{m},n}} \subseteq \mathbf{P}_{\mathfrak{d}_g^{\mathfrak{m}}}$, and thus $p_f^n \in \mathbf{P}_{\mathfrak{d}_g^{\mathfrak{m}}}$.

We claim that an e-th coordinate vanishes on p if and only if it vanishes on p_f^n, for each $e \in \mathbb{E}(G_g^{\mathfrak{m}}) \cap E^{\mathfrak{o}}$. Indeed, suppose first that $e \in \mathbb{E}(p)$. Then $\mathrm{x}_{e,\mathfrak{d}_g^{\mathfrak{m}}(e)}(p) = 0$. Now, $\mathfrak{d}_f^{\mathfrak{m},n}(e) = \mathfrak{d}_g^{\mathfrak{m}}(e) + 1/2$. Thus, $\mathrm{x}_{e,\mathfrak{d}_g^{\mathfrak{m}}(e)}(p_f^n) = 0$ as well. And if $\overline{e} \in \mathbb{E}(p)$, then $\mathrm{x}_{\overline{e},\mathfrak{d}_g^{\mathfrak{m}}(e)}(p) = 0$. Since $\mathfrak{d}_f^{\mathfrak{m},n}(e) = \mathfrak{d}_g^{\mathfrak{m}}(e) - 1/2$, it follows that $\mathrm{x}_{\overline{e},\mathfrak{d}_g^{\mathfrak{m}}(e)}(p_f^n) = 0$ as well. Finally, suppose e is not supported in $E(p)$, that is, $e \in \mathbb{E}(G_f^{\mathfrak{m},n})$. Then the e-th coordinates of p_f^n are nonzero. So are those of p: since $e \notin \mathbb{E}(p)$, we have $\mathrm{x}_{e,\mathfrak{d}_g^{\mathfrak{m}}(e)}(p) \neq 0$, and since $\overline{e} \notin \mathbb{E}(p)$, we have $\mathrm{x}_{\overline{e},\mathfrak{d}_g^{\mathfrak{m}}(e)}(p) \neq 0$.

Let G' be a connected component of $G_f^{\mathfrak{m},n}$. By what we saw above, all the e-th coordinates of p and p_f^n are nonzero for $e \in \mathbb{E}(G') \cap E^{\mathfrak{o}}$. In addition, for each oriented cycle γ in G', Equation (5.3) is satisfied on p and on p_f^n. But then the projections of p and p_f^n on the product

$$\prod_{e \in \mathbb{E}(G') \cap E^{\mathfrak{o}}} \mathbf{P}^1_{e,\mathfrak{d}_f^{\mathfrak{m},n}(e)}$$

lie on the open torus inside the toric subvariety given by Equations (5.3) for oriented cycles γ in G'. These two projections lie on the same orbit of the product by the action of $\mathbf{G}_{\mathbf{m}}^{|V(G')|-1}$, as observed in Subsection 5.1.

As the above holds for each connected component of $G_f^{\mathfrak{m},n}$, it follows that p is on the orbit of p_f^n by the action of $\mathbf{G}_{\mathbf{m}}^{|V|-1}$. □

Corollary 5.4. *Given $\alpha \in C^1(G, \frac{1}{2}\mathbb{Z})$, we have that $\mathbf{P}_\alpha^* \cap Y_{\ell,\mathfrak{m}}^{a,b} \neq \emptyset$ if and only if $\alpha = \mathfrak{d}_f^{\mathfrak{m},n}$ for certain $n \in \mathbb{N}$ and $f \in C^0(G,\mathbb{Z})$, and in this case $\mathbf{P}_\alpha^* \cap Y_{\ell,\mathfrak{m}}^{a,b}$ is an orbit of the action of $\mathbf{G}_{\mathbf{m}}^{|V|-1}$ on $Y_{\ell,\mathfrak{m}}^{a,b}$ which is dense in $\mathbf{P}_\alpha \cap Y_{\ell,\mathfrak{m}}^{a,b}$.*

Proof. If $\alpha = \mathfrak{d}_f^{\mathfrak{m},n}$ for certain n and f then $p_f^n \in \mathbf{P}_\alpha^* \cap Y_{\ell,\mathfrak{m}}^{a,b}$ by Theorem 5.3. Conversely, if $p \in \mathbf{P}_\alpha^* \cap Y_{\ell,\mathfrak{m}}^{a,b}$ then p is on the orbit of p_f^n for certain n and f by Theorem 5.3. Then $\mathbf{P}_\alpha^* \cap \mathbf{P}^*_{\mathfrak{d}_f^{\mathfrak{m},n}} \neq \emptyset$ and thus $\alpha = \mathfrak{d}_f^{\mathfrak{m},n}$. Notice that n and f are not unique, but $\mathfrak{d}_f^{\mathfrak{m},n}$ is and it determines p_f^n. It follows that $\mathbf{P}_\alpha^* \cap Y_{\ell,\mathfrak{m}}^{a,b}$ is the orbit of p_f^n.

It remains to show that $\mathbf{P}_\alpha^* \cap Y_{\ell,\mathfrak{m}}^{a,b}$ is dense in $\mathbf{P}_\alpha \cap Y_{\ell,\mathfrak{m}}^{a,b}$. Consider a point of $\mathbf{P}_\beta^* \cap Y_{\ell,\mathfrak{m}}^{a,b}$ for $\beta \in C^1(G, \frac{1}{2}\mathbb{Z})$ such that $\mathbf{P}_\beta \subseteq \mathbf{P}_\alpha$. As we have seen above, the point is on the orbit of p_h^q for certain $q \in \mathbb{N}$ and $h \in C^1(G,\mathbb{Z})$ such that $\beta = \mathfrak{d}_h^{\mathfrak{m},q}$. Let G_α (resp. G_β) be the spanning subgraph of G containing the supports of all the oriented edges $e \in \mathbb{E}$ for which $\alpha_e \in \mathbb{Z}$ (resp. $\beta_e \in \mathbb{Z}$). Since $\mathbf{P}_\beta \subseteq \mathbf{P}_\alpha$, we have that G_β is a subgraph of G_α.

Give an orientation D to the set of edges in G_α which are not in G_β by saying the $e \in \mathbb{E}(D)$ if $\beta(e) < \alpha(e)$. It follows from Lemma 4.1 that the orientation is acyclic. Furthermore, let $X_0, \dots, X_m$ be the connected components of G_β. Since $\alpha_e = \beta_e$ for each $e \in \mathbb{E}(X_i)$ for each i, it follows from Lemma 4.1 that we may assume the X_i are ordered in such a way that if $e = uv \in \mathbb{E}(D)$ connects $u \in X_i$ and $v \in X_j$ then $i > j$.

Consider now the character $c\colon C^0(G,\mathbb{Z}) \to \mathbf{G_m}(\kappa(t))$ that takes $v \in V(X_i)$ to t^i for each $i = 0, \dots, m$. Let it act on p_f^n for each nonzero $t \in \kappa$ and let t tend to 0. The limit is p_h^q. Thus p_h^q, and hence its orbit, is in the closure of the orbit of p_f^n. It follows that $\mathbf{P}_\beta^* \cap Y_{\ell,\mathfrak{m}}^{a,b}$ is in the closure of $\mathbf{P}_\alpha^* \cap Y_{\ell,\mathfrak{m}}^{a,b}$ for each $\beta \in C^1(G, \frac{1}{2}\mathbb{Z})$ such that $\mathbf{P}_\beta \subseteq \mathbf{P}_\alpha$, and hence $\mathbf{P}_\alpha^* \cap Y_{\ell,\mathfrak{m}}^{a,b}$ is dense in $\mathbf{P}_\alpha \cap Y_{\ell,\mathfrak{m}}^{a,b}$. □

6. Mixed toric tilings III: Equations

In this section, we work out the equations for the toric tiling $Y_{\ell,\mathfrak{m}}^{a,b}$ in $\mathbf{R}$. For this, we will first give "equations" to the toric tiling $\mathbf{R}$. The tiling $Y_{\ell,\mathfrak{m}}^{a,b}$ will then be described by a further set of equations, leading to the determination of all the supporting "hypercubes" $\mathbf{P}_\alpha$, and restricting to the equations determining the $\mathbf{P}_{\ell,\mathfrak{m},f}^{a,b}$ on these hypercubes.

6.1. **Equations for R.** In this subsection, we explain how to put coordinates on the toric tiling $\mathbf{R}$ associated to the tiling of $C^1(G,\mathbb{R})$ by the hypercubes $\square_\alpha$.

We fix an orientation $\mathfrak{o} : E \to \mathbb{E}$ of the edges of G. We set $E^{\mathfrak{o}} := \mathfrak{o}(E)$.

For each edge $e \in E^{\mathfrak{o}}$ and each integer $i \in \mathbb{Z}$, let $\mathbf{P}^1_{e,i}$ be a copy of the projective line with projective coordinates $(\mathrm{x}_{e,i} : \mathrm{x}_{\overline{e},i})$. Consider the product $\mathbf{P} = \prod_{e \in E^{\mathfrak{o}}} \prod_{i \in \mathbb{Z}} \mathbf{P}^1_{e,i}$. It can be equipped with a natural structure of a projective limit of schemes of finite type, or can be simply seen as the functor of sets it represents on the category of rings or schemes. For our purposes in the present article, we find it more convenient to just view $\mathbf{P}$ as a set (by looking at κ-points).

Define the subset $\mathbf{R}' \subset \prod_{e \in E^{\mathfrak{o}}} \prod_{i \in \mathbb{Z}} \mathbf{P}^1_{e,i}$ by the equations

(1) $\mathrm{x}_{e,i}\mathrm{x}_{\overline{e},j} = 0$ for all pairs $(e,i),(e,j) \in E^{\mathfrak{o}} \times \mathbb{Z}$ with $j > i$,

with the requirement that

(2) For each $e \in E^{\mathfrak{o}}$, there exist indices $i, j \in \mathbb{Z}$ such that $\mathrm{x}_{e,i} \neq 0$ and $\mathrm{x}_{\overline{e},j} \neq 0$.

Alternatively, for each $e \in E^{\mathfrak{o}}$, denote by 0_e (resp. ∞_e) the point of $\prod_{i\in\mathbb{Z}} \mathbf{P}^1_{e,i}$ with coordinates $(\mathrm{x}_{e,i} : \mathrm{x}_{\overline{e},i}) = (0:1)$ (resp. $(\mathrm{x}_{e,i} : \mathrm{x}_{\overline{e},i}) = (1:0)$) for every i. Set

$$\mathbf{P}_e := \prod_{i \in \mathbb{Z}} \mathbf{P}^1_{e,i} - \{0_e, \infty_e\}.$$

Then $\mathbf{R}'$ is the subset of $\prod_{e \in E^{\mathfrak{o}}} \mathbf{P}_e$ given by Equations (1).

Recall from Subsection 3.2 the definition of $\mathbf{R}_e$ as a doubly infinite chain of rational smooth curves and the characterization of the toric tiling $\mathbf{R}$ as the product of the $\mathbf{R}_e$ for $e \in E^{\mathfrak{o}}$.

Proposition 6.1. *The locally of finite type scheme* $\mathbf{R}_e$ *is naturally identified with the subset of* $\mathbf{P}_e$ *given by the equations* $\mathrm{x}_{e,i}\mathrm{x}_{\overline{e},j} = 0$ *for all* $i, j \in \mathbb{Z}$ *with* $j > i$*.*

Proof. For each $i \in \mathbb{Z}$, set $0_{e,i} := (0 : 1) \in \mathbf{P}^1_{e,i}$ and $\infty_{e,i} := (1 : 0) \in \mathbf{P}^1_{e,i}$, and put

$$\mathbf{P}^1_i := \prod_{j<i}\{0_{e,j}\} \times \mathbf{P}^1_{e,i} \times \prod_{j>i}\{\infty_{e,j}\}.$$

Note that we have a canonical identification $\mathbf{P}^1_i \simeq \mathbf{P}^1$, and under this identification, the point ∞ on $\mathbf{P}^1_i$ is the point $\prod_{j<i}\{0_{e,j}\} \times \prod_{j\geq i}\{\infty_{e,j}\}$, whereas the point 0 on $\mathbf{P}^1_i$ is $\prod_{j\leq i}\{0_{e,j}\} \times \prod_{j>i}\{\infty_{e,j}\}$. In particular, the two subsets $\mathbf{P}^1_i$ and $\mathbf{P}^1_{i+1}$ intersect at the point 0 on $\mathbf{P}^1_i$, which is also the point ∞ on $\mathbf{P}^1_{i+1}$. It follows that $\mathbf{R}_e$ is identified with the doubly infinite chain of $\mathbf{P}^1$ obtained by taking the union of the $\mathbf{P}^1_i$ in $\mathbf{P}_e$.

We need only show that the stated equations define the union of the $\mathbf{P}^1_i$. Clearly, each point on $\mathbf{P}^1_i$ for each $i \in \mathbb{Z}$ satisfies the equations. Conversely, since $\mathbf{P}_e$ contains neither 0_e nor ∞_e, the coordinates of a point on $\mathbf{P}_e$ satisfy $\mathrm{x}_{e,i} \neq 0$ for some $i \in \mathbb{Z}$ and $\mathrm{x}_{\overline{e},j} \neq 0$ for some $j \in \mathbb{Z}$. Moreover, if the stated equations are satisfied at the point, then there is a minimum i with $\mathrm{x}_{e,i} \neq 0$ and a maximum j with $\mathrm{x}_{\overline{e},j} \neq 0$. Thus $\mathrm{x}_{e,m} = 0$ for $m < i$ and $\mathrm{x}_{\overline{e},m} = 0$ for $m > j$ at the point. From the equations we get $\mathrm{x}_{\overline{e},m} = 0$, whence $\mathrm{x}_{e,m} \neq 0$ for $m > i$, and $\mathrm{x}_{e,m} = 0$, whence $\mathrm{x}_{\overline{e},m} \neq 0$ for $m < j$. Thus $\mathrm{x}_{e,m} \neq 0$ if and only if $m \geq i$ and $\mathrm{x}_{\overline{e},m} \neq 0$ if and only if $m \leq j$. So $j \geq i - 1$. But if $j \geq i + 1$, then $\mathrm{x}_{e,i}\mathrm{x}_{\overline{e},i+1} \neq 0$, contradicting the equations. Thus $j = i$ or $j = i - 1$. It follows that the point lies on $\mathbf{P}^1_i$. ☐

Proposition 6.2. *The toric tiling* $\mathbf{R}$ *associated to the decomposition of* $C^1(G, \mathbb{R})$ *by the hypercubes* $\square_\alpha$ *is naturally identified with* $\mathbf{R}'$*.*

Proof. This follows from the previous proposition. ☐

From now one we will write $\mathbf{R}$ for $\mathbf{R}'$.

6.2. **Equations for $Y^{a,b}_{\ell,\mathfrak{m}}$.** We keep the notations of the previous section. Let $\ell : E \to \mathbb{N}$ be a length function, $\mathfrak{m} \in C^1(G, \mathbb{Z})$ be a twisting, and a and b be two characters of $C^1(G, \mathbb{Z})$ and $H^1(G, \mathbb{Z})$.

For each $\alpha \in C^1(G, \mathbb{Z})$ and $\gamma \in H^1(G, \mathbb{Z})$, consider a copy $\mathbf{P}^1_{\alpha,\gamma}$ of the projective line with projective coordinates $(\mathrm{P}_{\alpha,\gamma} : \mathrm{Q}_{\alpha,\gamma})$. As usual, let $0_{\alpha,\gamma}$ be the point on $\mathbf{P}^1_{\alpha,\gamma}$ given by $\mathrm{P}_{\alpha,\gamma} = 0$ and $\infty_{\alpha,\gamma}$ that given by $\mathrm{Q}_{\alpha,\gamma} = 0$

Consider the product:

$$\mathfrak{Z} := \prod_{\substack{\alpha\in C^1(G,\mathbb{Z})\\ \gamma\in H^1(G,\mathbb{Z})}} \mathbf{P}^1_{\alpha,\gamma}.$$

For each $\mathfrak{z} \in \mathfrak{Z}$, let $Y_{\mathfrak{z}} \subset \mathbf{R} \simeq \mathbf{R} \times \{\mathfrak{z}\} \subset \mathbf{R} \times \mathfrak{Z}$ be the subscheme of $\mathbf{R}$ given by the equations

$$\forall \alpha \in C^1(G,\mathbb{Z}),\ \gamma \in H^1(G,\mathbb{Z}),$$

$$\mathrm{P}_{\alpha,\gamma} \prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e>0}} \mathrm{X}_{e,\alpha_e}^{\gamma_e} \prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e<0}} \mathrm{X}_{\bar{e},\alpha_e}^{-\gamma_e} = \mathrm{Q}_{\alpha,\gamma} \prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e>0}} \mathrm{X}_{\bar{e},\alpha_e}^{\gamma_e} \prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e<0}} \mathrm{X}_{e,\alpha_e}^{-\gamma_e}.$$

Theorem 6.3. *Let $\ell\colon E \to \mathbb{N}$ be a length function, $\mathfrak{m} \in C^1(G,\mathbb{Z})$ and let*

$$a\colon C^1(G,\mathbb{Z}) \to \kappa^* \quad \textit{and} \quad b\colon H^1(G,\mathbb{Z}) \to \kappa^*$$

be characters. Let $\mathfrak{z} \in \mathfrak{Z}$ with coordinates for each $\alpha \in C^1(G,\mathbb{Z})$ and $\gamma \in H^1(G,\mathbb{Z})$ satisfying:

- *If $\sum_{e\in E^{\mathfrak{o}}}(\alpha_e\gamma_e\ell_e - \mathfrak{m}_e\gamma_e) < 0$, then $(\mathrm{P}_{\alpha,\gamma} : \mathrm{Q}_{\alpha,\gamma}) = (1:0)$*
- *If $\sum_{e\in E^{\mathfrak{o}}}(\alpha_e\gamma_e\ell_e - \mathfrak{m}_e\gamma_e) > 0$, then $(\mathrm{P}_{\alpha,\gamma} : \mathrm{Q}_{\alpha,\gamma}) = (0:1)$;*
- *If $\sum_{e\in E^{\mathfrak{o}}}(\alpha_e\gamma_e\ell_e - \mathfrak{m}_e\gamma_e) = 0$, then $(\mathrm{P}_{\alpha,\gamma} : \mathrm{Q}_{\alpha,\gamma}) = (1 : b(\gamma)\prod_{e\in E^{\mathfrak{o}}} a_e^{\alpha_e\gamma_e})$.*

Then $Y_{\mathfrak{z}} = Y^{a,b}_{\ell,\mathfrak{m}}$ as subsets of $\mathbf{R}$.

The rest of this section is devoted to the proof of this theorem.

First we prove that the scheme $Y^{a,b}_{\ell,\mathfrak{m}}$ is included in $Y_{\mathfrak{z}}$. Let $f \in C^0(G,\mathbb{Z})$. Recall the definition of $\mathfrak{d}^{\mathfrak{m}}_f \in C^1(G,\mathbb{R})$ from Section 2: for each $e = uv$ in $\mathbb{E}$ we have that $\mathfrak{d}^{m}_f(e)$ is the ratio $\frac{f(v)-f(u)+\mathfrak{m}_e}{\ell_e}$ if integer; otherwise $\mathfrak{d}^{m}_f(e) = \lfloor\frac{f(v)-f(u)+\mathfrak{m}_e}{\ell_e}\rfloor + \frac{1}{2}$. Also, $G^{\mathfrak{m}}_f$ is the spanning subgraph of G supported on the set of edges at which $\mathfrak{d}^{\mathfrak{m}}_f$ is an integer. Recall from Subsection 3.2 the definition of

$$\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f} := \prod_{e\in E^{\mathfrak{o}}} \mathbf{P}^1_{e,\mathfrak{d}^{\mathfrak{m}}_f(e)} \subset \mathbf{R}.$$

Under the identification in Section 6.1, as a subset of $\prod_j \mathbf{P}^1_{e,j}$,

$$\mathbf{P}^1_{e,i} = \prod_{j<i}\{0_{e,j}\} \times \mathbf{P}^1_{e,i} \times \prod_{j>i}\{\infty_{e,j}\} \quad \text{if } i \in \mathbb{Z}$$

and

$$\mathbf{P}^1_{e,i} = \prod_{j<i}\{0_{e,j}\} \times \prod_{j>i}\{\infty_{e,j}\} \quad \text{if } i \in \frac{1}{2}\mathbb{Z} - \mathbb{Z}.$$

The toric arrangement $Y^{a,b}_{\ell,\mathfrak{m}}$ is supported in the union of the $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$ as f ranges in $C^0(G,\mathbb{Z})$. For a given f, the variety $Y^{\ell,\mathfrak{m}}_{a,b} \cap \mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$ is given by the equations

$$\forall \text{ oriented cycle } \gamma \text{ in } G^{\mathfrak{m}}_f,$$

$$\prod_{e\in\bar{\gamma}\cap E^{\mathfrak{o}}} b_e a_e^{\mathfrak{d}^{\mathfrak{m}}_f(e)} \prod_{e\in\gamma\cap E^{\mathfrak{o}}} \mathrm{X}_{e,\mathfrak{d}^{\mathfrak{m}}_f(e)} \prod_{e\in\bar{\gamma}\cap E^{\mathfrak{o}}} \mathrm{X}_{\bar{e},\mathfrak{d}^{\mathfrak{m}}_f(e)} = \prod_{e\in\gamma\cap E^{\mathfrak{o}}} b_e a_e^{\mathfrak{d}^{\mathfrak{m}}_f(e)} \prod_{e\in\bar{\gamma}\cap E^{\mathfrak{o}}} \mathrm{X}_{e,\mathfrak{d}^{\mathfrak{m}}_f(e)} \prod_{e\in\gamma\cap E^{\mathfrak{o}}} \mathrm{X}_{\bar{e},\mathfrak{d}^{\mathfrak{m}}_f(e)}.$$

(Here, abusing the notation, b denotes an extension of b to a character of $C^1(G,\mathbb{Z})$.)

We show first the following claim:

Claim 6.4. *We have* $Y^{a,b}_{\ell,\mathfrak{m}} \cap \mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f} = Y_{\mathfrak{z}} \cap \mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$ *for each* $f \in C^0(G,\mathbb{Z})$*. In particular,* $Y^{a,b}_{\ell,\mathfrak{m}} \subseteq Y_{\mathfrak{z}}$*.*

Proof. The second statement is a consequence of the first, as $Y^{a,b}_{\ell,\mathfrak{m}}$ is supported in the union of the $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$.

Fix $f \in C^0(G,\mathbb{Z})$. Let β be the element of $C^1(G,\mathbb{Q})$ given by $\beta_e = \frac{f(v)-f(u)+\mathfrak{m}_e}{\ell_e}$ for each oriented edge $e = uv \in \mathbb{E}$. Note that we have for each cycle $\gamma \in H^1(G,\mathbb{Z})$,

$$\sum_{e\in E^{\mathfrak{o}}} (\beta_e\gamma_e\ell_e - \mathfrak{m}_e\gamma_e) = 0. \tag{6.1}$$

The defining equations of $Y_{\mathfrak{z}}$ are obtained by taking $\alpha \in C^1(G,\mathbb{Z})$ and $\gamma \in H^1(G,\mathbb{Z})$, and looking at the sum $\sum_{e\in E^{\mathfrak{o}}}(\alpha_e\gamma_e\ell_e - \mathfrak{m}_e\gamma_e)$: the associated equation is

$$\mathrm{P}_{\alpha,\gamma} \prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e>0}} \mathrm{X}^{\gamma_e}_{e,\alpha_e} \prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e<0}} \mathrm{X}^{-\gamma_e}_{\overline{e},\alpha_e} = \mathrm{Q}_{\alpha,\gamma} \prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e>0}} \mathrm{X}^{\gamma_e}_{\overline{e},\alpha_e} \prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e<0}} \mathrm{X}^{-\gamma_e}_{e,\alpha_e}. \tag{6.2}$$

for the point $(\mathrm{P}_{\alpha,\gamma} : \mathrm{Q}_{\alpha,\gamma})$ given in the statement of the claim, according to the sign of the sum.

Suppose first that $\sum_{e\in E^{\mathfrak{o}}}(\alpha_e\gamma_e\ell_e - \mathfrak{m}_e\gamma_e) < 0$. In this case, the equation corresponding to (α,γ) is

$$\prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e>0}} \mathrm{X}^{\gamma_e}_{e,\alpha_e} \prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e<0}} \mathrm{X}^{-\gamma_e}_{\overline{e},\alpha_e} = 0.$$

From Equation (6.1), we infer the existence of $e \in E^{\mathfrak{o}}$ for which $\alpha_e\gamma_e < \beta_e\gamma_e$. Then $\gamma_e \neq 0$. If $\gamma_e > 0$ then $\alpha_e < \mathfrak{d}^{\mathfrak{m}}_f(e)$, and if $\gamma_e < 0$ then $\alpha_e > \mathfrak{d}^{\mathfrak{m}}_f(e)$. By definition, each point on $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$ has coordinates satisfying $\mathrm{X}_{\overline{e},i} = 0$ for $i > \mathfrak{d}^{\mathfrak{m}}_f(e)$ and $\mathrm{X}_{e,i} = 0$ for $i < \mathfrak{d}^{\mathfrak{m}}_f(e)$. So, in any case, the equation corresponding to (α,γ) is automatically verified on $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$. The case where $\sum_{e\in E^{\mathfrak{o}}}(\alpha_e\gamma_e\ell_e - \mathfrak{m}_e\gamma_e) > 0$ follows similarly.

It remains thus to treat the last case, where the sum $\sum_{e\in E^{\mathfrak{o}}}(\alpha_e\gamma_e\ell_e - \mathfrak{m}_e\gamma_e)$ vanishes. It follows from Equation (6.1) that two cases can happen:

(i) For every e in the support of γ, we have $\alpha_e = \beta_e$.
(ii) There exist two edges e_1 and e_2 in $E^{\mathfrak{o}}$ such that $\alpha_{e_1}\gamma_{e_1} > \beta_{e_1}\gamma_{e_1}$ and $\alpha_{e_2}\gamma_{e_2} < \beta_{e_2}\gamma_{e_2}$.

In Case (i), all the edges e in the support of γ belong to the subgraph $G^{\mathfrak{m}}_f$, and we have $\alpha_e = \mathfrak{d}^{\mathfrak{m}}_f(e)$. If γ is an oriented cycle, by the definition of the point $(\mathrm{P}_{\alpha,\gamma} : \mathrm{Q}_{\alpha,\gamma})$, the equation corresponding to (α,γ) is precisely the equation in $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$ associated to γ. At any rate, γ is a sum of oriented cycles supported in $G^{\mathfrak{m}}_f$, whence the equation corresponding to (α,γ) is a product of equations in $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$ associated to oriented cycles.

In Case (ii), from $\alpha_{e_1}\gamma_{e_1} > \beta_{e_1}\gamma_{e_1}$ we get:

- Either $\gamma_{e_1} > 0$, in which case $\alpha_{e_1} > \mathfrak{d}^{\mathfrak{m}}_f(e_1)$ and so $\mathrm{X}_{\overline{e}_1,\alpha_{e_1}} = 0$ on $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$;
- Or $\gamma_{e_1} < 0$, in which case $\alpha_{e_1} < \mathfrak{d}^{\mathfrak{m}}_f(e_1)$ and so $\mathrm{X}_{e_1,\alpha_{e_1}} = 0$ on $\mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$.

In either case,

$$\prod_{\substack{e\in E^{\mathbf{o}}\\ \gamma_e>0}} \mathrm{x}_{\overline{e},\alpha_e}^{\gamma_e} \prod_{\substack{e\in E^{\mathbf{o}}\\ \gamma_e<0}} \mathrm{x}_{e,\alpha_e}^{-\gamma_e} = 0 \quad \text{on } \mathbf{P}_{\mathfrak{d}_f^{\mathfrak{m}}}.$$

Analogously, from $\alpha_{e_2}\gamma_{e_2} < \beta_{e_2}\gamma_{e_2}$ we get

$$\prod_{\substack{e\in E^{\mathbf{o}}\\ \gamma_e>0}} \mathrm{x}_{e,\alpha_e}^{\gamma_e} \prod_{\substack{e\in E^{\mathbf{o}}\\ \gamma_e<0}} \mathrm{x}_{\overline{e},\alpha_e}^{-\gamma_e} = 0 \quad \text{on } \mathbf{P}_{\mathfrak{d}_f^{\mathfrak{m}}}.$$

In particular, once again Equation (6.2) corresponding to (α,γ) is automatically verified on $\mathbf{P}_{\mathfrak{d}_f^{\mathfrak{m}}}$. □

It follows from the above claim that to prove Theorem 6.3 we need only show that $Y_{\mathfrak{z}}$ is contained in the union of the $\mathbf{P}_{\mathfrak{d}_f^{\mathfrak{m}}}$. This will follow from a series of claims, culminating with Claim 6.9.

Claim 6.5. *Let $\alpha \in C^1(G,\mathbb{R})$ and $\gamma\in H^1(G,\mathbb{Z})$. Then:*

- *If $\sum_{e\in E^{\mathbf{o}}}(\alpha_e\gamma_e\ell_e - \mathfrak{m}_e\gamma_e) < 0$ then*
$$Y_{\mathfrak{z}} \subset \bigcup \mathbf{P}_\beta,$$
with the union over the $\beta\in C^1(G,\mathbb{Z})$ such that $(\alpha_e-\beta_e)\gamma_e<0$ for some $e\in E^{\mathbf{o}}$.
- *If $\sum_{e\in E^{\mathbf{o}}}(\alpha_e\gamma_e\ell_e - \mathfrak{m}_e\gamma_e) > 0$ then*
$$Y_{\mathfrak{z}} \subset \bigcup \mathbf{P}_\beta,$$
with the union over the $\beta\in C^1(G,\mathbb{Z})$ such that $(\alpha_e-\beta_e)\gamma_e>0$ for some $e\in E^{\mathbf{o}}$.

Proof. We treat only the first case; the second follows with a similar argument.

Suppose first $\alpha\in C^1(G,\mathbb{Z})$. In this case,

$$\prod_{\substack{e\in E^{\mathbf{o}}\\ \gamma_e>0}} \mathrm{x}_{e,\alpha_e}^{\gamma_e} \prod_{\substack{e\in E^{\mathbf{o}}\\ \gamma_e<0}} \mathrm{x}_{\overline{e},\alpha_e}^{-\gamma_e} = 0$$

on $Y_{\mathfrak{z}}$. Then the claim follows, since for each $i\in\mathbb{Z}$,

$$\mathrm{x}_{\overline{e},i}=0 \text{ defines } \bigcup_{\substack{\beta\in C^1(G,\mathbb{Z})\\ \beta_e<i}} \mathbf{P}_\beta \quad\text{and}\quad \mathrm{x}_{e,i}=0 \text{ defines } \bigcup_{\substack{\beta\in C^1(G,\mathbb{Z})\\ \beta_e>i}} \mathbf{P}_\beta.$$

In the general case, let $\tilde\alpha\in C^1(G,\mathbb{Z})$ satisfying, for each $e\in E^{\mathbf{o}}$,

$$\tilde\alpha_e = \begin{cases} \lceil\alpha_e\rceil & \text{if } \gamma_e\le 0\\ \lfloor\alpha_e\rfloor & \text{if } \gamma_e>0.\end{cases}$$

Clearly,

$$\sum_{e\in E^{\mathbf{o}}}\tilde\alpha_e\gamma_e\ell_e \le \sum_{e\in E^{\mathbf{o}}}\alpha_e\gamma_e\ell_e < \sum_{e\in E^{\mathbf{o}}}\mathfrak{m}_e\gamma_e.$$

We may thus apply what we have just proved to conclude that

$$Y_{\mathfrak{z}} \subset \bigcup \mathbf{P}_{\beta},$$

where the union is over the $\beta \in C^1(G,\mathbb{Z})$ such that $(\tilde{\alpha}_e - \beta_e)\gamma_e < 0$ for some $e \in E^{\mathfrak{o}}$. But $(\tilde{\alpha}_e - \beta_e)\gamma_e < 0$ if and only if $(\alpha_e - \beta_e)\gamma_e < 0$. □

Claim 6.6. *Let $\alpha \in C^1(G, \frac{1}{2}\mathbb{Z})$ such that $Y_{\mathfrak{z}}$ has a point on the interior of $\mathbf{P}_{\alpha}$. Then, for each $\gamma \in H^1(G,\mathbb{Z})$, we have*

$$\Big|\sum_{e\in E^{\mathfrak{o}}}(\alpha_e\gamma_e\ell_e - \mathfrak{m}_e\gamma_e)\Big| \leq \frac{1}{2}\Big(\sum_{\substack{e\in E^{\mathfrak{o}}\\ \alpha_e\notin\mathbb{Z}}} \ell_e|\gamma_e|\Big).$$

Proof. Suppose by contradiction that

$$\Big|\sum_{e\in E^{\mathfrak{o}}}(\alpha_e\gamma_e\ell_e - \mathfrak{m}_e\gamma_e)\Big| > \frac{1}{2}\Big(\sum_{\substack{e\in E^{\mathfrak{o}}\\ \alpha_e\notin\mathbb{Z}}} \ell_e|\gamma_e|\Big)$$

for a certain $\gamma \in H^1(G,\mathbb{Z})$. Suppose first $\sum_{e\in E^{\mathfrak{o}}}(\alpha_e\gamma_e\ell_e - \mathfrak{m}_e\gamma_e) \geq 0$. Then

$$\sum_{\substack{e\in E^{\mathfrak{o}}\\ \alpha_e\in\mathbb{Z}}}\alpha_e\gamma_e\ell_e + \sum_{\substack{e\in E^{\mathfrak{o}}\\ \alpha_e\notin\mathbb{Z}\\ \gamma_e\geq 0}}(\alpha_e - 1/2)\gamma_e\ell_e + \sum_{\substack{e\in E^{\mathfrak{o}}\\ \alpha_e\notin\mathbb{Z}\\ \gamma_e<0}}(\alpha_e + 1/2)\gamma_e\ell_e - \sum_{e\in E^{\mathfrak{o}}}\mathfrak{m}_e\gamma_e > 0.$$

Let $\tilde{\alpha} \in C^1(G,\mathbb{Z})$ satisfying, for each $e \in E^{\mathfrak{o}}$,

$$\tilde{\alpha}_e = \begin{cases} \alpha_e & \text{if } \alpha_e \in \mathbb{Z} \\ \alpha_e - \frac{1}{2} & \text{if } \alpha_e \notin \mathbb{Z} \text{ and } \gamma_e \geq 0 \\ \alpha_e + \frac{1}{2} & \text{if } \alpha_e \notin \mathbb{Z} \text{ and } \gamma_e < 0. \end{cases}$$

Applying Claim 6.5, we get that

$$Y_{\mathfrak{z}} \subset \bigcup \mathbf{P}_{\beta}$$

where the union is over the $\beta \in C^1(G,\mathbb{Z})$ such that $(\tilde{\alpha}_e - \beta_e)\gamma_e > 0$ for some $e \in E^{\mathfrak{o}}$. For any such β we have that either $\beta_e < \alpha_e - 1/2$ or $\beta_e > \alpha_e + 1/2$ for some $e \in E^{\mathfrak{o}}$. But in neither case there is a point on $\mathbf{P}_{\beta}$ lying on the interior of $\mathbf{P}_{\alpha}$, contradicting the hypothesis on $Y_{\mathfrak{z}}$.

The case where $\sum_{e\in E^{\mathfrak{o}}}(\alpha_e\gamma_e\ell_e - \mathfrak{m}_e\gamma_e) \leq 0$ is treated similarly. □

Claim 6.7. *Let $\alpha \in C^1(G,\mathbb{Z})$ be such that $Y_{\mathfrak{z}}$ has a point on the interior of $\mathbf{P}_{\alpha}$. Then there exists $f \in C^0(G,\mathbb{Z})$ such that $\alpha = \mathfrak{d}^{\mathfrak{m}}_f$.*

Proof. By Claim 6.6, since α is integer valued, we get

$$\forall\, \gamma \in H^1(G,\mathbb{Z}), \qquad \sum_{e\in E^{\mathfrak{o}}}(\alpha_e\gamma_e\ell_e - \mathfrak{m}_e\gamma_e) = 0.$$

Define $\beta \in C^1(G,\mathbb{Z})$ by setting $\beta_e := \alpha_e\ell_e - \mathfrak{m}_e$ for each $e \in \mathbb{E}$. Then

$$\sum_{e\in E^{\mathfrak{o}}} \beta_e\gamma_e = 0$$

for every $\gamma \in H^1(G,\mathbb{Z})$. This guarantees the existence of a function $f \in C^0(G,\mathbb{Z})$ such that $\beta_e = f(v) - f(u)$ for each oriented edge $e = uv$ in $\mathbb{E}$. Equivalently,

$$\alpha_e = \frac{f(v) - f(u) + \mathfrak{m}_e}{\ell_e}$$

for each $e = uv$ in $\mathbb{E}$, or $\alpha = \mathfrak{d}^{\mathfrak{m}}_f$. □

A weaker statement holds for the general case of an $\alpha \in C^1(G, \frac{1}{2}\mathbb{Z})$ which is not necessarily integral. For the claim we use the following lemma.

Lemma 6.8. *Let $G = (V,E)$ be a connected loopless graph and $\mathfrak{o}$ an orientation. Let $\beta \in C^1(G,\mathbb{Z})$ and $h\colon E \to \mathbb{Z}_{\geq 0}$. Suppose that for each $\gamma \in H^1(G,\mathbb{Z})$, we have*

$$\Big|\sum_{e\in E^{\mathfrak{o}}} \beta_e\gamma_e\Big| \leq \sum_{e\in E^{\mathfrak{o}}} h(e)|\gamma_e|.$$

Then there exists an element $\eta \in C^1(G,\mathbb{Z})$ that verifies the following two properties:

- *for each $e \in \mathbb{E}$, we have $|\eta_e| \leq h(e)$;*
- *for each $\gamma \in H^1(G,\mathbb{Z})$, we have*

$$\sum_{e\in E^{\mathfrak{o}}} \beta_e\gamma_e = \sum_{e\in E^{\mathfrak{o}}} \eta_e\gamma_e.$$

The proof of this lemma is given in Section 6.3. It will be used in the proof of Claim 6.9 below.

Claim 6.9. *Let $\alpha \in C^1(G, \frac{1}{2}\mathbb{Z})$ such that $Y_{\mathfrak{z}}$ has a point on the interior of $\mathbf{P}_\alpha$. Then there exists $f \in C^0(G,\mathbb{Z})$ such that $\mathbf{P}_\alpha \subseteq \mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$.*

Proof. By Claim 6.6, we have the inequality for each $\gamma \in H^1(G,\mathbb{Z})$:

$$\Big|\sum_{e\in E^{\mathfrak{o}}} (\alpha_e\gamma_e\ell_e - \mathfrak{m}_e\gamma_e)\Big| \leq \frac{1}{2}\Big(\sum_{\substack{e\in E^{\mathfrak{o}}\\ \alpha_e\notin\mathbb{Z}}} \ell_e|\gamma_e|\Big). \tag{6.3}$$

Define $\beta \in C^1(G,\mathbb{Z})$ by $\beta_e := 2\alpha_e\ell_e - 2\mathfrak{m}_e$ for each oriented edge $e \in \mathbb{E}$. Define the function $h : E \to \mathbb{Z}_{\geq 0}$ as follows:

$$\forall\, e \in E, \qquad h(e) = \begin{cases} \ell_e & \text{if } \alpha_e \notin \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}$$

By these definitions, Inequalities (6.3) can be rewritten in the form

$$\Big|\sum_{e\in E^{\mathfrak{o}}} \beta_e\gamma_e\Big| \leq \sum_{e\in E^{\mathfrak{o}}} h(e)|\gamma_e|.$$

Applying Lemma 6.8, we infer the existence of an element $\eta \in C^1(G,\mathbb{Z})$ verifying the following two properties:

- for each $e \in \mathbb{E}$, we have $|\eta_e| \leq h(e)$;
- for each $\gamma \in H^1(G,\mathbb{Z})$, we have
$$\sum_{e\in E^{\mathfrak{o}}} \beta_e\gamma_e = \sum_{e\in E^{\mathfrak{o}}} \eta_e\gamma_e.$$

This implies the existence of a function $\tilde{f} : V \to \mathbb{Z}$ such that for each oriented edge $e = uv$ in $\mathbb{E}$, we have
$$\beta_e - \eta_e = \tilde{f}(v) - \tilde{f}(u).$$
Write $\tilde{f} = 2f + \epsilon$ for $\epsilon : V \to \{0, 1\}$ and $f \in C^0(G,\mathbb{Z})$. We claim that $\mathbf{P}_\alpha \subseteq \mathbf{P}_{\mathfrak{d}^{\mathfrak{m}}_f}$.

Indeed, for each oriented edge $e = uv$ in $\mathbb{E}$, we have
$$2\alpha_e\ell_e - \eta_e - 2\mathfrak{m}_e = 2(f(v) - f(u)) + \epsilon(v) - \epsilon(u), \tag{6.4}$$
which implies that
$$\frac{f(v) - f(u) + \mathfrak{m}_e}{\ell_e} = \alpha_e - \frac{\eta_e - \epsilon(u) + \epsilon(v)}{2\ell_e}. \tag{6.5}$$
Then two cases can happen, depending on whether $\alpha_e \in \mathbb{Z}$ or not:

• If $\alpha_e \in \mathbb{Z}$, then $|\eta_e| \leq h(e) = 0$, and thus $\eta_e = 0$. In addition, from Equation (6.4) we get $\epsilon(v) - \epsilon(u) \equiv 0 \pmod 2$, which implies that $\epsilon(v) - \epsilon(u) = 0$, and hence
$$\alpha_e = \frac{f(v) - f(u) + \mathfrak{m}_e}{\ell_e} = \mathfrak{d}^{\mathfrak{m}}_f(e).$$

• If $\alpha_e \notin \mathbb{Z}$ then $\alpha_e = a + \frac{1}{2}$ for an integer a, and
$$\frac{f(v) - f(u) + \mathfrak{m}_e}{\ell_e} = a + \frac{\ell_e - \eta_e + \epsilon(u) - \epsilon(v)}{2\ell_e}.$$
Also, $|\eta_e| \leq h(e) = \ell_e$. We claim that
$$0 \leq \frac{\ell_e - \eta_e + \epsilon(u) - \epsilon(v)}{2\ell_e} \leq 1. \tag{6.6}$$
Indeed, since $|\epsilon(u) - \epsilon(v)| \leq 1$ this is obviously true if $|\eta_e| < \ell_e$. Now, if $\eta_e = \pm\ell_e$, from Equation (6.4) we get, since $\alpha_e = a + \frac{1}{2}$,
$$\epsilon(v) - \epsilon(u) \equiv \ell_e - \eta_e \equiv 0 \pmod 2.$$
This implies that $\epsilon(u) = \epsilon(v)$, and then
$$\frac{\ell_e - \eta_e + \epsilon(u) - \epsilon(v)}{2\ell_e} = 0 \text{ or } 1.$$
It follows that $\mathfrak{d}^{\mathfrak{m}}_f(e) = a + \frac{1}{2}$ if both inequalities in (6.6) are strict, whereas $\mathfrak{d}^{\mathfrak{m}}_f(e) = a$ or $\mathfrak{d}^{\mathfrak{m}}_f(e) = a + 1$ otherwise. Since $\alpha_e = a + \frac{1}{2}$, we get that either $\alpha_e = \mathfrak{d}^{\mathfrak{m}}_f(e)$ or we have $\mathfrak{d}^{\mathfrak{m}}_f(e) \in \mathbb{Z}$ and $\alpha_e = \mathfrak{d}^{\mathfrak{m}}_f(e) \pm \frac{1}{2}$.

In any case $\mathbf{P}_{e,\alpha_e} \subseteq \mathbf{P}_{e,\mathfrak{d}_f^{\mathfrak{m}}(e)}$ for each $e \in E^{\mathfrak{o}}$, and thus $\mathbf{P}_\alpha \subseteq \mathbf{P}_{\mathfrak{d}_f^{\mathfrak{m}}}$. □

The interiors of the $\mathbf{P}_\alpha$ for $\alpha \in C^0(G, \frac{1}{2}\mathbb{Z})$ form a stratification of $\mathbf{R}$. Thus the above claim finishes the proof of Theorem 6.3.

6.3. **Proof of Lemma 6.8.** In this section we prove Lemma 6.8. We proceed by induction on the quantity $\Sigma(h) := \sum_{e\in E} h(e)$. If $h = 0$, then the claim holds trivially for $\eta = 0$. Suppose now h is a nonzero function and assume the claim holds for all functions h' with $\Sigma(h') < \Sigma(h)$.

Note that for an oriented cycle γ we have

$$\sum_{e\in E^{\mathfrak{o}}} \beta_e\gamma_e = \sum_{e\in\gamma} \beta_e\gamma_e = \sum_{e\in\gamma} \beta_e \quad \text{and} \quad \sum_{e\in E^{\mathfrak{o}}} h(e)|\gamma_e| = \sum_{e\in\gamma} h(e)\gamma_e = \sum_{e\in\gamma} h(e).$$

Let A be the set of oriented edges $e \in \mathbb{E}$ for which there exists an oriented cycle γ in the graph which contains e and such that the equality holds:

$$\sum_{f\in\gamma} \beta_f = \sum_{f\in\gamma} h(f).$$

Note that it might happen for an oriented edge $e \in \mathbb{E}$ that both e and $\overline{e}$ belong to A. Let A^+ be the set of all oriented edges in A with $h(e) > 0$.

Claim 6.10. *For each oriented edge $e \in \mathbb{E}$ with $h(e) > 0$, the set A^+ contains at most one of the two oriented edges e and $\overline{e}$.*

Proof. For the sake of a contradiction, let e be an oriented edge with $h(e) > 0$, and let γ_1 and γ_2 be two oriented cycles such that

$$\sum_{f\in\gamma_i} \beta_f = \sum_{f\in\gamma_i} h(f),$$

and $e \in \gamma_1$ and $\overline{e} \in \gamma_2$. Let $\gamma = \gamma_1 + \gamma_2$. Adding the two equations above for γ_i, we get

$$\sum_{f\in E^{\mathfrak{o}}} \beta_f\gamma_f = \sum_{f\in\gamma_1} h(f) + \sum_{f\in\gamma_2} h(f).$$

Note however that $e \notin \mathrm{supp}(\gamma)$, whence

$$\sum_{f\in E^{\mathfrak{o}}} h(f)|\gamma_f| < \sum_{f\in\gamma_1} h(f) + \sum_{f\in\gamma_2} h(f).$$

Since by hypothesis

$$\sum_{f\in E^{\mathfrak{o}}} \beta_f\gamma_f \leq \sum_{f\in E^{\mathfrak{o}}} h(f)|\gamma_f|,$$

we get

$$\sum_{f\in E^{\mathfrak{o}}} h(f)|\gamma_f| < \sum_{f\in\gamma_1} h(f) + \sum_{f\in\gamma_2} h(f) = \sum_{f\in E^{\mathfrak{o}}} \beta_f\gamma_f \leq \sum_{f\in E^{\mathfrak{o}}} h(f)|\gamma_f|,$$

which is a contradiction. □

Suppose now there is an oriented edge e_0 with $h(e_0) > 0$ such that neither e_0 nor $\overline{e}_0$ belongs to A. Then for the function $\tilde{h} : E \to \mathbb{Z}_{\geq 0}$ defined by $\tilde{h}(e_0) := h(e_0) - 1$ and $\tilde{h}(e) := h(e)$ on edges $e \neq e_0$ of G, we still have the inequalities:

$$\forall \gamma \in H^1(G, \mathbb{Z}), \quad \Big|\sum_{e \in E^{\mathfrak{o}}} \beta_e \gamma_e\Big| \leq \sum_{e \in E^{\mathfrak{o}}} \tilde{h}(e)|\gamma_e|.$$

Indeed, for each $\gamma \in H^1(G, \mathbb{Z})$ write $\gamma = \sum \gamma_i$, where the γ_i are oriented cycles such that for each i and $f \in \mathbb{E}$, we have that $\gamma_{i,f} > 0$ only if $\gamma_f > 0$. Then

$$\Big|\sum_{e \in E^{\mathfrak{o}}} \beta_e \gamma_e\Big| \leq \sum_i \Big|\sum_{e \in \gamma_i} \beta_e\Big| \leq \sum_i \sum_{e \in \gamma_i} \tilde{h}(e) = \sum_i \sum_{e \in E^{\mathfrak{o}}} \tilde{h}(e)|\gamma_{i,e}| = \sum_{e \in E^{\mathfrak{o}}} \tilde{h}(e)|\gamma_e|,$$

where the existence of e_0 is used in the second inequality. Applying then the induction hypothesis to $\tilde{h}$, we conclude the proof of the theorem.

We may thus assume that $h(e) > 0$ for an oriented edge e if and only if $e \in A^+$ or $\overline{e} \in A^+$. Furthermore, by Claim 6.10, if $e \in A^+$ then $\overline{e} \notin A^+$.

The following claim completes the proof of our lemma.

Claim 6.11. *Let $\eta \in C^1(G, \mathbb{Z})$ defined by setting*

$$\eta_e := \begin{cases} h(e) & \text{if } e \in A^+, \\ -h(e) & \text{if } \overline{e} \in A^+, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{e \in E^{\mathfrak{o}}} \beta_e \gamma_e = \sum_{e \in E^{\mathfrak{o}}} \eta_e \gamma_e$$

holds for all $\gamma \in H^1(G, \mathbb{Z})$.

Proof. Notice first that

$$\sum_{e \in E^{\mathfrak{o}}} \eta_e \gamma_e = \sum_{e \in A^+} h(e) \gamma_e$$

for each $\gamma \in H^1(G, \mathbb{Z})$. Since every $\gamma \in H^1(G, \mathbb{Z})$ is a sum of oriented cycles, we need only prove the stated equality for an oriented cycle γ. Then we need to prove that

$$\sum_{e \in \gamma} \beta_e = \sum_{e \in A^+} h(e) \gamma_e.$$

Suppose by contradiction this is not the case. Up to replacing γ by $\overline{\gamma}$, if necessary, we may assume that

$$\sum_{e \in \gamma} \beta_e < \sum_{e \in A^+} h(e) \gamma_e.$$

Let $e_1,\dots,e_k$ be the set of all the oriented edges in $\gamma\cap A^+$. For each $i=1,\dots,k$, since $e_i\in A$, there is an oriented cycle γ_i with $e_i\in\gamma_i$ for which the equality

$$\sum_{e\in\gamma_i}\beta_e=\sum_{e\in\gamma_i}h(e)$$

holds. Let $\tilde\gamma=\gamma_1+\cdots+\gamma_k-\gamma$. We have

$$\begin{aligned}\sum_{e\in E^{\mathbf{o}}}\beta_e\tilde\gamma_e&=\Big(\sum_{i=1}^k\sum_{e\in\gamma_i}\beta_e\Big)-\sum_{e\in\gamma}\beta_e\\&=\Big(\sum_{i=1}^k\sum_{e\in\gamma_i}h(e)\Big)-\sum_{e\in\gamma}\beta_e.\end{aligned}$$

We may sum now over the $e\in\mathbb{E}$ such that $h(e)>0$, that is, those e such that $e\in A^+$ or $\overline{e}\in A^+$. By Claim 6.10, if $e\in A^+$ then the $\gamma_{i,e}$ are non-negative, whereas if $\overline{e}\in A^+$ then the $\gamma_{i,e}$ are nonpositive. It follows that

$$\sum_{i=1}^k\sum_{e\in\gamma_i}h(e)=\sum_{e\in A^+}h(e)\sum_{i=1}^k\gamma_{i,e}.$$

Thus

$$\sum_{e\in E^{\mathbf{o}}}\beta_e\tilde\gamma_e=\sum_{e\in A^+}h(e)\sum_{i=1}^k\gamma_{i,e}-\sum_{e\in\gamma}\beta_e=\sum_{e\in A^+}h(e)\tilde\gamma_e+\sum_{e\in A^+}h(e)\gamma_e-\sum_{e\in\gamma}\beta_e>\sum_{e\in A^+}h(e)\tilde\gamma_e.$$

In addition, if $e\in A^+$, since the $\gamma_{i,e}$ are non-negative, it follows that $\tilde\gamma_e\geq 0$ if $\gamma_e\leq 0$; but if $\gamma_e>0$ then $e=e_i$ for some i, whence $\tilde\gamma_e\geq 0$ as well. Thus

$$\sum_{e\in E^{\mathbf{o}}}h(e)|\tilde\gamma_e|=\sum_{e\in A^+}h(e)|\tilde\gamma_e|=\sum_{e\in A^+}h(e)\tilde\gamma_e.$$

It follows that

$$\sum_{e\in E^{\mathbf{o}}}\beta_e\tilde\gamma_e>\sum_{e\in E^{\mathbf{o}}}h(e)|\tilde\gamma_e|,$$

a contradiction. □

7. Mixed toric tilings IV: Degenerations of tori

The aim of this section is to prove that each arrangement of toric varieties $Y^{a,b}_{\ell,\mathfrak{m}}$ is an equivariant degeneration of the torus $\mathbf{G}_{\mathbf{m}}^{|V|-1}$.

Let $G=(V,E)$ be a connected graph without self loops

7.1. **Degenerations of $\mathbf{G_m}$.** Notations as in Sections 4 and 6. Fix $e \in \mathbb{E}$. In this section we construct a degeneration of $\mathbf{G_m}$ to the doubly infinite chain $\mathbf{R}_e$ of smooth rational curves. Furthermore, we construct "finite approximations" of this degeneration, which consist of degenerations of $\mathbf{G_m}$ to open subschemes $\mathbf{R}_e^{(n)} \subset \mathbf{R}_e$ for $n \in \mathbb{N}$, as defined below.

Let n be a nonnegative integer number. The scheme $\mathbf{R}_e^{(n)}$ is the open subscheme of $\mathbf{R}_e$ defined by

$$\mathbf{R}_e^{(n)} := \mathbf{R}_e - \bigcup_{\substack{i\in\mathbb{Z}\\ |i|\geq n+1}} \mathbf{P}^1_{e,i}.$$

Note in particular that for $n = 0$ we have $\mathbf{R}_e^{(0)} \simeq \mathbf{G_m}$.

Recall that we view $\mathbf{R}_e$ in

$$\mathbf{P}_e = \prod_{i\in\mathbb{Z}} \mathbf{P}^1_{e,i} - \{0_e, \infty_e\}$$

given by the equations $\mathrm{x}_{e,i}\mathrm{x}_{\overline{e},j} = 0$ for all $i < j$. We denoted by $(\mathrm{x}_{e,i} : \mathrm{x}_{\overline{e},i})$ the coordinates of $\mathbf{P}^1_{e,i}$ for each i, and by $0_{e,i}$ (resp. $\infty_{e,i}$) the point on $\mathbf{P}^1_{e,i}$ given by $\mathrm{x}_{e,i} = 0$ (resp. $\mathrm{x}_{\overline{e},i} = 0$). Also, 0_e (resp. ∞_e) is the point projecting to $0_{e,i}$ (resp. $\infty_{e,i}$) for each i.

Now, for each nonnegative integer n, let

$$\mathbf{P}_e^{(n)} := \Big(\mathbf{P}^1_{e,-n} - \{\infty_{e,-n}\}\Big) \times \prod_{|i|<n} \mathbf{P}^1_{e,i} \times \Big(\mathbf{P}^1_{e,n} - \{0_{e,n}\}\Big).$$

Then $\mathbf{R}_e^{(n)}$ is the closed subscheme of $\mathbf{P}_e^{(n)}$ given by the equations

$$\mathrm{x}_{e,i}\,\mathrm{x}_{\overline{e},j} = 0 \qquad \forall -n \leq i < j \leq n$$

We may view $\mathbf{R}_e^{(n)}$ as the open subscheme of $\mathbf{R}_e$ given by $\mathrm{x}_{\overline{e},-n} \neq 0$ and $\mathrm{x}_{e,n} \neq 0$, as a point on $\mathbf{R}_e$ satisfying these inequalities is a point on

$$\prod_{i<-n}\{0_{e,i}\} \times \Big(\mathbf{P}^1_{e,-n} - \{\infty_{e,-n}\}\Big) \times \prod_{|i|<n} \mathbf{P}^1_{e,i} \times \Big(\mathbf{P}^1_{e,n} - \{0_{e,n}\}\Big) \times \prod_{i>n}\{\infty_{e,i}\}$$

satisfying $\mathrm{x}_{e,i}\,\mathrm{x}_{\overline{e},j} = 0$ for all $-n \leq i < j \leq n$, and the above product can be identified with $\mathbf{P}_e^{(n)}$.

Proposition 7.1. *Let $B := \mathrm{Spec}(\kappa[[t]])$. For each $a \in \mathbf{G_m}(k)$, define $\mathfrak{X}_e(a) \subseteq \mathbf{P}_e \times B$ by*

($\star$) $\mathrm{x}_{e,i}\mathrm{x}_{\overline{e},j} = (at)^{j-i}\mathrm{x}_{\overline{e},i}\mathrm{x}_{e,j}$ *for all pairs of integers i, j with $i < j$,*

and $\mathfrak{X}_e^{(n)}(a) \subseteq \mathbf{P}_e^{(n)} \times B$ for each $n \in \mathbb{Z}_{\geq 0}$ by

($\star_n$) $\mathrm{x}_{e,i}\mathrm{x}_{\overline{e},j} = (at)^{j-i}\mathrm{x}_{\overline{e},i}\mathrm{x}_{e,j}$ *for all pairs of integers i, j with $-n \leq i < j \leq n$.*

Then $\mathfrak{X}_e(a)$ and $\mathfrak{X}_e^{(n)}(a)$ for each $n \in \mathbb{Z}_{\geq 0}$ are flat over B and satisfy:

1. *Their total spaces are regular.*
2. *Their generic fibers over B are isomorphic to $\mathbf{G_m}$.*
3. *Their special fibers over B are $\mathbf{R}_e$ and $\mathbf{R}_e^{(n)}$ for each $n \in \mathbb{Z}_{\geq 0}$, respectively.*

Proof. Property (2) follows from observing that for a point on the generic fiber, all the coordinates $\mathrm{x}_{e,i}$ and $\mathrm{x}_{\overline{e},i}$ are nonzero; once this has been proved, it follows from Equations ($\star$) and ($\star_n$) that the ratio $\mathrm{x}_{e,0}/\mathrm{x}_{\overline{e},0} \in \mathbf{G_m}$ determines all the other ratios $\mathrm{x}_{e,i}/\mathrm{x}_{\overline{e},i}$. But suppose by contradiction that for a point on the generic fiber of $\mathfrak{X}_e(a)$ we have $\mathrm{x}_{e,i} = 0$ for some i. Then it follows from Equations ($\star$) that also $\mathrm{x}_{e,j} = 0$ for every $j > i$. And, using the same equations with i and j exchanged, that $\mathrm{x}_{e,j} = 0$ for all $j < i$. This shows that all the coordinates $\mathrm{x}_{e,j}$ are zero. But the point 0_e does not lie on $\mathbf{P}_e$, a contradiction. The same argument works for when $\mathrm{x}_{\overline{e},i} = 0$, and for the scheme $\mathfrak{X}_e^{(n)}(a)$ for every n.

Property (3) follows easily, by observing that setting $t := 0$ in Equations ($\star$) (resp. ($\star_n$)) yields the equations defining $\mathbf{R}_e$ (resp. $\mathbf{R}_e^{(n)}$).

As for flatness and Property (1), it is enough to prove them at a point on the special fiber. For a point on the special fiber $\mathfrak{X}_e(a)$, equal to $\mathbf{R}_e$ by Property (2), this means a point with coordinates $\mathrm{x}_{e,m} = 0$ for all $m < \ell$ and $\mathrm{x}_{\overline{e},m} = 0$ for all $m > \ell$, for some ℓ. Furthermore, we may assume that $\mathrm{x}_{\overline{e},\ell} \neq 0$. The parameters at the point are thus the ratios $\mathrm{x}_{e,m}/\mathrm{x}_{\overline{e},m}$ for $m \leq \ell$, the ratios $\mathrm{x}_{\overline{e},m}/\mathrm{x}_{e,m}$ for $m > \ell$ and t. Equations ($\star$) become the following equations on the parameters:

$$\begin{aligned}
&\frac{\mathrm{x}_{e,m}}{\mathrm{x}_{\overline{e},m}} = (at)^{\ell-m}\frac{\mathrm{x}_{e,\ell}}{\mathrm{x}_{\overline{e},\ell}} \quad \text{for } m < \ell,\\
&\frac{\mathrm{x}_{\overline{e},m}}{\mathrm{x}_{e,m}} = (at)^{m-\ell-1}\frac{\mathrm{x}_{\overline{e},\ell+1}}{\mathrm{x}_{e,\ell+1}} \quad \text{for } m > \ell+1 \text{ and}\\
&\frac{\mathrm{x}_{e,\ell}}{\mathrm{x}_{\overline{e},\ell}}\frac{\mathrm{x}_{\overline{e},\ell+1}}{\mathrm{x}_{e,\ell+1}} = at.
\end{aligned}$$

We obtain that $\mathrm{x}_{e,\ell}/\mathrm{x}_{\overline{e},\ell}$ and $\mathrm{x}_{\overline{e},\ell+1}/\mathrm{x}_{e,\ell+1}$ are free parameters, the only ones, and thus $\mathfrak{X}_e$ is regular. Furthermore, the above last equation proves that $\mathfrak{X}_e$ is flat over B. The same argument works for $\mathfrak{X}_e^{(n)}(a)$, or by simply observing that $\mathfrak{X}_e^{(n)}(a)$ is the open subscheme of $\mathfrak{X}_e(a)$, given by $\mathrm{x}_{\overline{e},-n} \neq 0$ and $\mathrm{x}_{e,n} \neq 0$. Indeed, these inequalities impose no conditions on the generic fiber of $\mathfrak{X}_e(a)$ over B, whereas they extract the special fiber of $\mathfrak{X}_e^{(n)}(a)$ over B from that of $\mathfrak{X}_e(a)$. ☐

Observe that the families $\mathfrak{X}_e(a)$ and the $\mathfrak{X}_e^{(n)}(a)$ over B are $\mathbf{G_m}$-equivariant, in the sense that the natural action of $\mathbf{G_m}$ on $\mathbf{P}_e$, taking a point with coordinates $(x_{e,i} : x_{\overline{e},i})$ to that with coordinates $(cx_{e,i} : x_{\overline{e},i})$ for each $c \in \mathbf{G_m}(k)$, induces naturally actions on $\mathfrak{X}_e(a)$ and the $\mathfrak{X}_e^{(n)}(a)$ leaving invariant their fibers over B. Also, under the identification of the general fiber of any of these families with $\mathbf{G_m}$ stated in the proof of Proposition 7.1, the action on the general fiber is that given by the group structure of $\mathbf{G_m}$.

Given in addition $\ell \in \mathbb{N}$, define $\mathfrak{X}_e(a,\ell)$ (resp. $\mathfrak{X}_e^{(n)}(a,\ell)$) as the base change of $\mathfrak{X}_e(a)$ (resp. $\mathfrak{X}_e^{(n)}(a)$) by the map $B \to B$ given by $t \mapsto t^\ell$. Of course, also $\mathfrak{X}_e(a,\ell)$ and the $\mathfrak{X}_e^{(n)}(a,\ell)$ are $\mathbf{G_m}$-equivariant families over B.

7.2. **Degeneration of $\mathbf{G}_{\mathbf{m}}^{|V|-1}$ to $Y^{a,b}_{\ell,\mathfrak{m}}$.** Let $G=(V,E)$ be a connected graph without self loops, and $\ell : E \to \mathbb{N}$ a length function. Let $a\colon C^1(G,\mathbb{Z}) \to \kappa^*$ and $b\colon H^1(G,\mathbb{Z}) \to \kappa^*$ be characters, and $\mathfrak{m} \in C^1(G,\mathbb{Z})$. Fix an orientation $\mathfrak{o}$ of the edges of the graph. Recall the notation: $a_e := a(\chi_e - \chi_{\overline{e}})$ for each $e \in \mathbb{E}$.

Recall the subscheme $Y^{a,b}_{\ell,\mathfrak{m}}$ of $\mathbf{R}$ that we introduced in Section 4, and the action of $\mathbf{G}_{\mathbf{m}}^{|V|-1}$ on $\mathbf{R}$ that leaves $Y^{a,b}_{\ell,\mathfrak{m}}$ invariant in Section 5. The aim of this section is to describe a $\mathbf{G}_{\mathbf{m}}^{|V|-1}$-equivariant degeneration of $\mathbf{G}_{\mathbf{m}}^{|V|-1}$ to $Y^{a,b}_{\ell,\mathfrak{m}}$.

More precisely, let $B := \operatorname{Spec}(\kappa[[t]])$. Let

$$\mathbf{W} := \prod_{e\in E^{\mathfrak{o}}} \mathbf{P}_e \subset \prod_{e\in E^{\mathfrak{o}}} \prod_{i\in\mathbb{Z}} \mathbf{P}^1_{e,i}.$$

And let $\mathfrak{Y}^{a,b}_{\ell,\mathfrak{m}}$ be defined in the product $\mathbf{W}\times B$ by the following set of equations:

($\star_{e,i,j}$) For each edge $e \in E^{\mathfrak{o}}$ and each pair (i,j) of integers with $i<j$:

$$\mathrm{X}_{e,i}\mathrm{X}_{\overline{e},j} = (a_e t^{\ell_e})^{j-i}\mathrm{X}_{\overline{e},i}\mathrm{X}_{e,j}.$$

($\star_{\gamma,\alpha}$) For each $\gamma\in H^1(G,\mathbb{Z})$ and each $\alpha\in C^1(G,\mathbb{Z})$:

$$\Big(b(\gamma)\prod_{e\in E^{\mathfrak{o}}} a_e^{\gamma_e\alpha_e}\Big)\cdot \prod_{e\in E^{\mathfrak{o}}}\Big(\frac{\mathrm{X}_{\overline{e},\alpha_e}}{\mathrm{X}_{e,\alpha_e}}\Big)^{\gamma_e} = t^{\sum_{e\in E^{\mathfrak{o}}}\gamma_e\alpha_e\ell_e - \mathfrak{m}_e\gamma_e}.$$

While Equations ($\star_{e,i,j}$) are clear, by Equations ($\star_{\gamma,\alpha}$) we rather mean:

($\star'_{\gamma,\alpha}$) For each $\gamma\in H^1(G,\mathbb{Z})$ and each $\alpha\in C^1(G,\mathbb{Z})$:

- If $\sum_{e\in E^{\mathfrak{o}}}\gamma_e\alpha_e\ell_e - \mathfrak{m}_e\gamma_e \geq 0$ then
$$\Big(b(\gamma)\prod_{e\in E^{\mathfrak{o}}} a_e^{\gamma_e\alpha_e}\Big)\cdot \prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e>0}} \mathrm{X}_{\overline{e},\alpha_e}^{\gamma_e}\cdot \prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e<0}} \mathrm{X}_{e,\alpha_e}^{-\gamma_e} = t^{\sum_{e\in E^{\mathfrak{o}}}\gamma_e\alpha_e\ell_e-\mathfrak{m}_e\gamma_e}\prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e>0}} \mathrm{X}_{e,\alpha_e}^{\gamma_e}\cdot \prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e<0}} \mathrm{X}_{\overline{e},\alpha_e}^{-\gamma_e}.$$

- If $\sum_{e\in E^{\mathfrak{o}}}\gamma_e\alpha_e\ell_e - \mathfrak{m}_e\gamma_e < 0$ then
$$\Big(b(\gamma)\prod_{e\in E^{\mathfrak{o}}} a_e^{\gamma_e\alpha_e}\Big)\cdot t^{\sum_{e\in E^{\mathfrak{o}}}\mathfrak{m}_e\gamma_e-\gamma_e\alpha_e\ell_e}\cdot \prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e>0}} \mathrm{X}_{\overline{e},\alpha_e}^{\gamma_e}\cdot \prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e<0}} \mathrm{X}_{e,\alpha_e}^{-\gamma_e} = \prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e>0}} \mathrm{X}_{e,\alpha_e}^{\gamma_e}\cdot \prod_{\substack{e\in E^{\mathfrak{o}}\\ \gamma_e<0}} \mathrm{X}_{\overline{e},\alpha_e}^{-\gamma_e}.$$

For each $v\in V$ (resp. $e\in\mathbb{E}$), let $\mathbf{G}_{\mathbf{m},v}$ (resp. $\mathbf{G}_{\mathbf{m},e}$) be a copy of $\mathbf{G}_{\mathbf{m}}$. In Subsection 7.1 we defined an action of $\mathbf{G}_{\mathbf{m},e}$ on $\mathbf{P}_e$ for each $e\in\mathbb{E}$. It induces an action of $\mathbf{G}_{\mathbf{m}}^E := \prod_{e\in E^{\mathfrak{o}}}\mathbf{G}_{\mathbf{m},e}$ on $\mathbf{W}$ and on $\mathbf{W}\times B$.

There is a natural map of group schemes

$$\mathbf{G}_{\mathbf{m}}^V \to \mathbf{G}_{\mathbf{m}}^E, \tag{7.1}$$

where $\mathbf{G}_{\mathbf{m}}^V = \prod_{v\in V}\mathbf{G}_{\mathbf{m},v}$, defined by sending $(c_v\,|\,v\in V)$ to $(\mu_e\,|\,e\in E^{\mathfrak{o}})$, where $\mu_e := c_v/c_u$ for each $e=uv\in E^{\mathfrak{o}}$. Since G is connected, the kernel of the homomorphism is $\mathbf{G}_{\mathbf{m}}$, embedded diagonally in $\mathbf{G}_{\mathbf{m}}^V$. It thus follows that $\mathbf{G}_{\mathbf{m}}^V/\mathbf{G}_{\mathbf{m}}$ acts naturally on $\mathbf{W}$ and on $\mathbf{W}\times B$. Furthermore, it leaves $\mathfrak{Y}_{\ell,\mathfrak{m}}^{a,b}$ invariant.

Theorem 7.2. *$\mathfrak{Y}_{\ell,\mathfrak{m}}^{a,b}$ is naturally a subscheme of the fibered product $\prod_{e\in E^{\mathfrak{o}}}\mathfrak{X}_e(a_e,\ell_e)$ over B. It has generic fiber over B isomorphic to $\mathbf{G}_{\mathbf{m}}^V/\mathbf{G}_{\mathbf{m}}$, and special fiber equal to $Y_{\ell,\mathfrak{m}}^{a,b}$. Furthermore, the action of $\mathbf{G}_{\mathbf{m}}^V/\mathbf{G}_{\mathbf{m}}$ on $\mathfrak{Y}_{\ell,\mathfrak{m}}^{a,b}$ is identified with the action given by the group structure on the generic fiber and the action defined in Subsection 5.1 on the special fiber.*

Proof. Equations $(\star_{e,i,j})$ are those of the fibered product $\prod_{e\in E^{\mathfrak{o}}}\mathfrak{X}_e(a_e,\ell_e)$ over B, so the first statement is clearly true. As for the second statement, observe that setting $t=0$ in Equations $(\star_{e,i,j})$ and Equations $(\star'_{\gamma,\alpha})$ yield the equations for $Y_{\ell,\mathfrak{m}}^{a,b}$ in $\mathbf{W}$, as we proved in Theorem 6.3. This shows that the special fiber of $\mathfrak{Y}_{\ell,\mathfrak{m}}^{a,b}/B$ is indeed $Y_{\ell,\mathfrak{m}}^{a,b}$. Also, that the action of $\mathbf{G}_{\mathbf{m}}^V/\mathbf{G}_{\mathbf{m}}$ on the special fiber coincides with that defined in Subsection 5.1 on $Y_{\ell,\mathfrak{m}}^{a,b}$ is clear, by comparing the two definitions.

As for the generic fiber, when $t\neq 0$, Equations $(\star_{e,i,j})$ imply that all coordinates $\mathrm{x}_{e,i}$ and $\mathrm{x}_{\overline{e},i}$ are nonzero, and the ratios $\mathrm{x}_{e,i}/\mathrm{x}_{\overline{e},i}$ are determined by the ratios $\mathrm{x}_{e,0}/\mathrm{x}_{\overline{e},0}$ for $e\in E^{\mathfrak{o}}$, as we have seen in the proof of Proposition 7.1. Equations $(\star'_{\gamma,\alpha})$ are thus equivalent to Equations $(\star_{\gamma,\alpha})$. And those equations follow from the Equations $(\star_{\gamma,0})$, given the expressions for the ratios $\mathrm{x}_{e,i}/\mathrm{x}_{\overline{e},i}$ in terms of the ratios $\mathrm{x}_{e,0}/\mathrm{x}_{\overline{e},0}$ obtained from Equations $(\star_{e,i,j})$.

The ratios $\mathrm{x}_{e,0}/\mathrm{x}_{\overline{e},0}$ describe the generic fiber of $\mathfrak{Y}_{\ell,\mathfrak{m}}^{a,b}$ as the subscheme of $\mathbf{G}_{\mathbf{m}}^E\subset\prod_{e\in E^{\mathfrak{o}}}\mathbf{P}^1_{e,0}$ given by the equations:

$$\forall\,\gamma\in H^1(G,\mathbb{Z}),\qquad b(\gamma)\cdot\prod_{e\in E^{\mathfrak{o}}}\Big(\frac{\mathrm{x}_{\overline{e},0}}{\mathrm{x}_{e,0}}\Big)^{\gamma_e}=t^{-\sum_{e\in E^{\mathfrak{o}}}\mathfrak{m}_e\gamma_e}.$$

Consider the map of group schemes

$$\mathbf{G}_{\mathbf{m}}^E\longrightarrow\mathrm{Hom}(H^1(G,\mathbb{Z}),\mathbf{G}_{\mathbf{m}}),$$

which on the level of points is given by sending $\underline{\mu}=(\mu_e\,|\,e\in E^{\mathfrak{o}})$ of $\mathbf{G}_{\mathbf{m}}^E$ to the morphism $\phi_{\underline{\mu}}$ defined by

$$\phi_{\underline{\mu}}(\gamma):=\prod_{e\in E^{\mathfrak{o}}}\mu_e^{\gamma_e}\qquad\text{for each }\gamma\in H^1(G,\mathbb{Z}).$$

Its kernel is naturally identified with the image of the map of group schemes $\mathbf{G}_{\mathbf{m}}^V\to\mathbf{G}_{\mathbf{m}}^E$ given in (7.1). The kernel of the latter is isomorphic to $\mathbf{G}_{\mathbf{m}}$ via the diagonal embedding.

Thus the natural action (coordinate by coordinate) of $\mathbf{G}_{\mathbf{m}}^E$ on $\mathbf{G}_{\mathbf{m}}^E$ induces an action of the group scheme $\mathbf{G}_{\mathbf{m}}^V/\mathbf{G}_{\mathbf{m}}$ on $\mathbf{G}_{\mathbf{m}}^E$ which is transitive on the subscheme defined by the equations

$$\forall\,\gamma\in H^1(G,\mathbb{Z}),\qquad\prod_{e\in E^{\mathfrak{o}}}\mu_e^{\gamma_e}=b'(\gamma)$$

for every character $b' \colon H^1(G,\mathbb{Z}) \to \kappa^*$. It follows that the generic fiber of $\mathfrak{Y}^{a,b}_{\ell,\mathfrak{m}}$ is a split torsor over $\mathbf{G}^V_{\mathbf{m}}/\mathbf{G}_m$. $\square$

By substituting $\prod_{e\in E^{\mathfrak{o}}} \mathbf{P}_e^{(n_e)}$ for $\mathbf{W}$, for any given $n\colon E \to \mathbb{Z}_{\geq 0}$, and restricting the range of the i, j in Equations $(\star_{e,i,j})$ and the α_e in Equations $(\star'_{\gamma,\alpha})$ to the interval $[-n_e, n_e]$ for each $e \in E^{\mathfrak{o}}$, we obtain a degeneration of $\mathbf{G}_{\mathbf{m}}^{|V|-1}$ to a "finite approximation" of $Y^{a,b}_{\ell,\mathfrak{m}}$, namely to $Y^{a,b}_{\ell,\mathfrak{m}} \cap \mathbf{R}^{(n)}$, where $\mathbf{R}^{(n)} := \prod_{e\in E^{\mathfrak{o}}} \mathbf{R}_e^{(n_e)}$.

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CNRS - Centre de mathématiques Laurent Schwartz, École Polytechnique, France
E-mail address: omid.amini@polytechnique.edu

Instituto Nacional de Matemática Pura e Aplicada, Estrada Dona Castorina 110, 22460-320 Rio de Janeiro RJ, Brazil
E-mail address: esteves@impa.br